Getting the Ball Rolling: Voluntary Contributions to a Large-Scale Public Project

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Abstract

This paper studies dynamic voluntary contributions to large-scale projects. While agents can observe the progress of the project and their own costs of contribution, they have incomplete information about the contribution costs of others. I show that the equilibrium pattern of contributions is influenced by the interplay of two opposing incentives: First, agents prefer to free ride on others for contributions. However, second, agents also wish to encourage each other to contribute by increasing their own contributions. Main findings of the paper include: (1) Agents make concessions toward the completion of the project by increasing their contributions as the project moves forward, (2) as additional agents join the group, existing agents increase their contributions in some states and reduce it in others. In particular, agents increase their contributions in the initial stages of the project to increase its value to others and secure their future contributions as a result, (3) groups that are formed by more patient agents and that undetake larger projects tend to be larger, and (4) groups that rely on voluntary contributions tend to be too small compared to the social optimum. The empirical evidence on contributions to open source software projects supports my findings.

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1 Introduction

The successful completion of many large-scale projects depends on the continuous participation of multiple parties. Although binding contracts would be desirable in these situations, often times they are not feasible. There are typically noncontractible uncertainties associated with the projects themselves or with the participants' economic and political environments. Moreover, in the case of international projects, contracts are notoriously hard to enforce due to the lack of a supranational authority. Binding contracts, however, are not always necessary for the successful completion of large-scale long-term projects involving multiple parties.

There have been significant successes in the progress and completion of important projects that rely mostly on voluntary contributions. For instance, the U.S., Russia, Canada, Japan, Brazil along with 11 European countries began the assembly of the International Space Station in 1998, which is by far the largest international scientific and technological undertaking. When completed in about 8 years, the Station will be used by the contributing countries as a permanent laboratory and testing site. This large-scale project is expected to cost more than $10 billion, and although this amount has been initially divided among the participating countries [www.nasa.gov], changing economic and political conditions have sometimes made it hard for countries to live up to their obligations.[1] Another significant success is the development of open source software like Apache, Linux, and GNOME. These large-scale programs are developed by programmers scattered around the world, who voluntarily and freely contribute their valuable time and skills. As of late 2000, the web server Apache was used by more than 59% of active servers across all domains, and Linux as a server operating environment gained 17% market share [Johnson (2002), Lerner and Tirole (2001, 2002)].[2]

What determines the successful completion of a large-scale project that relies on voluntary contributions by several parties? While a full answer to this question is obviously quite complex and context specific, in this paper I construct a fairly general model intended to

[1] Indeed, in 1998 there was a heated discussion in the U.S. Congress regarding Russia's commitment to the project. It turned out the Russian Space Agency did not receive the funding from its government for 1997 and did not have a firm commitment for 1998, either. [www.cnn.com/allpolitics/1998/05/12]

[2] There are, of course, many other cases of large-scale projects that rely on voluntary contributions. A recent example is the current international effort to eradicate global terrorism. Similarly, countries undertake voluntary actions to restore earth's ozone layer [Murdoch and Sandler (1997)], and to clean up rivers crossing international boundaries [Sigman (2002)].

capture the following salient features of such projects: (1) They require voluntary contributions by participants, (2) they are usually divided into smaller subprojects that are to be completed in a prespecified order,[3] (3) participants possess private information about their key characteristics such as current financial and political status, or simply their outside opportunities, which fluctuate over time, and (4) the benefits of a completed project are enjoyed for an extended period of time.

The formal model builds on the discrete public good framework of Palfrey and Rosenthal (1988, 1991). There are N risk-neutral agents who work on a joint project whose full return is realized only after a sequence of stages have been completed. In each period, the agents simultaneously decide whether or not to contribute to advancing the project after commonly observing the state of the project and privately observing their own cost of contributing. The costs of contributing vary across agents and time. These costs reflect such key variables as a country's financial status, which fluctuates with market conditions, or a computer programmer's opportunity cost, which changes from time to time according to his outside options.

To address the main question, I first investigate participants' equilibrium incentives to contribute to a multi-stage project, and how these contributions change with the progress of the project and with the number of participants. I then ask what the optimal group size is for a given project, and whether the group size tends to be too small or too large from the social standpoint.

In general, contributions to a joint project suffer from the well-known free-rider problem. While agents benefit freely from others' contributions, they bear the full cost of their own efforts, leading them to rely "too much" on others. When projects require a sequence of contributions, however, a second effect, countervailing the free-riding incentive, emerges. To see this, suppose there is at first a single agent that is initially indifferent about contributing to a public project. Now, suppose this agent is informed that a second agent would be interested in contributing to the project if it were one step further along or sufficiently "mature". That is, the second agent's current valuation of the project is too small to attract his initial contribution. If the free-rider effect were the only force at play here, then the first agent would continue to remain indifferent about initiating the project. The presence of the second agent, however, can in fact break the first agent's indifference and

[3] For instance, the Space Station Project has been divided into three major phases with many subphases and the open source programs are composed of modules of smaller programs.

cause him to start the project in order to attract future contributions. In a sense, the first agent invests in the project in order to free ride on the second agent in the future. Using the terminology of Bolton and Harris (1999)[4], I call the latter effect the "encouragement" effect and examine how it interacts with the free riding incentive throughout the relationship.

The organization of the paper and a brief preview of the findings are as follows. In section 2, I present the basic model in which agents' contributions are perfect substitutes. If at least one agent contributes, then the project moves forward. Otherwise, it remains idle until the next contribution. I show that this dynamic game possesses a unique symmetric Markov Perfect Equilibrium (MPE). I find that early in the game agents make concessions toward the completion of the project through increased efforts, which highlights the presence of the positive encouragement effect. In general, there are two opposing effects on an agent's effort choice arising from increased contribution by others: On the one hand, it makes his effort less pivotal thereby facilitating the free-rider effect. On the other hand, it brings the future returns generated by greater future contributions closer thereby facilitating the encouragement effect. The latter effect mitigates the former and thus in equilibrium each agent raises his effort level.

In Section 3, I consider group size. In general, an increase in the group size has one direct and two strategic effects on agents' contribution decisions. The direct effect comes from the possible "congestion" in the utilization of the completed project such as the Space Station or clean international waters. The strategic effects are the encouragement and the well-known free riding incentives. When the project is a noncongestible public good, e.g., an open source program or the Earth's ozone layer, where one more user does not degrade individual returns, the direct effect is not present. In such cases, I find that equilibrium contributions across states are not monotonically affected by the group size. When one more agent joins the group, the original members may increase their contributions in some states and reduce it in others. In general, the encouragement effect is stronger in a larger group in the initial stages of the project leading agents to increase their contributions in response to an additional agent– simply because the gain from encouraging others is greater in a larger group. However, as the project nears completion, the encouragement effect loses strength and the free riding incentive becomes more dominant in a larger group, leading agents to lower their contributions. In fact, it is possible that an additional agent might –

[4] Although my focus and formal model are considerably different from those in Bolton and Harris (1999), both papers highlight similar dynamic effects ,as becomes clear in the analysis.

on net – slow down the progress of the project. Despite this nonmonotonicity, agents always strictly benefit from the inclusion of an additional group member. This is because the cost saving associated with an additional agent more than offsets the reduced pace of completion. Interestingly, as the group size tends to infinity, the equilibrium contributions approach the socially optimal level. This finding runs counter to that of Olson's (1965) intuition that as the group size increases, the inefficiency in public good provision also increases. Interestingly, the empirical evidence on programmers' contributions to open source software is consistent with my results on group size as I discuss at length.

When the project is a congestible public good, the congestion effect counteracts the net positive dynamic effects. In such cases, I consider the optimal group size that balances the congestion and the dynamic effects. I find that larger projects require a larger group, but – more interestingly – the optimal group size increases with agents' patience. This is because more patient agents are able to better control the free riding incentive, which allows them to include more members to expedite the completion of the project. Nonetheless, in Section 4, I show that groups that rely on voluntary contributions tend to be too small with respect to the "social optimum" where they can coordinate their actions.

Finally, in Section 5, I extend the model to incorporate complementaries in agents' efforts and I conclude in Section 6.

Related Work In a one-period incomplete information setting, Menezes et al.(2001) and Palfrey and Rosenthal (1988, 1991) consider private provision of a discrete public good. Part of my model can be considered as an extension of their analyses to a dynamic setting in which a sequence of complementary discrete public goods are produced. Palfrey and Rosenthal (1994) extend their one-period setting to an infinitely-repeated game where agents receive the full return in each period if the public good is produced in that period and they confront theoretical findings with experimental results. While my theoretical model is also infinite horizon, agents receive the full benefit only when the project is completed.[5] Bagnoli and Lipman (1989) and Palfrey and Rosenthal (1984) show the possibility of efficient provision of a discrete public good in a one-period complete information model. Admati and Perry (1991) consider a dynamic contribution game with complete information, where two agents take turns to contribute to a discrete public good with a fixed cost. Their point is that the free-rider incentive is so severe in many cases that the public good is not

[5] Also see Bac (1996), McMillan (1979), and Pecorino (1999) who consider the possibility of cooperation in infinitely-repeated game settings of public good provision under different assumptions.

provided. Marx and Matthews (2000) analyze a similar framework but , like mine, allow for simultaneous contributions within the same period and more than two agents. They show that allowing simultaneous contributions can yield more positive results than those of Admati and Perry. Fershtman and Nitzan (1991) develop a differential game model of continuous public good provision. They find that in steady state, agents not only free ride on the current contributions but also on the future ones, and therefore the MPE yields a lower level of public good than does the open-loop equilibrium. Somewhat surprisingly, I find that while agents free ride on current contributions, they view future contributions as complementary in my model. I elaborate on this point below.

2 The Model

Consider a dynamic extension of Palfrey and Rosenthal (1988, 1991). There are N agents who work on a joint project with H subprojects where H is a natural number. The progress of the project is observed by all agents. Let h be the state variable indicating that $(h-1)$ subprojects have been completed, and let $r(h)$ denote the corresponding *per period* return to each agent in state h. The project yields its full return, $v(N) > 0$, after all subprojects are completed. Namely,

$$Assumption\ 1 : r(h) = \begin{cases} v(N)\ for\ h > H \\ 0\ \ for\ h \leq H \end{cases}$$

where $v(N)$ can be decreasing, constant, or increasing in N, capturing different types of projects and how participants share the benefits of the project. For instance, in the Space Station project, each country's benefit would certainly decrease with an additional user, as there is a limited place to perform experiments in the Station. In such cases, I will assume $v(N)$ is decreasing, and call such projects "congestible". For the projects of software programs and the restoration of the ozone layer, it seems more reasonable to assume no congestion so that $v(N)$ is constant. It is also possible that the presence of more participants might increase individual payoff, $v(N)$, if there are positive externalities in utilizing the finished project.

Assumption 1 indicates that there are no immediate returns before completing the project. While certain projects might yield intermediate payoffs, as long as the cumulative payoff with the progress of the project weakly increases at a weakly increasing rate, the qualitative results in the paper will hold. Furthermore, by making each subproject *ex-ante* identical, I abstract from the latter complication to better highlight the dynamics in the model . Zero return, however, is pure normalization.

The interaction among agents is modeled as an infinite horizon Markov game and the MPE is the solution concept.[6] Informally, a MPE consists of strategies for each agent constituting a perfect equilibrium for all payoff relevant histories described by the state h. The Markov behavior requires that agents base their decisions only on the progress of the project. This behavior also has intuitive appeal in my setting as I wish to investigate the effects of the progress of the project on agents' incentives to contribute.[7] In each period, agents observe the state of the project and simultaneously decide whether or not to contribute. Like in Rosenthal and Palfrey's setting, contribution is a 0-1 decision denoted by the indicator variable s.[8] In the basic model, I assume as long as one agent contributes, the project moves to the next state. Otherwise, it waits for one period, and the state does not change.[9] The process repeats itself next period with the updated state and with new cost draws for agents. This type of production function assumes agents' contributions are perfect substitutes. Nonetheless, I will show that agents' equilibrium contributions exhibit some intertemporal complementarity and discuss a more general production function in Section 5. The cost of contribution or effort comes from a twice continuously differentiable cumulative distribution, $F(c)$, which is independently and identically distributed over time and across agents.[10] The support of distribution is $[0, \bar{c}]$ with $\bar{c} > 0$ and $F'(c) = f(c) > 0$. Agents discount the future returns and costs by $\delta \in (0, 1)$.[11] Let $W_i(h, c_i)$ be agent i's

[6] See, for example, Fudenberg and Tirole (1991) ch. 13 for a review.

[7] In theory, agents can use less "forgiving" strategies for a deviation than the Markov ones. However, as with the international projects, each country will have hard time to punish others if there is a defection, either because governments change, or because it is hard to prove defection in the presence of incomplete information. Furthermore, focusing on Markov strategies allows me to determine the minimal level of cooperation agents can achieve.

[8] In Appendix B, I also study a continuous contribution extension and show that the same qualitative results as in the discrete case would arise.

[9] This is clearly an approximation. In reality, working on a project, for instance, does not guarantee its immediate progress. A more realistic setting would be where given that someone works on the project, it moves to the next state with some probability. Conversely, it might also be that if no agent contributes, the project might depreciate. These are interesting extensions. However, I conjecture that as long as these transition probabilities remain constant over states, qualitative results in the paper continue to hold.

[10] This assumption, aside from being realistic, rules out the strategic learning among agents in a public good context, which is the topic of Bliss and Nalebuff (1984) and Gradstein (1992), among others. In these papers, agents have private information about their costs of contribution and these costs do not change over time. Thus, as time passes by, agents leak information to others often resulting in inefficient delays in contributions. Although this type of strategic learning is an important element of public good provision, here I abstract from this aspect to better focus on the effects of the progress of the project on agents' contribution pattern.

[11] Assuming that agents are ex-ante symmetric clearly does not fit the examples in the Introduction, where it seems more reasonable to consider heterogenous cost distributions and discount factors. While I leave the

value function when he realizes c_i and the project is in state h. Suppose that $\lambda^*_{-i}(h)$ is the equilibrium probability that at least one agent other than i contributes. Agent i, then, solves the following dynamic program:[12]

$$W_i(h, c_i) = \max_{s_i \in \{0,1\}} \left\{ \begin{array}{c} r(h) + s_i[-c_i + \delta W_i(h+1)] \\ +(1-s_i)\delta[\lambda^*_{-i}(h)W_i(h+1) + (1-\lambda^*_{-i}(h))W_i(h)] \end{array} \right\} \quad (1)$$

where $W_i(h) \equiv E_c[W_i(h,c)]$ and E_c is the expectation operator with respect to c.[13]

According to (1), agent i decides whether or not to contribute to the project upon observing the state of the project and his own current cost. If he contributes, i.e., $s_i = 1$, he bears the full cost of his effort in which case the project moves forward with certainty and he receives the discounted expected continuation value, $\delta W_i(h+1)$. However, if he decides not to, i.e. $s_i = 0$, then he will have to rely on others for contribution. In this instance, he conjectures the expected probability that at least one other agent will contribute, which in turn determines his discounted future expected continuation value, $\delta[\lambda^*_{-i}(h)W_i(h+1)+(1-\lambda^*_{-i}(h))W_i(h)]$. In each decision though, he receives the return, $r(h)$. Define the following cutpoint:

$$x_i(h) \equiv [1 - \lambda^*_{-i}(h)]\delta\Delta W_i(h) \quad (2)$$

where $\Delta W_i(h) \equiv W_i(h+1) - W_i(h)$.

Equation (1) reveals that agent i's equilibrium strategy simply has the following cut-off property:[14]

$$s^*_i(h,c) = \left\{ \begin{array}{c} 1, if\ c \leq x_i(h) \\ 0, otherwise \end{array} \right. \quad (3)$$

We say that as $x_i(h)$ increases, so does agent i's contribution or effort.[15][16] According to (2), this effort is monotonic in the likelihood of being the pivotal contributor and in the

generalization for future work, I believe the present model is rich enough to identify the main forces in the generalization.

[12] Note that agents incur the cost at the time they volunteer in this model. This might create *ex-post* inefficiency in equilibrium as contributions in excess of one are wasted. However, below we will see that agents take this possibility into account when contributing.

[13] Since time enters only through discounting and the cost distribution is stationary over time, I drop the time index throughout the analysis. Also, below I show that the value functions exist and are well-behaved.

[14] Since $c \in [0, \bar{c}]$, I adopt the convention that $x_i(h) = 0$ for $x_i(h) < 0$ and $x_i(h) = \bar{c}$ for $x_i(h) > \bar{c}$.

[15] This is in *ex-ante* contribution sense. As the cutpoint increases, agent i becomes more likely to make a contribution.

[16] Agent i is indifferent about whether or not to contribute if he draws a cost $c = x_i(h)$. Given the cost distribution is continuous, the probability of this event is zero and thus it has no effect on the analysis.

discounted net future gains. The net return, $\Delta W_i(h)$, can be also interpreted as the shadow value of the progress. The complicated dynamic interaction endogenously determines this shadow value and thus the effort levels in each state.

Since agents adopt payoff-relevant strategies only, it is clear from (1) that for $h > H$, $\Delta W_i(h) = 0$ and thus the unique equilibrium is $x_i(h) = 0$ for all i. That is, no agent would take action once the project is completed. This further implies the boundary condition that for $h > H$:

$$W_i(h) = \frac{v(N)}{1-\delta} \tag{4}$$

Also, since agents are *ex-ante* symmetric, it seems reasonable to focus on the symmetric equilibrium for $h \leq H$ as well. Suppose $x_i(h) = x(h)$ for all i in equilibrium. To determine the MPE, I first define the following function and record its properties:

$$B(x;\delta) \equiv \frac{1}{\delta}\frac{x}{[1-F(x)]^{N-1}} - \int_0^x [1-F(c)]dc \tag{5}$$

Lemma 1. *(a)For $h \leq H$, $\overline{W}_i(h+1) = B(x(h);\delta)$ and $\overline{W}_i(h) = B(x(h);1)$ where $\overline{W}_i(h) = (1-\delta)W_i(h)$.*

(b) $B'(x;\delta) > 0$ for $x \in (0, \overline{c}]$, and $B'(0;\delta) = \frac{1}{\delta} - 1$.

Proof. All proofs are contained in the appendix. ■

Eq.(5) and Lemma 1 are interpreted most easily by $\overline{W}_i(h) = B(x(h);1)$. Here, $\overline{W}_i(h)$ can be viewed as the per period "option" value of the project to agent i, if it were sold in state h. Furthermore, integrating the second term by parts and arranging terms, one can rewrite (5) as $B(x;1) \equiv \frac{x}{[1-F(x)]^{N-1}}[1-[1-F(x)]^N] - \int_0^x cdF(c)$. Thus, the equilibrium condition $\overline{W}_i(h) = B(x(h);1)$ is equivalent to

$$\overline{W}_i(h) + \int_0^{x(h)} cdF(c) = \delta\Delta W_i(h)[1-[1-F(x(h))]^N] \tag{6}$$

where $\frac{x}{[1-F(x)]^{N-1}} = \delta\Delta W_i(h)$ follows from (2).

Given all agents choose a cutoff x, the l.h.s. of eq.(6) is the forgone option value plus the expected cost of contribution for agent i. In equilibrium, agent i equates this total cost to the expected net benefit of moving to the next state on the r.h.s. of (6). Figure 1 further illustrates the backward induction described in Lemma 1. It graphs the functions $B(x;\delta)$ and $B(x;1)$. By equating the future average option value to $B(x;\delta)$, it first finds the current cutoff. It then evaluates $B(x;1)$ to determine the average option value as of the current state.

FIGURE 1 ABOUT HERE

Armed with Lemma 1, the following proposition characterizes the equilibrium:

Proposition 1. *There exists a unique symmetric MPE with the following properties: For* $h \leq H$,

(i) $x(h) > 0$ *for all* h.

(ii) $x(h)$ *is strictly increasing in* h.

(iii) $W_i(h)$ *is strictly increasing at an increasing rate in* h*. That is,* $\Delta W_i(h) > 0$ *and* $\Delta W_i(h) > \Delta W_i(h-1)$.

Part (i) has two implications: First, the project of any size takes off with positive probability despite no immediate returns. In equilibrium, agents are willing to take current losses for future returns. Second, the project is finished with some positive probability. In Section 4, I show that the equilibrium rate of completion is too slow from the social standpoint.

The intuition behind part (ii) is more involved. In equilibrium, the cutpoint for agent i given in (2) reduces to:

$$x_i(h) = [1 - F(x(h))]^{N-1}\delta\Delta W_i(h) \tag{7}$$

Everyone else's increasing effort has two opposing effects on one's decision summarized on the r.h.s. of (7): On the one hand, it makes his effort less pivotal and thus facilitates the free-rider effect. Formally, as $x(h)$ increases $[1 - F(x(h))]^{N-1}$ becomes smaller. On the other hand, it brings future returns generated by greater future efforts closer and creates an encouragement effect given by $\delta\Delta W_i(h)$. As part (ii) of Proposition 1 indicates, the latter effect mitigates the former and therefore each agent raises his effort as the project moves forward. Put differently, agents view their own efforts as strategic substitutes for others' current efforts and as strategic complements to the future efforts. This is so in equilibrium even though there are no complementaries in the production function. Interestingly, the observation runs counter to that of Fershtman and Nitzan (1991). They analyze the steady state provision of a public good in a complete information differential game context with linear Markov strategies. The main finding is that agents free ride not only on the current contributors but also on the future ones. Namely, in Fershtman and Nitzan's setting, agents consider their contributions as strategic substitutes not only for current contributions of others but also for future contributions. Therefore, the MPE yields a lower steady state level of public good than does the open-loop equilibrium in which agents commit to a pattern of contributions and not respond to the accumulated amount. As Wirl (1996) indicates, the main source of this finding is not that agents use Markov strategies but they are restricted to use linear strategies. He concludes that if other smooth nonlinear strategies are considered, one will see that there are multiple steady state equilibria, some of which yield a higher level of public good than does the open-loop equilibrium.[17]

Since the present model is stationary, we can easily determine the average waiting time in each state. Note that the equilibrium probability that the project will move from state h to $h+1$ is given by $\lambda^*(h) \equiv 1 - [1 - F(x(h))]^N$. Thus, the average waiting time in state h is $w(h) = \frac{1}{\lambda^*(h)}$. Part (ii) of Proposition 1 implies that this waiting time shrinks as the project moves forward.

The last part of Proposition 1 reveals that the value agents' attach to the project increases at an increasing rate over states. In other words, for $h \leq H$, the shadow value of progress is strictly positive and it increases with the progress.

Proposition 1 also has another implication. In the model, I assume that the progress or state is publicly observable. However, for some projects, it might be privately known only by the current contributors. Proposition 1 then implies that contributors have a strict incentive to inform others about the progress as soon as they contribute–both to avoid duplication of efforts and to secure greater future efforts. Thus, as long as contributions can be costlessly communicated with noncontributors, the results with observable state hold. For consistency, I will continue to assume that the state is publicly observed.

[17] The observation in Proposition 1 accords with Bolton and Harris (1999), who consider a continuous time two-armed bandit model of strategic experimentation where agents privately choose whether to invest in a safe or risky asset each period. While they also identify free-riding and encouragement effects in this social learning environment, my focus and model are significantly different from theirs. In particular, I consider the issue of private contributions to a large-scale public project, and also deal with the optimal group size and complementary contributions.

3 The Effects of Group Size

In general, an increase in the group size has three effects on agents' contribution decisions: one direct effect due to the (possible) change in the payoff after the project is completed, and two strategic effects due to the dynamics before the project is completed. To better understand these effects and distinguish between different types of projects, I first consider noncongestible projects for which the direct effect is not present, and then consider other projects for which it is present.

3.1 Noncongestible Projects

Imagine that participants are contributing toward projects like open source software or restoration of the ozone layer, where an additional user does not degrade one's final payoff, i.e., $v(N) = v$ for all N. In this case, an increase in the group size has two opposing strategic effects on agents' contributions. First, it worsens the well-known free-rider incentive as there are now more agents to free ride on. However, second, it also strengthens the encouragement incentive, as by increasing one's own contribution today, an agent can attract greater future contributions in a larger group. Furthermore, since there are more future contributions to be made in the initial stages of the project, it is intuitive that the latter incentive will be stronger in those stages while losing its steam with the progress of the project. Thus, it is reasonable to conjecture that an increase in the group size results in an increase in equilibrium contributions in the initial stages of the project and a decrease in the mature stages. I confirm this conjecture in

Proposition 2. *Suppose* $v(N) = v$ *for all* N. *For* $N \geq 1$,

(i) $x(H; N+1) < x(H; N)$,

(ii) For sufficiently large H, *there exists* h_0 such that for $h < h_0$, $x(h; N+1) > x(h; N)$,

(iii) For $h \leq H$, $\lim_{N\to\infty} x(h; N) = 0$, *and* $\lim_{N\to\infty} \lambda^*(h; N) = 1$.

First two parts of Proposition 2 imply that agents' contributions are not uniformly affected by an increase in group size. Part (i) reveals that each agent reduces his equilibrium contribution to the last subproject in a larger group. Intuitively, since no future contributions are needed in the final stage of the project, the encouragement incentive vanishes and the free-rider effect leads to a reduction in contributions. Part (ii) of Proposition 2 shows that when the project is sufficiently large, the encouragement incentive becomes sufficiently strong in the early stages of the project and dominates the free-rider incentive. As a result, each agent increases his own contribution in a larger group in the initial stages of the projects. To gain intuition about how contributions change in the intermediate states with the group size, I consider a numerical example as depicted in Figure 2.

FIGURE 2 ABOUT HERE

In the example, there are 18 subprojects, i.e., $H = 18$.[18] The cost of effort is distributed uniformly in $[0, 3]$ and the project yields a value $v = 2$ to each agent upon completion. Also, agents discount future returns by $\delta = .9$.

Refer to Figure 2. When the group consists of a single agent, obviously there are no free riding or encouragement incentives. Note that in this case, the agent would finish the project with certainty in the penultimate state, $h = 18$, i.e., $x(18) = \bar{c} = 3$. However, the project virtually does not take off for the states below $h = 6$, i.e., $x(h) \approx 0$ for $h \leq 6$. If there is one more agent in the group, agents encourage each other for the initial ten states by choosing higher cutpoints than the single-agent case. Thus, the project is more likely to take off with a larger group in these states. However, for the remaining eight states, the free-rider effect becomes stronger resulting in lower effort levels. Note that one more agent in the group does not mean the project will move at a faster rate in every state. For instance, in the penultimate state, the probability that the project will be finished is strictly less than 1 whereas each agent would finish it with certainty if he were alone in that state. As the group size grows further to $N = 3$, the critical state below which agents put higher efforts in a larger group goes down. This is because with more agents, the project matures faster and the free-rider effect becomes more dominant starting in an earlier state. The example also shows that agents' effort levels stay relatively steady in a larger group.

According to the parts (i) and (ii) of Proposition 2, and the example in Figure 2, it is clear that each agent's contribution in a given state is ambiguous in the group size. Even so, as the group size tends to infinity, each agent expects someone in the group will draw a cost close to 0 and thus chooses a cutpoint close to 0 as a result. Interestingly, this holds in all states as recorded in part (iii). This does not mean, however, the project would never move forward. In contrast, I show in the Appendix A that the equilibrium probability that the project moves from h to $h + 1$, i.e. $\lambda^*(h; N)$ converges to 1. The increase in the number of agents more than offsets the reduction in the cutpoint. In fact, in Section 4, I demonstrate that as $N \to \infty$, agents' welfare approaches to the socially efficient level.

[18] Figure 2 shows only the states 1 through 18 since we already know that $x(h) = 0$ for $h > 18$.

This last observation coincides with Bliss and Nalebuff's who also find in a war of attrition model of public good provision that even though the expected time of supplying the good is ambiguous in the group size, it shrinks to 0 as the group size tends to infinity. Hence, they conclude for sufficiently large population the free-rider problem disappears completely. However, the result might seem puzzling in light of Mailath and Postlewaite (1990), who, in a one-period mechanism design setting, conclude that as the group size tends to infinity, the probability of supplying the public good converges to 0, even though it should be provided with probability 1. While both Mailath and Postlewaite's and my model predict that as $N \to \infty$, agents would feel less pivotal and reduce their contributions, the difference stems from the fact that Mailath and Postlewaite assume the exogenous cost of providing the public good increases in group size faster than the free-riding incentive.

The predictions in Proposition 2 are widely consistent with the empirical studies of voluntary contributions by programmers to open source software projects, e.g. Hertel et al. (2000) for the Linux project, Koch and Schneider (2002) for GNOME project, Mockus et al. (2000) for the Apache project. These papers report an increase in the average contributions with the number of programmers, especially, in the early stages of the projects, and a decline in the mature stages.[19] The fact that more programmers need not reduce the pace of a software project was first argued by Raymond (1999), who made the assertion known as the Linus' Law that "Given enough eyeballs, all bugs are shallow." This is indeed surprising, because previously it was believed that Brooks' Law [Brooks (1995)] that states "adding people to a late project makes it later...the fact that a woman can have a baby in nine months does not imply that nine women can have a baby in one month." holds. Though the two assertions seem conflicting, they are not. In his book, Brooks also suggested an exception to his law by pointing out "... if a project's tasks are partitionable, you can divide them further and assign them to ... people who are added late to the project.", which is also the building block of Raymond's argument.

Despite the ambiguous effects of group size on agents' effort levels, agents do strictly benefit from an additional group member as I note in

Proposition 3. *Suppose $v(N) = v$ for all N. Then, for $h \leq H$, $W_i(h, N+1) > W_i(h, N)$.*

Proposition 3 shows that an increase in the number of agents is a Pareto improvement

[19] Note that the average contribution is $F(x(h))$ in my model.

in the sense that each agent has a higher welfare in each state. This seems intuitive for the states in which agents increase their effort levels as this improves the pace of the project resulting in a higher value.[20] However, when the project matures, an additional agent exacerbates the free-rider incentive and might even slow down the progress of the project. Even so, the cost saving associated with one additional agent more than offsets this reduced pace. Proposition 3 thus implies that the optimal group size for noncongestible projects is infinity. Once again this is consistent with the fact that open source projects accept contributions from programmers all over the world, rather than restricting the group to just a few.

In light of Proposition 3, I should also note that the qualitative results pertaining to congestible projects trivially hold when there are positive externalities in the utilization of the project, i.e., $v(N)$ increases in N. In particular, both the direct and the net dynamic effects are positive. However, this is not the case in what follows with congestible project.

3.2 Congestible Projects and the Optimal Group Size

When the returns of completed projects such as the Space Station and clean international waters are congestible, the direct congestion effect of the group size counteracts the net positive effect identified in Proposition 3. How the group balances the two effects depends on its ability to restrict entry into the group. While it would be hard to ban countries from the use of neighboring waters on the grounds of congestion, it is perfectly plausible that contributing countries have the sole access to the Space Station. In the latter case, it is then important to ask what the optimal group size is. To put the analysis into perspective, I assume that the congestion is sufficiently severe for large groups.

$$\textit{Assumption } 2: \lim_{N\to\infty} v(N) = 0.$$

From Assumption 2, it easily follows that when the entry into the group cannot be restricted, the payoff will be driven down to 0 after the project is completed. In such situations, the project will not be undertaken. This is the extreme case of congestion. However, suppose that the access can be restricted or simply the project is an excludable public good. In these cases, I will assume that the optimal group size is the one that maximizes each member's expected net present value of the project, $\overline{W}_i(1, N|\delta, H)$, before

[20] This is clearly seen from Lemma 1 where $B(x;1)$ is stricly increasing in x.

the project starts.[21] Let $N^*(H,\delta)$ be the optimal group size for a project with H subprojects and an individual discount factor of δ. More formally, since from Lemma 1, $\overline{W}_i(1,N|\delta,H) = B(x^*(N,\delta);1|N,H)$, we have

$$N^*(H,\delta) = \arg\max_N B(x^*(N,\delta);1|N,H) \tag{8}$$

The result in Proposition 3 for noncongestible projects along with Assumption 2 imply that $N^*(H,\delta)$ exists and finite. In what follows, I will futher assume it is unique. The following result records how the group size changes with the project size and the agents' discounting.

Proposition 4. *Suppose that the project is an excludable public good and Assumption 2 holds. Then, the optimal group size (weakly) increases as the project becomes larger; and as agents become more patient.*

To see the intuition behind the link between the group size and project scale, note that having the same number of participants that is optimal for a smaller project cannot be excessive for a larger project. Given the same group size, the return from a completed project remains the same across different size projects. But this means agents would prefer more entry into the group in order to finish a larger project in a shorter time[Proposition 3].

The intuition behind the second part of the proposition is a little more involved. For a fixed group size, an increase in the discount factor has three effects on agent i's contribution: First, since he now cares more about the future, he is willing to take larger current losses and thus increases his cutpoint in (7). Second, given that others contribute more in response to a higher δ, agent i's contribution becomes less pivotal, i.e., $[1-F(x(T;\delta))]^{N-1}$ is smaller, and thus he tends to free ride and decrease his contribution. Third, others' increasing contributions brings the future returns generated by future contributions closer and thus encourages agent i. Overall, I show in the Appendix A that agent i's contribution increases with δ. This implies that more patient agents are able to control the free-riding incentive more effectively, which in turn allows them to expand the group and expedite the completion

[21] While I am not explicitly modeling how the group actually forms, one can imagine that one agent offers memberships to other ex-ante identical agents, each of whom then either accepts or declines the offer. There are however no side payments. In this case, it is clear that the number of offers the first agent makes will maximize his expected discounted value of the project. Notice also that N^* maximizes the ex-ante average *per period* payoff, rather than just the discounted payoff. This allows me to isolate the strategic effect of discounting.

of the project.[22] Next, I investigate whether or not the group tends to be too large or too small with respect to the social optimum.

4 Benchmark: Social Optimum

There are two sources of inefficiencies in the noncooperative contribution game I have analyzed: (1) Agents simultaneously choose their cut-off points, i.e., $x_i(h)$, and (2) agents have incomplete information about others' costs of contribution. The first-best would occur if neither of these problems existed. Namely, in a first-best situation, a social planner would maximize the total welfare given that he fully knows agents' current costs of contribution. The optimal strategy in the first-best situation would be to assign the project to the lowest cost agent in each state unless this cost is too high. However, given the informational constraints, this solution is infeasible in the present setting, which relies on voluntary contributions. Therefore, the second-best situation where the social planner coordinates agents' cutpoints before the costs of contribution are realized seems more appropriate as a benchmark.

Suppose a social planner maximizes the total welfare, or equivalently the average welfare per agent in the group by choosing a cutpoint, $x_i^{**}(h)$, for agent i.[23] Let $W^{**}(h)$ be the optimal *average* payoff per agent in state h. It is clear that once the project is completed, the social planner will choose $x_i^{**}(h) = 0$ for $h > H$ and thus have no agent contribute. This implies the boundary condition: For $h > H$,

$$W^{**}(h) = \frac{v(N)}{1-\delta}$$

For $h \leq H$, however, $W^{**}(h)$ satisfies the following dynamic program:

$$W^{**}(h) = \frac{1}{1-\delta} \max_{\{x_i(h)\}} \left\{ -\frac{1}{N} \sum_{i=1}^{N} \int_0^{x_i(h)} c dF(c) + \left[1 - \prod_{i=1}^{N} (1 - F(x_i(h))) \right] \delta \Delta W^{**}(h) \right\} \quad (9)$$

[22] The fact that more patient agents can control free riding incentive more effectively is true despite there are no reputational concerns in my setting, which is in contrast to several other dynamic models of public good provision, e.g., Bac (1996), Marx and Matthews (2000), McMillan (1979) and Pecorino (1999), where agents adopt trigger strategies. In such settings, as the discount factor becomes larger and agents become more patient, the punishment becomes more severe for any deviation, which helps sustain the cooperative outcome. Here, I focus on the Markov behavior which excludes such trigger strategies. Nonetheless, the relationship grows to be more cooperative in response to a higher discount factor.

[23] Alternatively, suppose agents are able to write a one-period contract on their cutoffs before costs of contribution are realized.

The first-order condition for $x_i(h)$ implies that

$$x_i^{**}(h) = [1 - \lambda_{-i}^{**}(h)]N\delta\Delta W^{**}(h) \tag{10}$$

where again $\lambda_{-i}^{**}(h)$ represents the optimal probability that at least one agent other than i will contribute.[24] Comparing (10) and (2), we see that when determining the cut-off for agent i, the social planner takes the effect of i's contribution on the total welfare rather than just on i's own welfare. Just like before, I will assume that the social planner optimally assigns symmetric cut-off points to all agents. That is, in equilibrium $x_i^{**}(h) = x^{**}(h)$. Now define the following function:

$$B^{**}(x) = \frac{x}{\delta N[1-F(x)]^{N-1}} - \int_0^x [1-F(c)]dc + \left(1 - \frac{1}{N}\right)x[1-F(x)] \tag{11}$$

A variant of Lemma 1 determines how the social planner finds the optimal cut-off points for the group.

Lemma 2. *(a) For* $h \leq H, \overline{W}^{**}(h+1) = B^{**}(x^{**}(h);\delta)$ *and* $\overline{W}^{**}(h) = B^{**}(x^{**}(h);1)$ *where* $\overline{W}^{**}(h) = (1-\delta)W^{**}(h)$.

(b) $B^{**\prime}(x;\delta) > 0$ *for* $x \in (0, \overline{c}]$, *and* $B^{**\prime}(0;\delta) = \frac{1}{N}(\frac{1}{\delta} - 1)$.

Part (a) of Lemma 2 indicates that the backward induction for the second-best solution works exactly the same way as the noncooperative case. Not surprisingly, when there is only one agent, the noncooperative solution is also the social optimum. Since the qualitative properties of $B^{**}(x;\delta)$ coincides with that of $B(x;\delta)$, the existence of a unique social optimum easily follows by applying the proof of Proposition 1. Here we are first interested in knowing whether the coordination of efforts strictly benefit agents in all states, and more importantly whether the project progresses at the socially optimal rate. The following proposition answers these questions:

Proposition 5. *For* $N > 1$, and $h \leq H$,

(i) $W_i(h) < W^{**}(h)$,

(ii) $x(h) < x^{**}(h)$,

(iii) $\lim_{\delta\to 1} x(h) = x_1 < \lim_{\delta\to 1} x^{**}(h) = x_1^{**}$,

According to part (i) of Proposition 5, agents strictly benefit from coordinating their efforts. This is because the equilibrium cutpoints are in the social planner's choice set. Part (ii) implies that agents contribute too infrequently to the project and thus the project

[24] It is easy to verify that the maximized function is strictly quasi-concave.

progresses too slowly from the social standpoint. This means the positive encouragement effect cannot fully mitigate the negative free-rider effect in our model. This inefficiency remains even when the discounting is negligible as recorded in part (iii).

Next, I investigate the degree of inefficiencies caused by an increase in the group size when the project is a noncongestible public good, and compare the optimal group size with the benchmark when it is a congestible public good. Let $N^{**}(H, \delta)$ be the socially optimal group size for a congestible project. Then, we have

Proposition 6.

(i) If the project is a noncongestible public good, i.e., $v(N) = v$, then $\lim_{N\to\infty} W_i(h) = \lim_{N\to\infty} W^{**}(h) = \frac{\delta^{H+1-h}}{1-\delta}v$.

(ii)If the project is congestible and Assumption 2 holds, then $N^*(H;\delta) \leq N^{**}(H, \delta)$.

The first part of Proposition 6 reveals that as the group size grows without bound for a noncongestible project, the inefficiencies disappear and agents' expected payoffs converge to the socially optimal one. Together with part (iii) of Proposition 2, this result goes against our intuition from Olson (1965), who asserts "the larger the group, the farther it will fall short of providing an optimal amount of collective good."(p. 35) The main difference comes from the fact that the technology in my model is that of a "best-shot" one, which requires only one contribution in each stage. Thus, in both the incomplete information and the second best cases, as the group size becomes sufficiently large, the contribution is made by the lowest cost agent and the expected probability of having such an agent converges to 1 in the limit.

The second part implies that groups that rely on voluntary contributions tend to be too small with respect to the social optimum. The intuition is clear: In the social optimum, agents are able to coordinate their strategies so that they can control the free riding incentive more easily. This allows the group to include more members in order to finish the project in a shorter time. Conversely, when the coordination of strategies is not feasible, the group controls the free-riding incentive by being small.

5 Complementary Contributions

Until now, I have assumed that agents' contributions are perfect substitutes in that in a given period if at least one of them contributes, the project moves forward. Although this is the case for open source software, or projects that require only the financial contributions for example, other projects might require certain degree of complementarity in agents' efforts. To capture this possibility, suppose that at least $m \in \{1, ..., N\}$ agents need to contribute in a given period for the project to move forward. We say that a higher m corresponds to a higher degree of complementarity in production with $m = 1$ and $m = N$ being the two polar cases representing perfect substitutes and perfect complements, respectively. An increase in the degree of complementarity has two opposing effects on the group interaction: (1) It reduces the free-rider problem by making agents' contributions less substitutable, and (2) it exacerbates the coordination problem by requiring others' contributions to make one's own worthwhile.

Let $\lambda^*_{-i}(m, h)$ be the equilibrium probability that at least m agents other than i contribute in state h. Like in perfect substitute case, agent i solves the following dynamic program to decide whether or not to contribute in state h:

$$W_i(h, c_i) = \max_{s_i \in \{0,1\}} \left\{ \begin{array}{c} r(h) + s_i[-c_i + \delta[\lambda^*_{-i}(m-1, h)W_i(h+1) + (1 - \lambda^*_{-i}(m-1, h))W_i(h)]] \\ +(1 - s_i)\delta[\lambda^*_{-i}(m, h)W_i(h+1) + (1 - \lambda^*_{-i}(m, h))W_i(h) \end{array} \right\} \tag{12}$$

It is clear from (12) that agent i adopts a similar cut-off strategy to (2) with the following cutpoint:

$$x_i(h) = [\lambda^*_{-i}(m-1, h) - \lambda^*_{-i}(m, h)]\delta\Delta W_i(h) \tag{13}$$

Note that the expression, $\lambda^*_{-i}(m-1, h) - \lambda^*_{-i}(m, h)$, is the probability that agent i's contribution is pivotal. That is, it is the probability that exactly $m - 1$ agents other than i contribute in state h. Thus, the intuition behind (13) is the same as in the substitute case. In equilibrium, agents optimally choose zero contributions once the project is completed, i.e., $x_i(h) = 0$ for $h > H$. For the intermediate states, the zero contribution continues to be an equilibrium representing a complete coordination failure whenever there is complementarity, $m \geq 2$–given that at least one agent does not contribute, it is best for agent i not to contribute either. However, there might be other equilibria in which all agents do contribute and which are clearly Pareto improving over the zero-contribution equilibrium. I will assume that agents coordinate on the most favorable equilibrium in each state.[25] Below, I demonstrate that despite agents' best efforts to overcome coordination problem, some projects may never start. Since the coordination problem gets worse in a larger group, I focus on

[25] Palfrey and Rosenthal (1988, 1991) also consider complementary contributions case and note the multiplicity of equilibria. They use the notion of an equilibrium being *globally expectationally stable*, which can be used in my setting to eliminate the zero-contribution equilibrium whenever another equilibrium exists.

noncongestible projects to abstract from further negative effect due to congestion. But I should note that the qualitative results below are strengthened by congestion. Furthermore, to simplify the formal exposition, I only consider the perfect complement case where the project requires full participation of agents in each stage, and assume $F(c) = \left(\frac{c}{\overline{c}}\right)^{\alpha}, \alpha > 0$. However, the essential intuition for the general case remains the same.[26] To determine the symmetric equilibrium, define the following function:

$$B^C(x;\delta) = \frac{1-\delta}{\delta}\frac{x}{F(x)^{N-1}} + \int_0^x F(c)dc$$

Lemma 3. *For* $h \leq H$, $\overline{W}_i(h+1) = B^C(x(h);\delta)$ *and* $\overline{W}_i(h) = B^C(x(h);1)$ *where* $\overline{W}_i(h) = (1-\delta)W_i(h)$.

The following proposition characterizes the equilibrium:

Proposition 7. *There exists a unique symmetric MPE with the following properties:*

i) If $1 - \alpha(N-1) \leq 0$, *then sufficiently large projects or projects with a sufficiently low final value,* $v < v_{\min}$ *where* $v_{\min} \equiv B^C(x_{\min};\delta)$ *and* $x_{\min} = \left\{\frac{1-\delta}{\delta}[\alpha(N-1)-1]\right\}^{\frac{1}{\alpha N}}\overline{c}$ *never start and all other projects start with positive probability.*

ii) If $1 - \alpha(N-1) > 0$, *then all projects start with positive probability.*

iii) The projects that start have the following properties: For $h \leq H$,

- $x(h)$ *is strictly increasing in* h.
- $W_i(h)$ *is strictly increasing at an increasing rate in* h.

Part (i) of Proposition 7 implies that when agents' contributions are complementary, large-scale projects may not start at all. This is unlike the substitute case where with positive probability all projects take off [Part (i) of Proposition 1]. The fact that agent i's contribution is valuable only when other agents contribute creates a coordination problem and raises his "effective" cost of contribution. This cost may prove to be too high compared to the expected value of the project. Like Figure 1, Figure 3 demonstrates the backward induction described in Lemma 3 when there are complementaries in agents' inputs. Note however that unlike Figure 1, there are no positive $x(h)$'s for projects longer than three stages. This means projects longer than three stages fail to take off.

[26] In particular, for a given degree of complementarity, the project may not start, depending crucially on the cost distribution and group size.

FIGURE 3 ABOUT HERE

The conditions in (i) and (ii) provide intuition regarding the roles of the discount factor, group size and the cost distribution in coordination failure. As δ increases, it is more likely that the project of any size will start. In fact, as $\delta \to 1$, all projects start. The coordination failure worsens as group size increases and high costs become more likely. For a given distribution, the maximum group size that would satisfy the condition in part (ii) is $N^* = \frac{1}{\alpha}$. For instance, when $\alpha = \frac{1}{2}$, a 2-agent group will start any project. If the group size exceeds 2, some projects may not take off. This suggests that projects with complementary efforts might require outside help for a number stages to insure that the rest will be finished through voluntary contributions.

This finding has similar flavor to that of Andreoni (1998), who considers the role of fund-raising activities in inducing subsequent voluntary contributions. Within a one-shot contribution game with complete information, Andreoni assumes contributions need to exceed some exogenous threshold level to yield any return. He notes that when contributions are perfect substitutes, the zero-contribution equilibrium exists if this threshold is too high. Therefore, an outside help, e.g. an initial pledge or a government grant is needed to break this –rather inefficient– equilibrium and generate future voluntary contributions. The non-convexity in my model is due to the complementaries in agents' contributions creating a complete coordination failure when agents' values of the project are very low.

Part (iii) indicates that once the project starts, the relationship among agents grows like in the substitute case.

6 Concluding Remarks

I have considered in some detail how participants voluntarily contribute to a large-scale project that consists of a sequence of subprojects. I discover that (1) for noncongestible projects, more participation improves the performance of the project. (2) For congestible projects, there is an optimal group size, which increases with the project size and with participants' patience. More patient agents are able to control the free riding incentive better and thus able to include more members to expedite the completion of the project. (3) The optimal group size tends to be too small from the social perspective, in which agents can coordinate their strategies. This means even partial commitments among agents improve the performance of the project not only by reducing free riding incentive, but by improving groups' ability to include more members. And (4) when agents' contributions are complementary, while the free riding incentive is reduced, there is a danger that the project may not take off due to the coordination problem. Such projects might require an initial outside help.

In conclusion, I should note some possible extensions to the current analysis. For one, I have not considered heterogeneity in agents' key variables such as their cost distributions and discount factors. For instance, one country's financial fluctuation reflected by its cost distribution or political stability summarized by its discount factor can be different from another. How does this heterogeneity affect cooperation? Another extension is related to the optimal division of a large-scale project into subprojects. My analysis takes this division as exogenous, but provides intuition into the dynamics.

7 APPENDIX A

Proof of Lemma 1: Using (1) and (3) in the text, take the following expectation:

$$\begin{aligned} W_i(h) &= E_c[W_i(h,c)] \\ &= \int_0^{x_i(h)} [-c + \delta W_i(h+1)]dF(c) \\ &+ \int_{x_i(h)}^{\overline{c}} \delta[\lambda^*_{-i}(h)W_i(h+1) + (1-\lambda^*_{-i}(h))W_i(h)]dF(c) \end{aligned} \tag{A1}$$

Integrating the first integral by parts yields:

$$\begin{aligned} W_i(h) &= -x_i(h)F(x_i(h)) + \delta W_i(h+1)F(x_i(h)) + \int_0^{x_i(h)} F(c)dc \\ & +\delta[\lambda^*_{-i}(h)W_i(h+1) + (1-\lambda^*_{-i}(h))W_i(h)][1 - F(x_i(h))] \end{aligned} \tag{A2}$$

Recall that

$$x_i(h) = [1 - \lambda^*_{-i}(h)]\delta\Delta W_i(h) \tag{A3}$$

Using (A3) successively, (A2) reduces to

$$W_i(h) = \delta W_i(h+1) - \int_0^{x_i(h)} [1 - F(c)]dc \tag{A4}$$

(A3) implies that

$$W_i(h) = W_i(h+1) - \frac{x_i(h)}{\delta[1 - \lambda^*_{-i}(h)]} \tag{A5}$$

Together with (A5), (A4) reveals that

$$\begin{aligned} \overline{W}_i(h+1) &= (1-\delta)W_i(h+1) \\ &= B(x(h);\delta) \end{aligned} \tag{A6}$$

where I make use of the symmetric equilibrium assumption so that $x_i(h) = x(h)$ for all i in equilibrium and

$$B(x;\delta) \equiv \frac{1}{\delta}\frac{x}{[1-F(x)]^{N-1}} - \int_0^x [1-F(c)]dc \tag{A7}$$

as defined in (5) in the text.

Substituting for $W_i(h+1)$ from (A5) into (A4) also implies that

$$\begin{aligned} \overline{W}_i(h) &= (1-\delta)W_i(h) \\ &= B(x(h);1) \end{aligned} \tag{A8}$$

completing the proof of part (a) of Lemma 1.

Now differentiate $B(x;\delta)$ with respect to x to find:

$$B'(x;\delta) = \frac{1}{\delta}\left[\frac{1}{(1-F(x))^{N-1}} + \frac{(N-1)xf(x)}{(1-F(x))^N}\right] - (1-F(x)) \tag{A9}$$

Since for $x \in [0,\overline{c}]$, $\frac{1}{(1-F(x))^{N-1}} - (1-F(x)) \geq 0$, we have $B'(x;\delta) > 0$. Also, (A9) implies that $B'(0;\delta) = \frac{1}{\delta} - 1$. Q.E.D.

Proof of Proposition 1: I use backward induction.

Note first that since the payoff does not change for states $h > H$, we have $\Delta W_i(h) = 0$. Then, (A3) implies that the unique equilibrium is $x(h) = 0$ for $h > H$, yielding the boundary condition in (4): For $h > H$,

$$\overline{W}_i(h) = v \tag{A10}$$

Consider $h = H$. From (A6), $x(H)$ solves the following equation:

$$v = B(x(H);\delta) \tag{A11}$$

For $N = 1$, if $v \geq B(\overline{c};\delta)$, then $x(H) = \overline{c}$. However, if $v < B(\overline{c};\delta)$, then since $v > 0$ and $B(0;\delta) = 0$, the Intermediate Value Theorem implies there exists $x(H) \in (0,\overline{c})$ that solves (A11). For $N \geq 2$, since $v > 0$, $B(0;\delta) = 0$, and $B(\overline{c};\delta) = \infty$, there exists $x(H) \in (0,\overline{c})$ that solves (A11). Moreover, $x(H)$ is unique due to $B'(x;\delta) > 0$.

Also from (A8),

$$\begin{aligned} \overline{W}_i(H) &= B(x(H);1) \\ &< B(x(H);\delta) \leq v \end{aligned} \tag{A12}$$

Thus, we have $\overline{W}_i(H) < \overline{W}_i(H+1)$.

Now suppose for some $j \geq 0$, there exists a unique $x(H-j) \in (0, \overline{c})$ and $\overline{W}_i(H-j) < \overline{W}_i(H-j+1)$. Again, from (A6), $x(H-j-1)$ solves the equation:

$$\overline{W}_i(H-j) = B(x(H-j-1); \delta) \tag{A13}$$

Since $\overline{W}_i(H-j) > 0$, using a similar argument above, there exists a unique $x(H-j-1) \in (0, \overline{c})$ that solves (A13). Furthermore,

$$\begin{aligned} \overline{W}_i(H-j-1) &= B(x(H-j-1); 1) \\ &< B(x(H-j-1); \delta) = \overline{W}_i(H-j) \end{aligned} \tag{A14}$$

Since $x(H-j)$ uniquely solves $\overline{W}_i(H-j+1) = B(x(H-j); \delta)$ and $x(H-j-1)$ solves (A13), given that $\overline{W}_i(H-j) < \overline{W}_i(H-j+1)$ by the induction hypothesis and $B'(x; \delta) > 0$, we have

$$x(H-j-1) < x(H-j) \tag{A15}$$

Hence, there exists a unique symmetric MPE where for $h \leq H$, $x(h)$ is strictly positive and increasing in h.

To prove the last part of Proposition 1, note that in equilibrium (A3) can be written as:

$$x(h) = [1 - F(x(h)]^{N-1} \delta \Delta W_i(h) \tag{A16}$$

Since $x(h)$ is strictly increasing for $h \leq H$, we must have

$$\Delta W_i(h-1) < \Delta W_i(h) \tag{A17}$$

Q.E.D.

Lemma A1: *For sufficiently large H, there exists* h_0 such that for $h < h_0$, $\lim_{H\to\infty} x(h) = 0$.

Proof of A1. Recall from Proposition 1 that the sequence $\{x(h)\}_{h=1}^{h=H}$ is strictly increasing or by reordering, the sequence $\{x(h)\}_{h=H}^{h=1}$ is strictly decreasing. Since, by definition of $x(h)$, the latter sequence is bounded below by 0, it converges to some $x_l \geq 0$; and so does its subsequence $\{x(h+1)\}_{h=H}^{h=1}$. Using Lemma 1, this implies that for sufficiently large H, there exists h_0 such that for $h < h_0$, $\lim_{H\to\infty} B(x(h+1); 1) = \lim_{H\to\infty} B(x(h); \delta)$. Since $B(.)$ is continuous in x, this further implies that $B(x_l; 1) = B(x_l; \delta)$, whose unique solution is $x_l = 0$. Q.E.D.

Proofs of Proposition 2 and 3. I start showing by induction that for any $N \geq 1$, $W_i(h, N+1) > W_i(h, N)$ for any $h \leq H$.

Suppose in equilibrium that $W_i(H, N+1) \leq W_i(H, N)$. Since $W_i(H+1, N+1) = W_i(H+1, N) = \frac{v}{1-\delta}$, we have $\Delta W_i(H, N+1) \geq \Delta W_i(H, N)$. Then, (A16) implies $x(H, N+1) \geq x(H, N)$. Since $B(x; 1)$ is strictly increasing in x and N, (A8) reveals that $W_i(H, N+1) > W_i(H, N)$, which is a contradiction. Hence, $W_i(H, N+1) > W_i(H, N)$.

To complete the induction argument, suppose $W_i(h, N+1) > W_i(h, N)$ for some $h \leq H$. Suppose, on the contrary, that $W_i(h-1, N+1) \leq W_i(h-1, N)$. Since $W_i(h, N)$ is strictly increasing in h for any given N, we have

$$W_i(h-1, N+1) \leq W_i(h-1, N) < W_i(h, N) < W_i(h, N+1) \tag{A18}$$

(A18) implies that $\Delta W_i(h-1, N+1) > \Delta W_i(h-1, N)$. Again, (A16) implies that $x(h-1, N+1) \geq x(h-1, N)$, which, in turn, implies $W_i(h-1, N+1) > W_i(h-1, N)$, yielding a contradiction. Hence, $W_i(h-1, N+1) > W_i(h-1, N)$. This completes the proof of Proposition 3.

Now I prove part (a) of Proposition 2.

Suppose by way of contradiction that $x(H, N+1) \geq x(H, N)$. From the above result, we know that $W_i(H, N+1) > W_i(H, N)$. Again, given that $W_i(H+1, N+1) = W_i(H+1, N) = \frac{v}{1-\delta}$, we have $\Delta W_i(H, N+1) < \Delta W_i(H, N)$. From here, (A16) implies $x(H, N+1) < x(H, N)$, a contradiction. Hence, $x(H, N+1) < x(H, N)$.

To prove the last part of Proposition 2, note the following first-order Taylor expansion for x sufficiently close to 0.

$$\begin{aligned} B(x;\delta) &\approx B(0;\delta) + B'(0;\delta)x \\ &\approx \left(\frac{1}{\delta} - 1\right) x \end{aligned} \tag{A19}$$

Since, from Lemma A1, we know that for sufficiently large H, there exists h_0 such that for $h < h_0$, $x(h, N)$ is arbitrarily close to 0, (A6) and (A19) imply that

$$\begin{aligned} \overline{W}_i(h, N+1) &\approx \left(\frac{1}{\delta} - 1\right) x(h, N+1) \\ \overline{W}_i(h, N) &\approx \left(\frac{1}{\delta} - 1\right) x(h, N) \end{aligned}$$

Furthermore, since $\overline{W}_i(h, N+1) > \overline{W}_i(h, N)$ from Proposition 3, we must have $x(h, N+1) > x(h, N)$.

To prove part (iii), first recall from part (i) that $\{x(H;N)\}_{N=1}^{N=\infty}$ is a strictly decreasing sequence. Since it is bounded below by 0, it must converge to some $\alpha \geq 0$. Suppose $\alpha > 0$. For any N, $x(H;N)$ uniquely solves the equation $B(x(H;N)) = v$. Given $\lim_{N\to\infty} x(H;N) = \alpha > 0$, we have $\lim_{N\to\infty} B(x(H;N)) = \infty \neq v$. Thus, $\alpha = 0$.

Now take any $h < H$. Since $x(h;N)$ is strictly decreasing in h and it is nonnegative, we have

$$0 \leq x(h;N) \leq x(H;N)$$

This implies

$$0 \leq \lim_{N\to\infty} x(h;N) \leq \lim_{N\to\infty} x(H;N) = 0$$

Thus, $\lim\limits_{N\to\infty} x(h;N) = 0$.

To complete the proof, suppose $\lim_{N\to\infty} [1 - F(x(h;N))]^N > 0$. But given that $\lim\limits_{N\to\infty} x(h;N) = 0$, we have $\lim_{N\to\infty} B(x(h;N)) = 0 \neq v > 0$. Thus, it must be that $\lim_{N\to\infty} [1 - F(x(h;N))]^N = 0$. Q.E.D.

Before proving Proposition 4, I note the following two useful lemmas.

Lemma A2. For a fixed group size, $x^*(\delta, N)$ and $B(x^*(\delta, N), \delta)$ both increase in δ.

Proof. Take any $\delta_1$and δ_2 such that w.l.o.g. $0 < \delta_1 < \delta_2 < 1$. I proceed by induction. Suppose by way of contradiction that $x^*(H-1;\delta_1) \geq x^*(H-1;\delta_2)$. Since $B(x;\delta)$ is increasing in x and decreasing in δ, (A6) implies $B(x^*(H-1;\delta_1);\delta_1) = v(N) > v(N) = B(x(H-1;\delta_2);\delta_2)$, which is a contradiction. Thus, $x^*(H-1;\delta_1) < x^*(T^H-1;\delta_2)$.

To complete the induction argument, suppose, for some $1 \leq k \leq H-1$, that $x^*(H-k;\delta_1) < x^*(H-k;\delta_2)$ and on the contrary that $x^*(H-k-1;\delta_1) \geq x(H-k-1;\delta_2)$. Since $B(x;1)$ is increasing in x, (A8) yields:

$$\overline{W}_i(H-k;\delta_1) < \overline{W}_i(H-k;\delta_2) \tag{A20}$$

. Note from (A8) that $x^*(H-k-1;\delta_1)$ and $x^*(H-k-1;\delta_2)$ uniquely solve the following equations:

$$\overline{W}_i(H-k;\delta_1) = B(x^*(H-k-1;\delta_1);\delta_1) \tag{A21}$$

$$\overline{W}_i(H-k;\delta_2) = B(x(H-k-1;\delta_2);\delta_2) \tag{A22}$$

Again, since $B(x;\delta)$ is increasing in x and decreasing in δ, (A21) and (A22) imply that $\overline{W}_i(T^H-k;\delta_1) \geq \overline{W}_i(T^H-k;\delta_2)$, which contradicts (A20). Hence, $x^*(H-k-1;\delta_1) < x(H-k-1;\delta_2)$. Q.E.D.

Proof of Proposition 4. For a fixed δ, let $N^*(H)$ and $N^*(H+1)$ be the optimal group size for a project with H and $H+1$ subprojects, respectively. Suppose $N^*(H) > N^*(H+1)$. By definition in (8), $\overline{W}_i(1, N^*(H+1)|\delta, H+1) > \overline{W}_i(1, N^*(H)|\delta, H+1)$. Furthermore, from Proposition 3, since $N^*(H) > N^*(H+1)$, we also have $\overline{W}_i(1, N^*(H)|\delta, H+1) > \overline{W}_i(1, N^*(H+1)|\delta, H+1)$, which is a contradiction. Thus, $N^*(H) \leq N^*(H+1)$.

To prove the second part, take any δ_0, δ_1 such that w.o.l.g. $\delta_0 < \delta_1$. For a given size project, let N_0^* and N_1^* be the optimal group sizes, respectively. To save on notation, let $\widetilde{B}(x^*(N,\delta), N) \equiv B(x^*(N,\delta), 1|N, H)$. By definition, we have $\widetilde{B}(x^*(N_1^*, \delta_0), N_1^*) \leq \widetilde{B}(x^*(N_0^*, \delta_0), N_0^*)$ and $\widetilde{B}(x^*(N_0^*, \delta_1), N_0^*) \leq \widetilde{B}(x^*(N_1^*, \delta_1), N_1^*)$. Thus, using Lemma A2, we have

$$\widetilde{B}(x^*(N_1^*, \delta_0), N_1^*) \leq \widetilde{B}(x^*(N_0^*, \delta_0), N_0^*) \leq \widetilde{B}(x^*(N_0^*, \delta_1), N_0^*) \leq \widetilde{B}(x^*(N_1^*, \delta_1), N_1^*)$$

This implies

$$\widetilde{B}(x^*(N_1^*, \delta_1), N_1^*) - \widetilde{B}(x^*(N_1^*, \delta_0), N_1^*) \geq \widetilde{B}(x^*(N_0^*, \delta_1), N_0^*) - \widetilde{B}(x^*(N_0^*, \delta_0), N_0^*)$$

or

$$\int_{x^*(N_1^*,\delta_0)}^{x^*(N_1^*,\delta_1)} \widetilde{B}_x(x, N_1^*)dx \geq \int_{x^*(N_0^*,\delta_0)}^{x^*(N_0^*,\delta_1)} \widetilde{B}_x(x, N_1^*)dx$$

Now, suppose, on the contrary, that $N_1^* < N_0^*$. Since $\widetilde{B}_x(.), \widetilde{B}_{xN}(.) > 0$, we must have $x^*(N_1^*, \delta_1) - x^*(N_1^*, \delta_0) > x^*(N_0^*, \delta_1) - x^*(N_0^*, \delta_0)$, or

$$\int_{\delta_0}^{\delta_1} \frac{\partial x^*(\delta, N_1^*)}{\partial \delta} d\delta > \int_{\delta_0}^{\delta_1} \frac{\partial x^*(\delta, N_0^*)}{\partial \delta} d\delta$$

Given $\frac{d}{dc}\left[\frac{f(c)}{1-F(c)}\right] \geq 0$, it is tedious but straighforward to show that $\frac{\partial^2 x^*(\delta,N)}{\partial\delta\partial N} > 0$. Together with $N_1^* < N_0^*$ by hypothesis, we have a contradiction. Hence, $N_1^* \geq N_0^*$. Q.E.D.

Proof of Lemma 2. Lemma 2 is proved in exactly the same way as in the proof of Lemma 1 by noting the optimal $x_i(h) = x^{**}(h)$ in (9) and (10). Q.E.D.

Lemma A3. For $N > 1$ and $x \in (0, \overline{c}]$, $B(x;\delta) > B^{**}(x;\delta)$.

Proof of Lemma A3. Take any $N > 1$ and $x \in (0, \overline{c}]$. Suppose $B(x;\delta) \leq B^{**}(x;\delta)$. That is,

$$\frac{x}{\delta\left[1-F(x)\right]^{N-1}} - \int_0^x [1-F(c)]dc \quad \leq \quad \frac{x}{N\delta\left[1-F(x)\right]^{N-1}} - \int_0^x [1-F(c)]dc + \left(1-\frac{1}{N}\right)x\left[1-F(x)\right]$$

which implies that

$$0 \leq \left(1 - \frac{1}{N}\right) x \left[(1 - F(x)) - \frac{1}{\delta\,(1 - F(x))^{N-1}}\right] < 0,$$

a contradiction. Thus, $B(x;\delta) > B^{**}(x;\delta)$. Q.E.D.

Proof of Proposition 5. First note the following comparative static by applying the Envelope Theorem on (9):

$$\frac{\partial W^{*}(h)}{\partial \Delta W^{**}(h)} = \frac{\delta}{1-\delta}\left[1 - (1 - F(x^{**}(h))^{N}\right] > 0 \tag{A23}$$

Now consider a backward induction and suppose $W_i(H) \geq W^{**}(H)$. Since $W_i(H+1) = W^{**}(H+1) = \frac{v}{1-\delta}$, we must have $\Delta W_i(H) \leq \Delta W^{**}(H)$. Suppose for a moment that $\Delta W_i(H) = \Delta W^{**}(H)$. In equilibrium, (2), (10), and $N > 1$ imply that $x^{**}(H) > x(H)$, in particular $x^{**}(H) \neq x(H)$. However, choosing $x_i = x(H)$ is feasible in (9) and it is not chosen. Then, we must have $W_i(H) < W^{**}(H)$, contradicting our original supposition. Thus, $W_i(H) < W^{**}(H)$. The same argument applies for $W_i(H) > W^{**}(H)$ due to (A23).

To complete the induction argument, suppose, for some $j \geq 0$, that $W_i(H-j) < W^{**}(H-j)$ and, on the contrary, that $W_i(H-j-1) \geq W^{**}(H-j-1)$. This implies $\Delta W_i(H-j-1) < \Delta W^{**}(H-j-1)$. Applying a similar argument to that above, we conclude that $W_i(H-j-1) < W^{**}(H-j-1)$,completing the proof of part (i).

To prove part (ii), take any $h \leq H$ and assume that $x(h) \geq x^{**}(h)$. This implies

$$\overline{W}^{**}(h) = B^{**}(x^{**}(h);1) \leq B^{**}(x(h);1) < B(x(h);1) = \overline{W}_i(h)$$

where the first inequality follows from $B^{**}(x;1)$ being strictly increasing in x and the second follows from Lemma A3. However, this contradicts part (i).

To prove part (iii), let x_1 be the unique solution to $B(x_1;1) = v$. Recall that $x(H)$ uniquely solves $B(x(H);\delta) = v$. Since $B(x;\delta)$ is continuous in both x and δ, $x(H) \to x_1$ as $\delta \to 1$. Now suppose for some $j \geq 0$, $x(H-j) \to x_1$ as $\delta \to 1$. Thus, as $\delta \to 1$, we have $\overline{W}_i(H-j) \to v$. Since $x(H-j-1)$ solves $\overline{W}_i(H-j) = B(x(H-j-1);\delta)$, we also have $x(H-j-1) \to x_1$ as $\delta \to 1$. Using the same arguments, one can also show that for $j \geq 0$, $\lim_{\delta\to 1} x^{**}(H-j) = x_1^{**}$ where $B^{**}(x_1^{**};1) = v$. Lemma A3 further implies that $x_1 < x_1^{**}$. Q.E.D.

Proof of Proposition 6. To prove the first part, recall that $x(H;N)$ uniquely solves $B(x(H;N);\delta) = v$. Since $\lim_{N\to\infty} x(H;N) = 0$ and $\overline{W}_i(H) = B(x(H;N);\delta)$, we have

$\lim_{N\to\infty}\overline{W}_i(H)=\delta v$. Using this argument inductively, we find that for $h\leq H$, $\lim_{N\to\infty}\overline{W}_i(h)=\delta^{H+1-h}v$ or $\lim_{N\to\infty}W_i(h)=\frac{\delta^{H+1-h}}{1-\delta}v$. A similar argument for the benchmark case shows $\lim_{N\to\infty}W_i^{**}(h)=\frac{\delta^{H+1-h}}{1-\delta}v$.

To prove the second part, first note that for a given (x,N), we have $B^{**}(x,1|N,H)-B(x,1|N,H)=\left(1-\frac{1}{N}\right)x\left[(1-F(x))-\frac{1}{\delta(1-F(x))^{N-1}}\right]$. This implies $0<B_x^{**}(.)<B_x(.)$ and $B_N^{**}(.)>B_N(.)>0$. Now let N^* be the solution to (8) and $x^*=x(.,N^*)$ be the cutoff for a given project. Treating N as a continuous variable, N^* solves $B_x(.)\frac{\partial x}{\partial N}+B_N(.)=0$. Using this, $0<B_x^{**}(.)<B_x(.)$ and $B_N^{**}(.)>B_N(.)>0$, we find $B_x^{**}(.)\frac{\partial x}{\partial N}+B_N^{**}(.)\Big|_{N=N^*,x=x^*}=[B_x^{**}(.)-B_x(.)]\frac{\partial x}{\partial N}++B_N^{**}(.)-B_N(.)\Big|_{N=N^*,x=x^*}>0$. This implies that $N^{**}\geq N^*$. Q.E.D.

Proof of Proposition 7. Given that $F(c)=\left(\frac{c}{\overline{c}}\right)^{\alpha}$, $B^C(x;\delta)=\frac{1-\delta}{\delta}x^{1-\alpha(N-1)}\overline{c}^{\alpha(N-1)}+\frac{x^{\alpha+1}}{\alpha+1}\overline{c}^{-\alpha}$. First, note that $B^C(x;1)$ is strictly increasing in x with $B^C(0;1)=0$. For $1-\alpha(N-1)\leq 0$, $B^C(x;\delta)$ is strictly convex in x and attains its unique minimum when $\frac{d}{dx}B^C(x_{\min};\delta)=0$ where $x_{\min}=x_{\min}=\left\{\frac{1-\delta}{\delta}[\alpha(N-1)-1]\right\}^{\frac{1}{\alpha N}}\overline{c}$. The proofs of the existence of a unique MPE and part (i) follow the same lines as the proof of Proposition 1. However, here there may be multiple and positive solutions to $x(h)$, satisfying $\overline{W}_i(h+1)=B^C(x;\delta)$. Since I assume that agents coordinate on the most favorable equilibrium and since $B^C(x;1)$ is increasing in x, they will coordinate on the largest $x(h)$. But this means the relevant part of $B^C(x;\delta)$ is its increasing part and therefore the rest of the proposition for this case follows the proofs in the substitute case. Define $v_{\min}=B^C(x_{\min};\delta)$. If $\overline{W}_i(h)<v_{\min}$, then $\overline{W}_i(h+1)=B^C(x;\delta)$ has no real solution. This occurs for sufficiently long projects or projects with sufficient small final value, v.[See Figure 3] In these cases, agents turn to zero contribution equilibrium.

For $1-\alpha(N-1)>0$, $B^C(x;\delta)$ is increasing in x everywhere with $B^C(0;\delta)=0$. This means there is a unique positive solution to $\overline{W}_i(h+1)=B^C(x;\delta)$ in each state. The rest of the proposition for this case follows like in the substitute case as well. Q.E.D.

8 Appendix B

A Dynamic Contribution Model with Continuous Contributions and Incomplete Information

Here, I develop a dynamic contribution model with noncongestible projects much like the one presented in the text but with continuous contributions. I show without detailing the full formal argument that the same qualitative results, in particular the encouragement effect can be obtained in such a model.

Suppose agent i contributes a continuous amount $y_i \geq 0$. Agent i's marginal cost of contribution, c_i,comes from a distribution $F(c_i)$. I assume that the project moves from state h to $h+1$ with probability $\Pi = \Pi(\sum_{j=1}^{N} y_j)$ where $\Pi'(.) > 0$, $\Pi''(.) < 0$, and $\Pi(0) = 0$. That is, agents' contributions are perfect substitutes[27] and the production function exhibits diminishing marginal returns. The stochastic progress can easily be interpreted within a threshold public good model, where the threshold is uncertain. [see Nitzan and Romano (1990)]. Suppose, for instance, the project moves forward if and only if $\sum y_j \geq M$, where M is a random threshold, that comes iid from a distribution $G(M)$. Then, as long as $g'(.) < 0$, the model can be considered as a dynamic extension of Nitzan and Romano (1990). However, the stochastic progress can admit other interpretations.

It is clear that when the project is completed, no agent will contribute and thus in equilibrium:

$$\overline{W}_i(H) = v \tag{B1}$$

For states $h < H$, let $y_i = y_i(c_i, h)$ denote i's equilibrium contribution as a function of his current cost and the state of the project. For convenience, define $\mathbf{c}_{-i} \equiv (c_1, ..., c_{i-1}, c_{i+1}, ..., c_N)$ and $y_{-i}(\mathbf{c}_{-i}, h) \equiv \sum_{j \neq i}^{N} y_j(c_j, h)$. Agent i solves the following dynamic program to determine his contribution in state h:

$$W_i(h, c_i) = \max_{y_i} \left\{ -cy_i + \delta W_i(h) + E_{c_{-i}} \Pi(y_i + y_{-i}(c_{-i}, h)) \delta \Delta W_i(h) \right\} \tag{B2}$$

Maximizing (B2) requires[28]

$$-c_i + E_{c_{-i}} \Pi'(y_i + y_{-i}(c_{-i}, h)) \delta \Delta W_i(h) \leq 0 \quad (= 0 \text{ if } y_i > 0) \tag{B3}$$

[27] Agents' contributions could be imperfect substitutes, where $\Pi = \Pi(y_1, .., y_N)$ with Π increasing in y_i and strictly concave in $(y_1, .., y_N)$. This is not essential for what follows. The perfect substitute case makes the argument more clear.

[28] The second order conditions are satisfied given that $\Pi''(.) < 0$ and $\Delta W_i(T) > 0$ in equilibrium.

Note that whenever $y_i > 0$, $y_i(c_i, h)$ is decreasing in c_i. That is, agent i's contribution decreases with his cost. Let $x_i(h)$ be i's cut-off such that

$$y_i(c_i, h) \begin{cases} > 0 & \text{if } c_i < x_i(h) \\ = 0 & \text{otherwise} \end{cases} \tag{B4}$$

where

$$x_i(h) = E_{c_{-i}} \Pi'(y_{-i}(c_{-i}, h)) \delta \Delta W_i(h) \tag{B5}$$

(B3) and (B5) imply that for $y_i > 0$,

$$E_{c_{-i}} \Pi'(y_i + y_{-i}(c_{-i}, h)) = \frac{c_i}{x_i(h)} E_{c_{-i}} \Pi'(y_{-i}(c_{-i}, h)) \tag{B6}$$

This means $y_i(c_i, h)$ is increasing in $x_i(h)$. Now, taking the expectation of both sides of (B2) with respect to c_i, we obtain

$$\begin{aligned} W_i(h) &= \int\limits_0^{x_i(h)} \left\{ -c_i y_i(c_i, h) + \delta W_i(h) + E_{c_{-i}} \Pi(y_i + y_{-i}(\mathbf{c}_{-i}, h)) \delta \Delta W_i(h) \right\} dF(c_i) \\ &+ \int\limits_{x_i(T)}^{\overline{c}} \left\{ \delta W_i(h) + E_{c_{-i}} \Pi(y_{-i}(\mathbf{c}_{-i}, h)) \delta \Delta W_i(h) \right\} dF(c_i) \end{aligned} \tag{B7}$$

Integrating the first term on the l.h.s. of (B7) by parts and using (B3) reduce (B7) to:

$$\overline{W}_i(h) = \frac{E_{c_{-i}} \Pi(y_{-i}(\mathbf{c}_{-i}, h))}{E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h))} x_i(h) + \int\limits_0^{x_i(h)} y_i(c_i, h) F(c_i) dc_i \tag{B8}$$

Also, since $x_i(h) = E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h)) \delta \Delta W_i(h)$ from (B5), we have

$$\begin{aligned} \overline{W}_i(h+1) &= \frac{1-\delta}{\delta} \frac{1}{E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h))} x_i(h) + \frac{E_{c_{-i}} \Pi(y_{-i}(\mathbf{c}_{-i}, h))}{E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h))} x_i(h) \\ &+ \int\limits_0^{x_i(h)} y_i(c_i, h) F(c_i) dc_i \end{aligned} \tag{B9}$$

Now, assume symmetric equilibrium so that $x_i(h) = x(h)$ for all i in equilibrium, and define the following function:

$$\begin{aligned} \widetilde{B}(x; \delta) &= \frac{1-\delta}{\delta} \frac{1}{E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h); x)} x + \frac{E_{c_{-i}} \Pi(y_{-i}(\mathbf{c}_{-i}, h); x)}{E_{c_{-i}} \Pi'(y_{-i}(\mathbf{c}_{-i}, h); x)} x \\ &+ \int\limits_0^{x} y_i(c_i, h; x) F(c_i) dc_i \end{aligned} \tag{B10}$$

Note that given the properties of $\Pi(.)$, $\widetilde{B}(x;\delta)$ is strictly increasing in x with $\widetilde{B}(0;\delta) = 0$. The following pair of recursive equations determine the unique equilibrium sequence of $x(h)$:

$$\begin{aligned} \overline{W}_i(h+1) &= \widetilde{B}(x(h);\delta) \\ \overline{W}_i(h) &= \widetilde{B}(x(h);1) \end{aligned} \tag{B11}$$

It is clear from (B11) that a proposition like Proposition 1 in the text can be stated here. This also highlights the presence of the encouragement effect in this continuous contribution model. To see this, suppose every agent but i increases his contribution in a given state by choosing a higher cutpoint. This has two opposing effects on agent i's decision: First, from (B5), it reduces the marginal return on agent i's contribution (due to $\Pi''(.) < 0$) and thus facilitates the free-riding incentive. However, second this also brings the future returns closer and encourages agent i to further his contribution. The latter incentive mitigates the former and therefore agent i increases his contribution.

To fix the intuition further, below I construct an example that allows for a closed form equilibrium contribution function $y_i(c_i, T)$.

<u>EXAMPLE:</u>

Let $\Pi(\sum y_j) = 1 - e^{-\sum y_j}$. Since $\Pi'(\sum y_j) = e^{-\sum y_j}$, (B6) reveals that

$$y_i(c_i, h) = \begin{cases} \ln(\frac{x_i(h)}{c_i}), & \text{if } c_i < x_i(h) \\ 0, & \text{otherwise} \end{cases} \tag{B12}$$

Assume symmetric equilibrium where $x_i(h) = x$ for all i. Since $y_i(c_i, h)$ is a piecewise function, when computing $E_{c_{-i}}\Pi(y_{-i}(\mathbf{c}_{-i}, h); x)$ and $E_{c_{-i}}\Pi'(y_{-i}(\mathbf{c}_{-i}, h); x)$, one needs to consider all possible cases where k out of N agents contribute and the rest do not. Define the probability function p as:

$$p(x) = \frac{\int_0^x [1 - F(c)]dc}{x} \tag{B13}$$

Then, we have the following expressions:

$$E_{c_{-i}}\Pi(y_{-i}(c_{-i}, h); x) = 1 - p(x)^{N-1} \text{ and } E_{c_{-i}}\Pi'(y_{-i}(c_{-i}, h); x) = p(x)^{N-1}. \tag{B14}$$

(B10) and (B14) imply

$$\begin{aligned} \widetilde{B}(x;\delta) &= \frac{1-\delta}{\delta}\frac{x}{p(x)^{N-1}} + \frac{1-p(x)^{N-1}}{p(x)^{N-1}}x + \int_0^x \ln(\frac{x}{c})F(c)dc \\ &= \frac{1}{\delta}\frac{x}{p(x)^{N-1}} - \int_0^x [1 - \ln(\frac{x}{c})F(c)]dc \end{aligned} \tag{B15}$$

Note that $0 < p(x) < 1$, $p'(x) < 0$ and $\lim_{x \to 0} p(x) = 1$. Thus, $\widetilde{B}(x;\delta)$ is increasing in x, decreasing in N and $\widetilde{B}(0;\delta) = 0$. This means that $\widetilde{B}(x;\delta)$ possesses the same qualitative properties as $B(x;\delta)$ defined in (5) in the text. This further means all the results in the text can be generated in this continuous contribution model. The details are available upon request.

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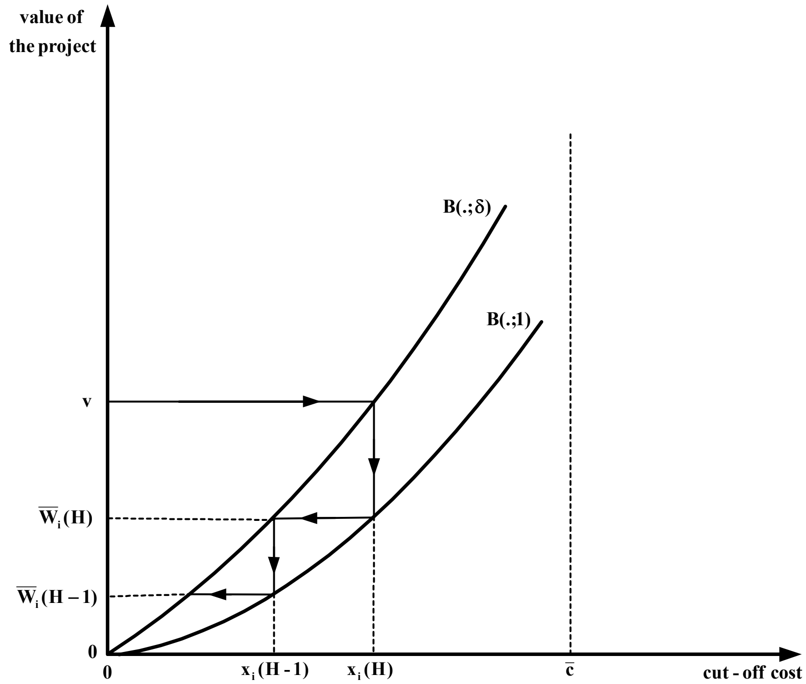
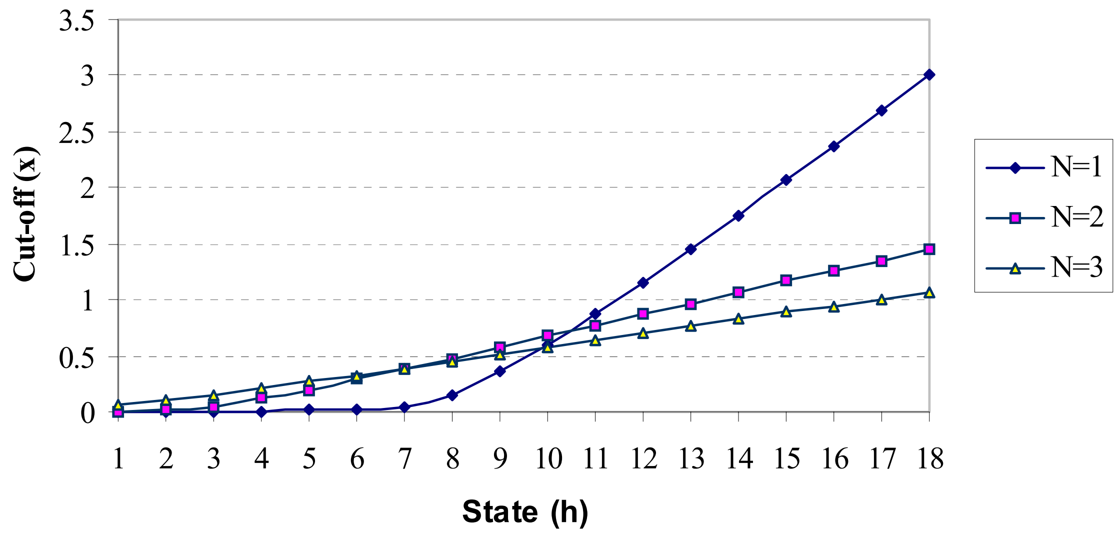

value of
the project
B(.;δ)
B(.;1)
v
$\overline{W}_i(H)$
$\overline{W}_i(H-1)$
0
0
$x_i(H-1)$
$x_i(H)$
$\bar{c}$
cut - off cost

Figure 1

Figure 2



3.5
3
2.5
2
1.5
1
0.5
0
Cut-off (x)
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
State (h)
N=1
N=2
N=3

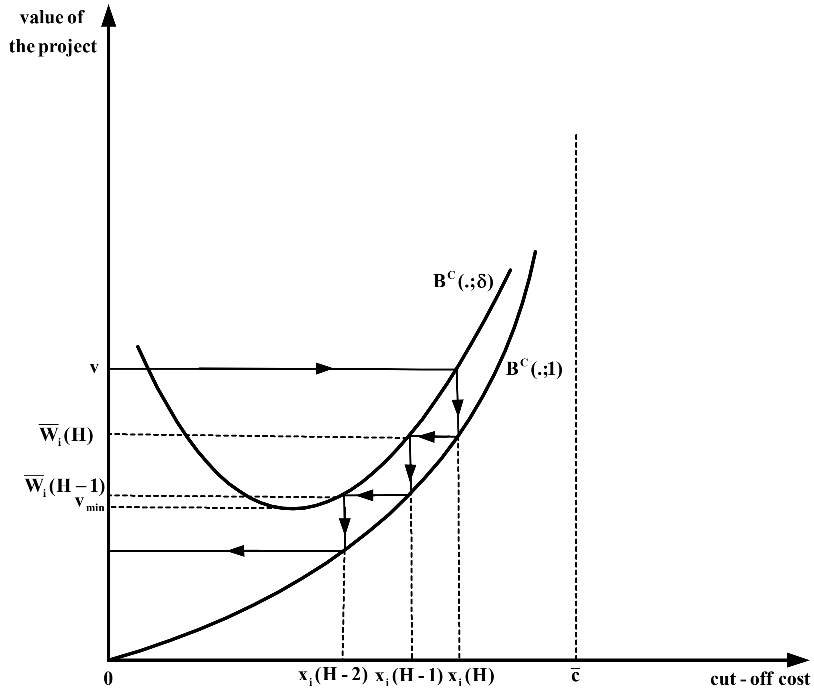

value of
the project
v
$\overline{W}_i(H)$
$\overline{W}_i(H-1)$
v_{min}
$B^C(.;\delta)$
$B^C(.;1)$
0
$x_i(H-2)$
$x_i(H-1)$
$x_i(H)$
$\bar{c}$
cut - off cost

Figure 3