Mixed Strategy Equilibrium in Tennis Serves

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Abstract

A mixed strategy is a random choice among available strategies, with each strategy being chosen a set percentage of the time. In many games that require unpredictable play, game theory predicts that a mixed strategy equilibrium, a situation where each player uses an optimal mixed strategy, will result. Economists have tested whether people play according to the mixed strategy equilibrium in laboratory experiments with two player zero-sum games—subjects in these experiments generally do not play in accordance with game theory's predictions. Recently, economists have published papers examining mixed strategy equilibrium play using professional sports as a natural experiment. This paper builds upon Walker and Wooders (2001), which examines mixed strategy play in the locations of serves in professional tennis matches. Walker and Wooders (2001) find that professional tennis players are closer to game theory's predictions than subjects in laboratory settings, but still "switch their serves up" more than is consistent with game theory's predictions. My hypothesis is that this result can be explained by a short-term timing effect where a serve that has just been hit is, ceteris paribus, less effective on the next point. I construct a model incorporating this timing effect and work out the theoretical implications of my model. I then estimate the magnitude of this timing effect and determine if optimal play under this model is consistent with the results obtained by Walker and Wooders. My conclusion is that the model accounts for a little under half of the deviation from game theory's predictions found in the data from professional tennis matches. This suggests that professional tennis players play closer to game theory's predictions when tested using a model designed to account for more of the complexities of tennis than the Walker and Wooders model, but they still do not play in complete accordance with those predictions.

I. Introduction

Rationality is one of the basic tenets of economics, but recent papers have challenged this assumption. Smith (1991) says that 30 years of experimental research rejects the assumption that individual rationality is a cognitive, calculating process of maximization. One of the tests for rationality is the ability to play (or learn to play) a game in accordance with mixed strategy equilibrium. Many games require unpredictable play—the classic example is rock, paper, scissors. No matter which of the three options you choose, if your opponent knew which one you were going to play, you would lose. In a game that requires unpredictable play, no set of pure strategies can be an equilibrium, defined as a set of strategies such that neither player has an incentive to deviate from his current strategy given the strategy his opponent is currently playing. The only equilibrium strategy in a game like rock, paper, scissors is not a pure strategy—a simple choice of rock, paper, scissors—but rather a mixed strategy. A mixed strategy is a random choice among the available strategies, with each strategy being chosen a set percentage of the time.

Mixed strategy play is one of the key theoretical insights into understanding strategic situations where unpredictability is important. While some strategic games have no equilibrium in pure strategies, every strategic game has at least one equilibrium when mixed strategies are included (Nash, 1950). For rock, paper, scissors, the mixed strategy equilibrium is for both players to choose each of rock, paper, and scissors a third of the time. In two-person zero-sum games von Neumann's Minimax Theorem states that each player should choose the strategy that maximizes his minimum payoff—this is equivalent to mixed strategy equilibrium play in these games. Rational play calls for players to play in accordance with this mixed strategy equilibrium.

The theory of mixed strategy play has not fared well in experiments involving human subjects. Human subjects do not necessarily play by their appropriate mixed strategies. Additionally, human subjects make choices that are serially dependent (based on previous choices)—they "switch up" their choices more than is consistent with random play. For example O'Neill (1987) tests minimax play in a laboratory experiment that involves subjects playing a repeated zero-sum game.[1] Brown and Rosenthal (1990) examine O'Neill's results and find that the frequency with which each player uses each card is not consistent with minimax play—they also find strong evidence of serial correlation in players' choices.

If game theorists were unable to find any situations in which people make choices according to mixed strategy equilibrium, the value of the theory would be significantly reduced. Recently, several papers have examined situations in professional sports to test the predictions of mixed strategy play. Professional athletes are very experienced at playing their sports and have large amounts of money at stake, factors that make them more likely to play according to mixed strategy equilibrium.

Professional sports calls for an incredible amount of unpredictable play. A football team that always goes for the same play on third down will find that it rarely works. Wee Willie Keeler's famous advice for how to be a good hitter in baseball was "Hit 'em where they ain't." Part of what made Pete Sampras' serve so effective was that it was hard to "read." Most tennis players toss the ball in a slightly different place based on where they are going to hit it. An opponent who picks up on these subtleties is said to

[1] In O'Neill's (1987) experiment, each player had four cards to choose from (the minimax strategy for each player was to pick the joker with probability 0.4 and to pick each of the other cards with probability 0.2). Players started with $2.50 and were paired against a single opponent for 105 trials. The winner of each trial won $0.05 from the loser and at the end of the trials each player kept the amount of money he or she had.

“read” the serve. Sampras, however, threw the ball in the same place no matter where he was going to hit the serve, preventing opponents from having a sense of where the ball was going.

Mark Walker and John Wooders (2001) examine mixed strategy equilibrium in the context of the directions that players hit serves in professional tennis matches. Their conclusion is that tennis players play in accordance with most of the predictions of mixed strategy equilibrium; however, tennis players “switch up” their service choices more than is consistent with purely random play. This paper constructs a model designed to explain why tennis players might play optimally by “switching it up” too much. I propose that there is short-term timing effect where the returner gets better at returning a serve that he has just seen and introduce a “timing variable” into the model used by Walker and Wooders (2001) to account for this effect. I then estimate the magnitude of the timing variable and test to see if the model can explain the results in the tennis data. My conclusion is the model presented here is not sufficient to explain the “switching it up” observed in the tennis data—it accounts for slightly less than half of the amount that players “switch it up” more than would be predicted by random play.

Section II is a review of the relevant literature concerning testing mixed strategy equilibrium using professional sports as a natural experiment. Section III introduces the “point game” model of tennis used by Walker and Wooders and then modifies it to include the short-term timing effect that I propose (the “timing variable”). Section IV details the data (which includes all of the data from Walker and Wooders (2001)), discusses how it will be used in the empirical specification section and runs some preliminary analysis. Section V estimates the magnitude of the timing variable in the model and then tests the results of the model against the actual data from the tennis matches (in particular through the use of the "run test" which tests whether the serving strategies found in the data are consistent with serially independent play). Section VI is a brief conclusion.

II. Literature Review

Since mixed strategy equilibrium is an important concept in game theory, recent papers have tried to test mixed strategy equilibrium using natural experiments. Recently, papers have been testing mixed strategy equilibrium using sporting events—in particular, in the locations of soccer penalty kicks and tennis serves. While situations requiring mixed strategies are common in sports, the ability to isolate decisions under uniform starting conditions make these events particularly useful in empirical research. While Walker and Wooders (2001) examine the location of tennis serves, two other papers, Chiappori, Levitt and Groseclose (2002) and Palacios-Huerta (2002) examine the location of soccer penalty kicks.

The results for professional tennis players are in contrast to the results obtained for soccer penalty kicks by Chiappori et al. (2002) and Palacios-Huerta (2002). Both papers on soccer find that the direction of soccer penalty kicks is in accordance with mixed strategy equilibrium—based on both the scoring percentages and serial independence of choices.[2] Unlike Walker and Wooders' (2001) result that there is serial correlation of serves in tennis, these papers on soccer find that knowing where a player

[2] Soccer penalty kicks have the advantage that the choices of both players, the direction the ball is kicked and the direction the goalkeeper moves, can both be observed. Thus, in analyzing soccer penalty kicks the choices of the goalkeeper can be examined as well.

kicked a ball the previous time tells you nothing about where he will kick it the next time—the location of kicks is serially independent.

One obvious difference exists between soccer penalty kicks and tennis serves: a given player shoots soccer penalty kicks days or weeks apart, while tennis serves are hit on every point of the entire match.[3] My hypothesis is that tennis serves are not serially independent because there is a slight short-term timing effect—a serve that you have just hit is, ceteris paribus, less effective the next time—the returner is better at returning the serve than he used to be. This effect, called the "timing variable" is represented in the model by a specific functional form and a parameter that represents the magnitude of the effect.

When thinking about sports, it is reasonable to suppose that players learn through doing.[4] Baseball pitchers with uncommon sidearm deliveries are often relief pitchers who only pitch an inning at a time. This ensures that they do not face batters more than once in a game or more than a handful of times a season. In amateur tennis, some serves like the "American Twist" serve (which has very weird spin) tend to be rare and are used only on important points. If returning "twist" serves were only a matter of anticipation, its use should be part of the standard mixed strategy equilibrium. One explanation for its use on key points is that the strength of the twist serve decreases as it is used more during the match, because opponents start to time it. Consequently, players play optimally by saving it for a crucial point.

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[3] Chiappori et al. (2002) mentions that soccer penalty kick "shootouts", where each team kicks a number of penalty kicks in a row, may be a better place to test for serial correlation.

[4] I use "learning through doing" to discuss situations where the future payoffs of a game change because of the current actions of the players. This corresponds to how I assume timing works in tennis. This is different from learning about the underlying payoffs of a game, or learning to play correctly given a set of known payoffs.

The existence of a timing variable implies that points are not independent and identically distributed—this idea is supported elsewhere in the literature. Klaassen and Magnus (1998) analyze almost 90,000 points from four years at Wimbledon and conclude that players' chances to win each point have a small but statistically significant deviation from the assumption of independence and identical distribution.

If timing a serve is taken as a very short term effect, points at the start of a game should be different from points played later in a game. Since players only serve every other game, the returner's timing should become worse at the start of a game due to not having seen the serve recently. The server, too, may be rusty due to all of the time not serving. In professional tennis, where players hit hundreds of serves a week, this effect would likely be smaller than the returner's timing problems (the returner, after all, does not play his current opponent very often). As the game continues, the returner will re-time the serve, so this effect, if it exists, would appear most strongly at the start of each game.

Klaassen and Magnus (1996) examine the dominance of the serve over time. In both the men's (and women's) singles, they find that service dominance decreases in a small but statistically significant way through the end of the third (second) set. This result is in accord with the idea that the returner learns to time the serve as the match progresses, although it does not imply the existence of a short term timing effect.[5] Klaassen and Magnus conclude that the returner does not immediately learn to hit his

[5] Klaassen and Magnus reject the idea that this decrease in service dominance could have been caused by the server getting tired. Their reasoning, although not spelled out explicitly, seems to be that professional tennis players should not get tired as early as the second set of a match and if players did tire this early, they would continue to tire over the course of the match and the effect would be more pronounced in the later sets.

returns better at the start of the match. Thus, any specification for how tennis players time serves should not have a particularly pronounced effect at the start of a match.

While Klaassen and Magnus (1996) provides evidence for some kind of effect where players learn by doing, there is no theoretical model for a short term timing effect in tennis and how it affects the match (including how it might affect service choices). I modify the "point game" model from Walker and Wooders (2001) to incorporate a timing variable to represent this effect.

Walker and Wooders (2001) model tennis as a 2x2 matrix game where the server chooses where to hit the ball and the returner chooses where to anticipate the serve. The server, if he knew where the returner was anticipating, would always choose to serve to the other direction. Conversely, the returner, if he knew where the server was going to serve, would always anticipate that direction. These conditions lead to a unique equilibrium in mixed strategies. Walker and Wooders assume that points are independent and identically distributed; the side of the court and which player is serving determine the corresponding "point game" that governs each player's payoffs. Walker and Wooders analyze ten tennis matches between top professional players in the finals and semifinals of major tournaments.

A problem with any analysis of service strategy in tennis is that only the server's choice is observed—the serve that the returner anticipates is unobservable. Additionally, the model assumes that professional tennis players have complete information about the underlying payoffs of the game. While professional tennis players do not have complete information, the best professional tennis players generally know each other's games (often based on previous matches) and can be expected to have a good sense of the payoffs in the model.[6]

One of the primary predictions of mixed strategy equilibrium is that the server is expected to win an equal percentage of the points, no matter where he hits his first serve. Walker and Wooders find that the percentage of points won on serves to each side is consistent with the assumption that professional players are playing according to mixed strategy equilibrium. However, Walker and Wooders conclude that professional tennis players vary the direction of their serves ("switch it up") more than is consistent with completely random play.

In general, inexperienced players of a repeated game with a mixed strategy equilibrium do not play it well. Their choices are serially dependent and their win percentages are often inconsistent with equilibrium play.[7] Professional tennis players, according to Walker and Wooders, are much better than most people in laboratory experiments at randomizing their play, but are still not perfect because of the serial dependence of service choices.

My hypothesis is that Walker and Wooders reject serial independence and thus perfect mixed equilibrium play because their model assumes serial independence of points. I relax the assumption that each point is an identical point game and introduce a "timing variable" designed to measure the short-term influence of previous serves on future points. Specifically, the more the server hits to a spot the better a returner is (he gets in a groove in athletic jargon). When testing whether a server is randomizing one

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[6] Players also do not have perfect information—that is, they do not remember the entire sequence of serves that opponents have hit during the match. It may be that players do not have sufficient recall to punish opponents whose choices exhibit only a small amount of serial correlation.
[7] In general, people "switch it up" too much when trying to generate random sequences (Wagenaar , 1972).

needs to account for the fact that he will serve less often to an overused side. Accounting for this timing effect could give the result that professional tennis players play in accordance with the mixed strategy equilibrium model that includes this timing effect.

It is reasonable to suppose that some kind of learning through doing exists in tennis. However, this does not imply that a short term timing effect explains that the serial correlation found by Walker and Wooders is in fact optimal mixed strategy equilibrium play. If the magnitude of the timing variable is zero, service choices in mixed strategy equilibrium should be serially independent (because this model will be identical to that in Walker and Wooders (2001)). If the magnitude of the timing variable is positive, optimal play should require negative serial dependence of service choices (and the larger the timing variable, the more negative serial dependence will be required).

My research determines if the results of Walker and Wooders (2001) that professional tennis players' service choices are serially dependent can be explained by the addition of a timing variable to their model. I construct a model for the timing variable and find its implications on optimal mixed equilibrium play. Using these predictions, I estimate the magnitude of the timing variable empirically and find it to be positive. I then test to see if the model is able to explain the serial dependence observed in the tennis data.

III. Theoretical Framework

Serving in the Game of Tennis

The serve in tennis is hit, and only hit, at the start of each point and must go into the service box on the opposite side of the court. A player has two chances to hit a serve in—if he misses the first serve, he gets a second serve. If a player misses the second serve it is a double-fault, and he loses the point. A point has two possible outcomes (either the server wins or the returner wins) that are completely observable.

The server may win the point on the serve, may lose the point through a double-fault, or the serve may be returned successfully in which case the point is played out from there. The serve plays a dominant role in men's tennis—Klaassen and Magnus (1996) analyze data from four years at Wimbledon (the major tournament where the serve dominates the most) and find that players win 65% of the points on their service. Players with ineffective serves do not rise to the top of men's professional tennis.

In professional tennis, first serves are generally hit as far to one side of the service box as possible, making it relatively easy to group service location into one of three categories—left, right, and occasionally center. Since professional players are highly skilled, even if a serve is out I can assume the direction in which the ball was going is the direction in which they intended to serve.[8]

The second serve raises other questions. A first serve is hit on every point, while a second serve is only hit when the first serve is missed; if the second serve is missed, the server loses the point. Consequently, second serves are less aggressive than first serves and are aimed closer to the center of the court. The combination of a smaller data set and less variation in service locations cause Walker and Wooders (2001) to exclude study of second serves—I do the same.

Serves are not all hit from the same side of the court. The first point of each game is played serving to the deuce (right side) court, and points alternate between the deuce

[8] Thus, it should be assumed that serves to the center generally reflect that the server aimed to the center. On the first serve, though, hitting the serve to the center is rare—only 6% of serves are to the center in the Walker and Wooders (2001) data. They do not consider center serves for this reason—I do the same.

and ad (left side) courts. The service also alternates, with each player serving every other game. The scoring system of tennis is as follows: each game within a match is won when a player has four or more points and a lead of at least two points. The first player to six games with a lead of at least two games wins a set (a tiebreaker is played if the score reaches six games apiece), and the winner of three out of five sets wins the match.

The Point Game Model

The point game model for tennis serves is a 2x2 normal (matrix) form game based on the server's choice of where to hit a first serve. The server chooses to hit this serve to the left or the right. Simultaneously, the returner anticipates that the serve will be either to the left or to the right. The winner of the point is then determined, possibly by the serve itself (if the returner fails to return it)—alternately, the first serve might miss and the point will be decided after the server hits a second serve.

The Point Game (a "point game matrix") (outcomes are probability that the server wins the point)

Server \ Receiver	L	R
L	π_{LL}	π_{LR}
R	π_{RL}	π_{RR}

Since this is a constant sum game (one player or the other wins each point), for each point the server tries to maximize his payoff while the returner tries to minimize the server's payoff (and thereby maximize his own chance to win the point). I assume that the following inequalities (the "mixed strategy equilibrium conditions") hold: $\pi_{LL} < \pi_{RL}$ and $\pi_{RR} < \pi_{LR}$ (the server wins the point more often if he serves to the location the returner has not anticipated) as well as $\pi_{LL} < \pi_{LR}$ and $\pi_{RR} < \pi_{RL}$ (the returner is more likely to win the point if he correctly anticipates where the serve will be hit). Given these assumptions, there is a unique equilibrium in strictly mixed strategies.[9]

In the mixed strategy equilibrium, the returner should choose to anticipate to the left with a probability that makes the server indifferent between hitting the serve to the left and to the right—therefore, the server should win an equal percentage of points when he serves to the left as when he serves to the right. Likewise, the server should choose to serve left with a probability that makes the returner indifferent between anticipating to the left and to the right. This is equivalent to von Neumann's Minimax Theorem, which states that in a two player zero sum game each player should choose the strategy that maximizes his minimum payoff. In a repeated game, the Minimax Theorem requires players to use this mixed strategy each time—knowing what a player has done in the past should tell you nothing about what he is going to do now (serial independence).

In equilibrium, the server's probability of hitting the serve to the left, S_L, can be calculated to be $S_L = (\pi_{RL} - \pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})$. Since the server's choices are only left or right, the probability that the server hits to the right is $(1-S_L)$. The returner's chance to anticipate a serve to the left, although it cannot be observed directly, is $A_L = (\pi_{LR} - \pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})$. Due to the mixed strategy equilibrium assumptions, S_L and A_L must be in the interval (0,1).

[9] This description of the point game is based on the model constructed by Walker and Wooders (2001). Chiappori et al (2002) use a similar 3x3 matrix to analyze soccer penalty kicks.

The server's chance to win the point when both players use their equilibrium strategies is $V = (\pi_{LR}\pi_{RL} - \pi_{LL}\pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})$. This is the general solution for a point game—each specific point game has its own matrix of payoffs which determine each player's equilibrium mixed strategies and the server's chance to win a point.

There are four point games in a tennis match, distinguished by which player is serving and whether the point is on the deuce-court or ad-court. If points are independent identically distributed, each point's payoffs are fully determined by its respective point game. Optimal play is then for each player to maximize the chance to win each point as if it was the only point by playing the appropriate mixed strategy equilibrium.[10] Thus, a player's service choices in tennis should not depend on his previous service choices.

Adding a timing effect to the model in the form of a timing variable entails assuming that points are not independent. As a result, the state of the game may play a role in choosing each player's optimal strategy, since a player's goal is to win the game, not simply to maximize his chance to win the current point. Under what conditions do tennis players still play optimally by maximizing their chance to win each point as if it was the only point?

For this to be the case two conditions must hold: first, each game within the tennis match must have a finite maximum length; second, the server's chance to win the next point (given that it will be played by both players as if it is the only point), must be independent of his choice of where to serve on this point. This second condition is not a reasonable assumption to make in my model—in particular, this condition holds if and

[10] Walker and Wooders (2000) prove this intuitive result for tennis as part of a general class of binary Markov games where a player's goal is to win a match consisting of point games.

only if the point matrix is symmetric (i.e., $\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$).. A symmetric point game matrix implies that the server will hit half of his serves to the left, which is clearly not the case in general.[11] Thus, optimal strategies in this model will depend on the state of the game (but probably only to a small extent). I assume that the optimal strategy for each player remains maximizing his chance to win each point as if it were the only point.

Including a Parabolic Timing Variable

The model of a timing variable used here is that on the deuce court only the serve hit on the previous deuce court point affects the payoff matrix for the current point. The same would be true of ad court points, so deuce court points would not affect ad court points and vice versa. Additionally, the timing effect disappears between games—the last point of the previous service game does not affect the first point of the next service game.[12]

The timing effect is incorporated into the point game by introducing a variable based on whether the previous serve was hit to the left or the right. Since players "lose their timing" of the opponent's serve between games, the model without the timing variable is a sufficient description of the first points of any game.[13] I assume that if the serve on the current point is hit to the left, for the next point the timing variable affects only the returner's chance to win the point if the serve is again hit to the left. The

[11] See the appendix for proofs of these assertions.
[12] Given that a player usually goes at least three or four minutes between the end of one of his service games and the start of his next service game it makes sense that an effect that only lasts for a single point within the game should not carry over during this time period.
[13] "First point" is used to describe the first point played on the deuce court as well as the first point played on the ad court in each service game. Thus, the first two points of each service game are called "first points" in this model.

magnitude of timing variable should be relatively small—specifically, the mixed strategy equilibrium conditions must still hold when the timing variable is included.

The model with no timing variable represents the first point played on the deuce (or ad) court in a given game. The models that include a timing variable describe subsequent points on the deuce (ad) court in the game.[14] In the data, I specify which points in each point game are the first points played in each game and which points are played after that. Since the timing variable is assumed to apply differently to these points, the server's chance to serve to the left and his win percentage could both vary on these points. Service choices on the first points are unaffected by the timing variable (I call the chance to serve left on a first point $S_{L,0}$ to distinguish it from later points) .

In order to distinguish the timing variable from servers being bad at randomizing their serves (and to make the model tractable), it should have the same functional form across the entire sample. The simplest case would be if the timing effect is constant, i.e., the server's chance to win a point if he hits his serve to the left (right) is a constant amount lower if his previous serve was to the left (right). This is not a particularly interesting model because in this case the timing variable has no implications for the

[14] It is possible to use the law of iterated expectations to construct a matrix representation for the server's chance to serve left on each point of the game, even if you do not know any of his service choices. The chance for the server to serve to the left on each point converges quickly to an equilibrium value as the length of the game increases (if the timing variable affects serial correlation of serves this chance is different from the chance to serve left on the initial point). This result is not needed here because the data set has information about the service location for each point in each service game and in the model the previous point's service choice is a sufficient description of the next point's service choices. I use this procedure to estimate the bias that would be introduced by using points later in the service game to estimate $S_{L,0}$, the server's chance to serve to the left on the first point of a game.

serial dependence of serves. Under this model, the timing variable would change the returner's anticipation of serves but these decisions cannot be observed.[15]

In trying to estimate a functional form for the timing variable, consider the case when the server's chance to win a point (not including the timing variable) is either very high or very low. For example, think of what would happen if Pete Sampras played an amateur player. The chance for Sampras to win a point when serving would be close to 100%. Even if the amateur timed Sampras's serve from having seen it a few times, it should have almost no effect on the percent of points won by each player since Sampras is so superior to the amateur.

This example illustrates that a linear probability model for the effects of timing a serve would not be correct at extremes. A different model, such as using the functional form of probit would be more appropriate in these situations—however, probit is too complicated to use as a model for a timing variable in tennis. It is possible to come up with a model that has the same key property as probit, though—namely, that the effect of timing is greatest when the chance to win the point is close to _ and that it falls to zero when the chance to win the point is close to 0 or 1. From this, I propose a model where the timing variable for each matrix entry, π, is a function of both π and a constant magnitude (across the entire data set) **T**—the timing variable is assumed to equal **T***π*(1-π).

This function is a parabola and has the properties described above, reaching its maximum for π= _ and its minimum (zero) for π=1 or π=0. While descriptively the

15 This is a general and unintuitive result in mixed strategy equilibrium— the server's strategy is affected by changing the payoffs from the returner's choices while the returner's strategy is affected by changing the payoffs from the server's choices.

effect of the timing variable is not necessarily highest for π= _, it must fall to zero for π close to 0 or 1, making this interpolation a reasonable assumption. The new point games are as follows:

The Point Game, given that the last serve to this side was to the *left* (outcomes are the probability that the server wins the point):

Server \ Receiver	L	R
L	$\pi_{LL} - T(\pi_{LL})(1- \pi_{LL})$	$\pi_{LR} - T(\pi_{LR})(1- \pi_{LR})$
R	π_{RL}	π_{RR}

The Point Game, given that the last serve to this side was to the *right* (outcomes are probability that the server wins the point)

Server \ Receiver	L	R
L	π_{LL}	π_{LR}
R	$\pi_{RL} - T(\pi_{RL})(1- \pi_{RL})$	$\pi_{RR} - T(\pi_{RR})(1- \pi_{RR})$

The chance to hit this serve to the left, given that you hit your previous serve to the left, is $S_{L,L} = (\pi_{RL-}\pi_{RR})/\{\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR} + \mathbf{T[\pi_{LL}(1- \pi_{LL}) - \pi_{LR}(1- \pi_{LR})]}\}$.

What properties does this expression have when T>0? By symmetry, it is easy to see that

$S_{L,L}= S_{L,0}$ if $\pi_{LL} + \pi_{LR} = 1$. Using the mixed equilibrium condition $\pi_{LR}> \pi_{LL}$, it can be shown that if $\pi_{LL} + \pi_{LR} >1$, then $S_{L,L}< S_{L,0}$. Also, if $\pi_{LL} + \pi_{LR} <1$, then $S_{L,L}> S_{L,0}$.

In the case when the previous serve is hit to the right the same calculations apply and the result is that $S_{L,R}= (\pi_{RL}- \pi_{RR} - \mathbf{T[(\pi_{RL})(1-\pi_{RL}) - (\pi_{RR})(1-\pi_{RR})]}) / \{\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR} + \mathbf{T[(\pi_{RR})(1-\pi_{RR}) - (\pi_{RL})(1-\pi_{RL})]}\}$. When T>0, A set of similar conditions hold as in the previous case, but reversed. Namely, $S_{L,R}= S_{L,0}$ if $\pi_{RL} + \pi_{RR} = 1$, $S_{L,R}>S_{L,0}$ if $\pi_{RL} + \pi_{RR} >1$, and $S_{L,R}< S_{L,0}$ if $\pi_{RL} + \pi_{RR} <1$.

What magnitude should **T** have? In general, the timing variable should not change the mixed equilibrium conditions. Two of these four mixed strategy equilibrium conditions are $\pi_{LL}< \pi_{LR}$ and $\pi_{RR}< \pi_{RL}$. Thus, if $\pi_{LL}< \pi_{LR}$, it must also be the case that $[\pi_{LL} - T(\pi_{LL})(1- \pi_{LL})] < [\pi_{LR} - T(\pi_{LR})(1- \pi_{LR})]$. While this requirement is matrix-specific, if $\pi - T^*\pi^*(1-\pi)$ is increasing for all values of π then this will hold for any matrix chosen.[16] This makes sense as a general requirement and is equivalent to the condition T<1. Since the descriptive analysis of tennis leads me to believe that the timing variable is relatively small, a correct model for the timing variable should have T<1 in any case.[17]

The case where $S_{L,R}> S_{L,0} > S_{L,L}$ is a particularly interesting case, since it is when the server's chance to serve to the left is lowest when his previous serve was to the left, is

[16] Depending on the point-game matrix, the other two mixed strategy equilibrium conditions ($\pi_{LR}> \pi_{RR}$ and $\pi_{RL}< \pi_{LL}$) may result in a more strict condition on T than T<1. This happens because the returner's decision where to anticipate can become a dominant strategy if the timing variable is sufficiently large. If this happened, the server would take the returner's dominant strategy into account and would always serve in the same direction.

[17] T being relatively small means that players do not get a lot better at returning a serve from seeing it. Since professional tennis players have plenty of practice at hitting serve returns in general, it makes sense that the effects of timing should not be particularly large.

highest when his previous serve was to the right, and is in between when there is no previous point (at the start of a game). This occurs when ($\pi_{LL} + \pi_{LR} > 1$) and ($\pi_{RL} + \pi_{RR} > 1$). When these conditions hold it can be shown that the server's chance to win the point (not including the timing variable) must be greater than _.[18] The converse, however, is not true—if the server's chance to win the point is greater than _ at least one but not necessarily both of ($\pi_{LL} + \pi_{LR} > 1$ and $\pi_{RL} + \pi_{RR} > 1$) hold. There is no value for the server's overall win percentage that is sufficient to establish that both ($\pi_{LL} + \pi_{LR} > 1$ and $\pi_{RL} + \pi_{RR} > 1$) must hold.[19] It is possible to gain further insight by making additional assumptions about the form of the point-game matrices.

Further Assumptions About Point-Game Matrices

What is it reasonable to assume the matrices in question look like? In matches between professional tennis players the numbers in the matrices should not be extremely close to 0 or 1. Tennis players do not anticipate the way that soccer goalkeepers do, by leaping to a side. Thus, we would expect that, unlike in soccer, the effects of anticipating the ball correctly in tennis should not be extremely large.[20] This has important results for the form of the matrix.

The difference in the server's chance to win the point, given that he hit the serve in a particular direction and the returner anticipated correctly versus incorrectly is limited by the percentage of first serves that are made in the court. This is because when the server misses a serve (it "goes out"), the result of the point should not depend on whether

[18] See Proposition 3 in the appendix for a proof of this assertion.

[19] A matrix of the form [x 1; x+_ 0], for example, can yield an arbitrarily high win percentage for the server as x→1, but only $\pi_{LL} + \pi_{LR} > 1$ holds in this matrix ($\pi_{RL} + \pi_{RR} < 1$ does not hold).

[20] In soccer, for example, if the goalkeeper stays in the center of the goal and the kicker kicks it to the center, it will be blocked (essentially) every time. Palacios-Huerta (2002) finds that if the kicker kicks it to his "natural side" and the goalkeeper guesses the opposite way, the kick is successful 95% of the time.

the returner anticipated that serve correctly or incorrectly. Specifically, if the server makes a first serve to the left in (it "goes in") with probability, $\mathbf{In_{Left}}$, then the maximum value for $\pi_{LR} - \pi_{LL}$ should be $\mathbf{In_{Left}}$. That is, the maximum value for the difference between the server's chance to win the point, given that the serve was to the left and the returner anticipated *incorrectly* and the server's chance to win the point, given that the server was to the left and the returner anticipated *correctly*, is the percent of serves aimed to the left that go in.[21]

Given that the first serve is hit in, it does not make sense to assume that the server wins every point if the returner guesses incorrectly, but never wins the point if the returner guesses correctly. The effect of guessing correctly may be large, but I will assume that given that the first serve is in, the difference in the chance to win the point if the returner guesses incorrectly and the chance to win the point if the returner guesses correctly is at most 50%. Thus, the maximum value for $(\pi_{LR} - \pi_{LL})$ is then _* $\mathbf{In_{Left}}$. Klaassen and Magnus (1996) analyze matches played at Wimbledon and find that the percentage of first serves made is just under 60%. For the data in Walker and Wooders (2001), the percentage of first serves made is slightly over 60%. These observations (and the assumption that serves to the left go in with the same frequency as serves to the right) lead to the approximation that in a typical point game matrix, $0 < (\pi_{LR} - \pi_{LL}) \leq 0.3$ (_ *0.6, from above), and $0 < (\pi_{RL} - \pi_{RR}) \leq 0.3$.

The restrictions above are a specific case of the general constraint $0 < (\pi_{LR} - \pi_{LL}) \leq \mathbf{M}$ and $0 < (\pi_{RL} - \pi_{RR}) \leq \mathbf{M}$, using the value $\mathbf{M} = 0.3$. What additional properties should hold in a matrix where this property holds for some generalized **M**?

[21] The same logic applies for serves to the right as well.

This restriction affects the "symmetry" of the matrix.[22] In a symmetric matrix ($\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$), the two serial correlations $S_{L,R}> S_{L,0}$ and $S_{L,0}> S_{L,L}$ hold identically.[23] For $0< (\pi_{LR} - \pi_{LL}) \leq \mathbf{M}$ and $0< (\pi_{RL} - \pi_{RR}) \leq \mathbf{M}$ ($0 <\mathbf{M}\leq 1$), as **M** decreases the matrix is forced to look increasingly symmetric because of the equilibrium conditions $\pi_{LL} < \pi_{RL}$ and $\pi_{RR} < \pi_{LR}$. Thus, as $\mathbf{M}\rightarrow 0$ $S_{L,R}> S_{L,0}$ and $S_{L,0}> S_{L,L}$ hold identically if and only if the server's chance to win the point is greater than _.

In fact, for any value of **M,** it can be shown that if the chance for the server to win the point is greater than $(1+\mathbf{M})/2$, this is sufficient to establish both $S_{L,R}> S_{L,0}$ and $S_{L,0}> S_{L,L}$.[24] In the case when $\mathbf{M}\rightarrow 1$, that is, when this restriction is no longer present, the result is the same as above, since the server must win above 100% of the points (not possible) to imply both negative serial correlations.

Using the previous estimate of **M**=0.3 in this equation, I can come up with the condition that in this model, if the server's win percentage is greater than 0.65, this implies $S_{L,R}> S_{L,0}$ and $S_{L,L}< S_{L,0}$. In my aggregate data, the server's chance to win each point is 64.7%, very close to the 65% threshold value. This provides a way to test this model—if the negative serial correlations are not found, the model is probably incorrect. The parabolic timing variable model is simple, intuitively appealing, and allows for a range of possible results, making it the best theoretical basis for the project.[25]

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[22] An "asymmetric" matrix such as [0 1; 1 0.9] is what this restriction is designed to prevent.

[23] For these serial correlations to hold identically, it must be the case that $\pi_{LL} + \pi_{LR} >1$ holds if and only if $\pi_{RL} + \pi_{RR} > 1$ also holds. In a symmetric matrix, since $\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$, $\pi_{LL} + \pi_{LR} = \pi_{RL} + \pi_{RR}$ and thus $\pi_{LL} + \pi_{LR} >1$ holds if and only if $\pi_{RL} + \pi_{RR} > 1$ also holds.

[24] This is equivalent to the assertion that if the server's chance to win the point is greater than $(1+\mathbf{M})/2$, this is sufficient to establish that both $\pi_{LL} + \pi_{LR} >1$ and $\pi_{RL} + \pi_{RR} > 1$ hold. See Proposition 4 in the appendix for a heuristic proof of this assertion.

[25] An alternate specification of the model would be to try to determine even more what the underlying matrix looks like. This would be done by adding the following (observable) variables: $\mathbf{In_{Left}}$, the chance for the server to make a first serve, given that it is hit to the left; $\mathbf{In_{Right}}$, the chance for the server to make a

IV. Data

Professional tennis matches are widely available in the form of videos of classic matches. Walker and Wooders (2001) use data from ten tennis matches between top players in the finals (or semifinals) of very important tennis tournaments. Top professional tennis players are used instead of amateurs because the top professional tennis players are much better at the sport than amateurs—while this is easy to see in the physical aspects of tennis, it almost certainly holds true for the strategic aspects as well. Since the matches are near the end of the biggest tournaments of the year, players will have the highest incentives to maximize their chances of winning which makes them more likely to play according to mixed strategy equilibrium. Additionally, the extremely high stakes (the winner stands to gain hundreds of thousands of dollars in prize money alone) mean that the players are certainly not experimenting with aspects of their game such as serving strategies (a possibility at smaller events).[26]

The tournaments in the data set are the four biggest tournaments of the year, the majors (the Australian Open, the French Open, Wimbledon, and the US Open) plus the

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first serve, given that it is hit to the right; **Z**, the chance for the server to win the point, given that he misses his first serve (making the assumption that the choice of first serve hit has no effect on the server's chance to win the point if the first serve is a fault). The four unknown quantities in the point game matrix change—they now correspond to the chance for the server to win the point, given that his serve goes in (to the left/right) and that the returner anticipated (to the left/right).

The benefit of this specification is that it allows examination of the two known quantities: the chance for the server to win the point, given that he hit his serve to the left (or right) and it went in. One of these equations is collinear, but using the other equation brings the quantities known about the matrix to three (the server's chance to serve left, his overall chance to win the point and the server's chance to win the point, given that he hit his serve to the left and it went in). With four unknown entries in the matrix, this would still not allow for direct estimation of the coefficients, but it does restrict one of the degrees of freedom. The problems that could arise from this are that increased complexity may make the model harder to work with and that dividing the sample further may cause small-data set issues.

[26] It could be physically tiring for players to hit the same serve too often, which causes players to "switch up" their serves too much. For example, a serve to the left may be faster than a serve to the right and thus harder on the server's shoulder. At the same time, a serve to the right may have more spin and thus be harder on the server's back. Since the players in the data set are top professionals and near (or at) the end of a major tournament, though, any such effect should be small.

Master's Cup, a large invitational tournament for the top eight players in the world at the end of the year. The earliest of these matches was Ken Rosewall-Stan Smith at Wimbledon in 1974 and the most recent (included in my data set but not Walker and Wooders (2001)) is Roger Federer-Marat Safin at the 2005 Australian Open.

The best professional tennis players would be expected to be very familiar with each other's abilities and to have a good idea of the payoffs within each point game. To increase the sample size of the data, Walker and Wooders use only men's tennis matches (played best three out of five sets—women's tennis matches are played best two out of three sets) and select in favor of longer matches. This may have the effect of selecting for matches where play corresponds more closely to mixed strategy equilibrium if matches where one player does not play according to the appropriate mixed strategies are shorter. Small deviations from optimal play could only lower a player's win percentage slightly (if the opponent noticed and took advantage of it), though, so small deviations from optimal play would not have a large impact on the length of the match.

The data consists of points from 11 tennis matches, which are divided into 44 point games (one point game for each server on the deuce court and one point game for each server on the ad courts, so four point games per match). Each point game has 50-100 points, for a total of 3308 data points in the sample after center serves are excluded. The data has a point by point score for the match. This includes which side of the court each point is played on, who is serving, the location of the first (and second) serves on each point (left or right, from the returner's perspective) and which player wins the point. My analysis includes this information, plus information on the previous point played on the same side of the court in the same service game (or whether the current point is the first point played this side of the court this game).

On a first serve, it is rare that an observer can tell where the returner was anticipating the serve—thus, for data collection purposes, it is effectively unobservable. On occasion, though, it is possible to observe the returner leaning towards one side. This supports the idea that the returner is anticipating the location of the serve. On the second serve this is even more apparent. A right-handed player with a stronger forehand than backhand, for example, will occasionally take several steps to the left immediately before the serve is hit in order to "run around" his backhand and hit a forehand instead.

The Walker and Wooders (2001) data includes information on serves to the center; these serves were not used by Walker and Wooders due to their rarity, being used only 6% of the time in their sample.[27] This provides a good justification for not including center serves, although it could have a small impact on the results.[28] In the ten matches in Walker and Wooders (2001), only one player was included more than twice (John McEnroe, who is in four of the matches)—none of the players consistently failed the tests for serial independence or equal win percentages. This indicates that the serial correlations observed are not the result of a single player alone.

The issue of excluding center serves became more complicated when I recorded the direction of the serves in a match myself. To supplement the data and to obtain a better idea of the challenges of collecting it, I watched the 2005 Australian Open Federer-Safin match and recorded data for the points. Deciding if a serve is "Center" (as opposed

[27] When I exclude center serves from the Walker and Wooders data, my summary statistics for the matches are almost the exact same as theirs (usually off by at most a single point).
[28] Pat Cash and Stan Smith (in two matches and one match in the sample, respectively) use center serves often—the other players use it rarely (sometimes on as few as 2% of the serves)

to “Left” or “Right”) is often difficult and can be particularly difficult when the serve misses in the net because the observer must extrapolate the path of the ball.

While a serve hit directly to the center (at the returner’s body instead of to one side of him) is relatively rare, hitting serves that are close to the center (so that it is too close for the returner to take a normal swing at the ball) is relatively common. Walker and Wooders (2001) used center to describe any serve that could not be identified as consciously chosen by the server to be distinctly left or right.[29] I coded the Federer-Safin match in the same way for consistency.[30]

The Matches in general:

Match	Total points played	Number of first serves made	% of first serves made	Points won by server	% of points won by server
1974 Wimbledon	342	226	66.1%	224	65.5%
1980 Wimbledon	374	234	62.6%	248	66.3%
1980 US Open	344	184	53.5%	212	61.6%
1982 Wimbledon	346	202	58.4%	225	65.0%
1984 French	296	174	58.8%	182	61.5%
1987 Australian	316	185	58.5%	196	62.0%
1988 Australian	325	220	67.7%	204	62.8%
1988 Masters	326	180	55.2%	214	65.6%
1995 US Open	236	130	55.1%	163	69.1%
1997 US Open	341	221	64.8%	232	68.0%
2005 Australian	322	182	56.5%	208	64.6%
Total	**3568**	**1956**	**59.9%**	**2308**	**64.7%**

Definitions:
Where serves are referred to, this analysis concerns only first serves. A player receives a second serve if he misses the first serve (if he misses the second serve, he loses the point)
A "first point" is the first point of the game played on each of the deuce and ad courts
$S_{L,0}$= the chance for the server to hit his serve to the left on a "first point"

[29] Personal correspondence with Mark Walker.
[30] I assume that the location of a serve (left, right, or center) is the factor explaining how well the returner timed future serves, but it may be that the returner times the serve not by the specific location of the serve but through the stroke that he hits on the return. A returner, then, times his forehand when his serve return is a forehand and times his backhand when his serve return is a backhand. For a right-handed returner, a serve to the right is a forehand and a serve to the left is a backhand (this is reversed for left-handed returners). Thus, would be reasonable to suppose that, for timing purposes, a serve hit slightly to the returner’s left is the same as a serve hit very far to the returner’s left, because the returner will hit the same stroke on both serves. Unfortunately, the Walker and Wooders (2001) data did not include the designations necessary to distinguish serves in this manner.

S(L after L)= $S_{L,L}$= the chance for the server to serve to the left on a point when his serve on the precious point was to the left								
S(L after R)= $S_{L,R}$ = the chance for the server to serve to the left on a point when his serve on the previous point was to the right.								
Match	$S_{L,0}$	$S_{L,L}$	$S_{L,R}$	Fits $S_{L,R}$> $S_{L,L}$?	Fits $S_{L,,R}$ > $S_{L,0}$ > $S_{L,L}$?	Overall % of serves hit to left	Overall win % for server	Number of points in this data set
1974 Wimbledon								
Rosewall, Deuce	88.9%	100.0%	100.0%	Equal	No	93.3%	70.7%	75
Rosewall, Ad	45.5%	66.7%	39.3%	No	No	50.0%	68.9%	74
Smith, Deuce	70.8%	58.5%	70.6%	Yes	No	64.6%	57.3%	82
Smith, Ad	100.0%	76.7%	100.0%	Yes	No	86.8%	68.4%	76
1980 US Open								
McEnroe, deuce	80.8%	59.5%	42.9%	No	No	61.9%	60.7%	84
McEnroe, ad	52.4%	53.6%	36.7%	No	No	46.8%	64.6%	79
Borg, deuce	20.8%	35.0%	47.2%	Yes	No	36.3%	57.5%	80
Borg, ad	37.0%	31.6%	43.3%	Yes	Yes	38.2%	61.8%	76
1980 Wimbledon								
McEnroe deuce	53.8%	41.2%	58.6%	Yes	Yes	50.6%	66.3%	89
McEnroe, ad	46.4%	48.3%	60.7%	Yes	No	51.8%	63.5%	86
Borg, deuce	25.0%	38.1%	44.0%	Yes	No	37.4%	67.7%	99
Borg, ad	17.9%	27.3%	20.8%	No	No	20.7%	66.3%	92
1982 Wimbledon								
McEnroe, deuce	50.0%	54.5%	32.3%	No	No	44.3%	68.4%	79
McEnroe, ad	48.0%	45.0%	42.3%	No	No	45.1%	66.2%	71
Connors, deuce	91.7%	77.2%	100.0%	Yes	Yes	83.5%	64.8%	91
Connors, ad	34.8%	26.1%	58.1%	Yes	Yes	41.6%	61.0%	77
1984 French Open								
McEnroe, deuce	69.6%	40.0%	73.7%	Yes	Yes	58.3%	56.9%	72
McEnroe, ad	58.3%	44.0%	72.2%	Yes	Yes	56.7%	61.2%	67
Lendl, deuce	30.0%	22.2%	48.5%	Yes	Yes	36.6%	70.4%	71
Lendl, ad	58.3%	33.3%	57.9%	Yes	No	49.3%	58.2%	67
1987 Australian								
Edberg, deuce	21.7%	23.5%	28.6%	Yes	No	25.3%	69.3%	75
Edberg, ad	72.0%	56.7%	85.7%	Yes	Yes	68.1%	59.4%	69
Cash, deuce	52.6%	55.6%	63.6%	Yes	No	57.4%	60.3%	68
Cash, ad	68.4%	39.3%	77.8%	Yes	Yes	58.5%	50.8%	65
1988 Australian Open								
Wilander, deuce	34.8%	23.1%	22.5%	No	No	26.3%	67.1%	76
Wilander, ad	47.8%	36.4%	56.5%	Yes	Yes	47.1%	66.2%	68
Cash, deuce	63.2%	58.6%	28.0%	No	No	49.3%	58.9%	73
Cash, ad	54.5%	69.2%	66.7%	No	No	63.5%	54.0%	63
1988 Masters Cup								

Becker, deuce	57.1%	60.5%	77.8%	Yes	No	63.1%	69.0%	84
Becker, ad	64.3%	61.8%	78.6%	Yes	Yes	65.8%	63.2%	76
Lendl, deuce	59.3%	51.6%	53.8%	Yes	No	54.8%	56.0%	84
Lendl, ad	65.5%	77.4%	75.0%	No	No	72.4%	76.3%	76
1995 US Open								
Sampras, deuce	47.4%	78.3%	35.3%	No	No	55.9%	71.2%	59
Sampras, ad	28.6%	43.8%	35.0%	No	No	35.1%	70.2%	57
Agassi, deuce	47.4%	66.7%	40.9%	No	No	50.8%	57.6%	59
Agassi, ad	60.0%	73.9%	83.3%	Yes	No	70.9%	76.4%	55
1997 US Open								
Sampras, deuce	57.1%	61.3%	44.1%	No	No	53.8%	63.4%	93
Sampras, ad	37.9%	50.0%	34.4%	No	No	39.8%	65.1%	83
Korda, deuce	57.1%	63.6%	71.4%	Yes	No	63.4%	69.5%	82
Korda, ad	77.4%	72.7%	70.0%	No	No	74.3%	78.4%	74
2005 Australian Open								
Safin, deuce	70.8%	65.7%	100.0%	Yes	Yes	72.9%	61.4%	70
Safin, ad	73.9%	45.2%	78.9%	Yes	Yes	63.0%	68.5%	73
Federer, deuce	45.5%	57.1%	36.4%	No	No	47.2%	66.7%	72
Federer, ad	63.2%	71.0%	57.1%	No	No	65.6%	67.2%	64

The most important statistics are the first three columns, $S_{L,0}$, $S_{L,L}$ and $S_{L,R}$. $S_{L,0}$ concerns "first points" when the timing variable does not apply so $S_{L,0}$ should be a function of the four entries (π_{LL}, π_{LR}, π_{RL}, π_{RR}) in the matrix for the specific point game in question but it should not depend on T. $S_{L,L}$ and $S_{L,R}$ measure the service choices for points when the server served left or right, respectively, on the previous point. These two statistics are generated by the four entries for the individual point game, (π_{LL}, π_{LR}, π_{RL}, π_{RR}), in addition to the constant T.

Preliminary Analysis and Use of (Slightly) Biased Estimators

What would we expect to find as the relationship between these coefficients? If my model is correct (with $T>0$) and the matrices follow the "not skewed" assumptions from earlier, then we should expect $S_{L,R} > S_{L,0} > S_{L,L}$. The fourth column measures part of this question: is $S_{L,L} < S_{L,R}$, i.e., is a server more likely to hit his current serve left when his previous serve was to the right than when it was to the left? We would expect $S_{L,R}>S_{L,L}$ to hold 50% of the time if there was no true difference between $S_{L,L}$ and $S_{L,R}$. The data shows no clear pattern, although there is a slight tendency for $S_{L,L}<S_{L,R}$ to hold ($S_{L,R}>S_{L,L}$ in 25 point games, $S_{L,R}<S_{L,L}$ in 18 point games and $S_{L,R}=S_{L,L}$ in a single point game). Excluding the single point game where $S_{L,R}=S_{L,L}$, , $S_{L,R} > S_{L,L}$ occurs in 58.1% of the point games in the data set.

The fifth column examines the full hypothesis from the theory, $S_{L,R}>S_{L,0}>S_{L,R}$. How often these conditions hold if the true values for $S_{L,R}$, $S_{L,0}$, and $S_{L,L}$ were all the same? All orders for the numbers would be equally likely and there are 3*2*1= 6 ordered arrangements, so $S_{L,R}>S_{L,0}>S_{L,L}$ should occur with probability 1/6 (16.7%). The actual percentage of point games where this condition holds is 29.5%, quite a bit higher than the 16.7% predicted if the true values of $S_{L,L}$, $S_{L,0,}$ and $S_{L,R}$ were equal. This supports my null hypothesis.[31]

I used the server's overall chance to serve to the left as an estimator for his chance to serve to the left on points at the start of a game. I also used his overall chance to win a point as an estimator of his chance to win a point at the start of a game.[32] Since the overall chance to serve to the left includes point not at the start of the game, this statistic is a biased estimator for the chance to serve to the left at the start of the game. However, since roughly _ of the points in the data set are not played at the start of a game, including the additional data triples the sample size. The bias introduced depends on the actual value for T—however it is probably quite small. For T=0, there is no bias since points at the start of a game are identical to points played after the start. Even when T is chosen to

[31] Detailed further in the empirical specification section.

[32] I choose reasonable values for (π_{LL}, π_{LR}, π_{RL}, π_{RR}) for each point game consistent with these chances for the server to serve left and for the server to win the point.

be large (greater than 0.5), the bias introduced here should still be small. The exact number depends both on the functional form of the matrix of the point game and the distribution of points within the match, but the bias is less than 0.05 for almost any specification of these parameters.

Minimizing the mean squared error of an estimator is equivalent to minimizing (_^{2} + ß^2) where _^{2} is the estimator's variance and ß is its bias. The bias introduced by including the additional points is small, while the variance of the sample decreases significantly with the inclusion of additional data (since the typical point game has 25 points at the start of the game and 50 points that are not played at the start of the game). Thus, using the additional data minimizes the mean squared error.[33]

In some of the matches, the chance to serve to the left on a given set of points is 100% (the divisions I use are: points at the start of a game, points after a serve to the left, and points after a serve to the right). If players are playing mixed strategies and there are a large number of points, this should not occur.[34] The 1974 Wimbledon Smith ad point game, for example, has the feature that Smith serves to the left the first point of all 26 games in the sample. This seems unlikely to happen randomly under the hypothesis of identical distribution of serves given that his chance to hit a serve to the left in the ad point game is only 86.8%.

Smith may use the serve to the left ("out wide") on the ad court as a way to focus at the start of a game, instead of hitting the serve to the right ("up the tee"). This is a

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[33] I performed the calculations for mean squared error with a few simplifying assumptions (mainly calculating standard errors with the assumption of independence and assuming that all biased points had equal bias) using representative matrices from the sample. The results were soundly in favor of using the additional data points. The simplifying assumptions should have had only a small effect on the results, so I did not revisit the calculations in further detail.

[34] Small sample size is one explanation of these results. If a typical point game contains 75 points (3300 points total/44 point games), then dividing the point game into three subdivisions (for $S_{L,0}$, $S_{L,R}$ and $S_{L,L}$) can make sample sizes for these subdivisions very small.

common tactic among amateur players, whose serves are less consistent than those of professional tennis players—this makes focusing devices more valuable.[35] However, personal correspondence with Ramsey Smith, the son of Stan Smith, indicates that this explanation is probably not correct—to quote Ramsey Smith, "My dad says that he has nothing against the serve up the tee!"

I performed some simple calculations to try to see if there is any change in winning percentages between the first deuce and ad points of a game and all subsequent points. For each match, I compared the win percentages for the server at the start of each game to the win percentages later in the game. The average match had the server winning 0.1% more of the points on the first points of a game than on subsequent points—a tiny change. There was no particular pattern to these results—most of the matches had small changes in the win percentage after the first point, about half increases and half decreases.

This is not consistent with my model (since the timing variable as presented so far decreases the server's chance to win the point but does not apply on the first points of a game, the server should win a higher percentage of points at the start of the game). This indicates an alternate model for the timing variable may provide better conclusions when the direction of serves and win percentage for the server are both considered.[36]

V. Empirical Specification:

[35] Additionally, an amateur player's opponents cannot watch a video of his previous matches (and do not have coaches who can do it for them), and may not be as adept as recognizing patterns in general.
[36] For example, a model in which the returner gets better at returning the serve he has just seen and worse at returning a serve that he has not seen probably would not have the property that the server wins a higher percentage of points at the start of the game.

The hypothesis that I am investigating is that the model of tennis with the timing variable accounts for the negative serial correlation in the server's choices found in Walker and Wooders (2001). That is, I am trying to determine if the data from professional tennis matches is consistent with servers playing optimally under the model including the timing variable.

When the matrices follow the "not skewed" assumptions and the timing variable is positive, my model predicts that $S_{L,L}<S_{L,0}<S_{L,R}$. That is, the server's chance to serve left is lowest when he has just hit a serve to the left, is highest when he has just hit a serve to the right, and is in between when it is the first point of a game (and there is no previous point). If $S_{L,L}$, $S_{L,0}$ and $S_{L,R}$ were all equal then this would occur with probability 1/6—in the data this condition holds in 29.5% of the point games. I performed a statistical test on the null hypothesis "The event $S_{L,L}<S_{L,0}<S_{L,R}$ occurs with probability 1/6" using a Bernoulli random variable. The resulting p-value is 0.022, so I can reject the null hypothesis at the 5% level.[37]

To test my model, I need some idea about what the payoffs in the matrices of each point game are. From my earlier arguments, the entries in a single row of each matrix should be within 0.3 of each other. Beyond that, only the server's chance to serve left and the server's chance to win the point can be observed but there are four variables (the entries in the matrix) that determine these values. Consequently, some form of guessing is unavoidable here.[38] Walker and Wooders (2001) estimate a power function for the test of equal winning percentages using a hypothetical point game matrix for their aggregate

[37] This is perhaps unsurprising given that Walker and Wooders (2001) find that players switch their serves up too much to be consistent with random play.
[38] It should be possible to use the maximum likelihood procedure used so far to estimate the underlying matrix for each point game simultaneously with T. However, this calculation would be very complicated and it is not clear if the results of the estimation would make sense.

data. That they did this suggests that the invisibility of the point games should not invalidate results based on specific assumptions about the matrix for a given point game.[39]

For example, for the point game with Federer serving on the ad court in the 2005 Australian Open Match, I estimated the matrix as follows (entries are Federer's chance to win the point). This matrix is consistent with the overall results of the match; namely, when Safin and Federer both play their equilibrium mixed strategies Federer wins 67.2% of the points and serves to the left with probability 65.2%. I also follow the approximation $0<(\pi_{LR} - \pi_{LL})< 0.3$ and $0<(\pi_{RL} - \pi_{RR})< 0.3$ established earlier and chose $(\pi_{LR} + \pi_{LL})> 1$ and $(\pi_{RL} + \pi_{RR}) >1$ so that if T>0, then $S_{L,R}> S_{L,0} > S_{L,L}$ holds and there is negative serial correlation of service choices.[40]

	Safin Receiving	
	L	R
Federer Serving		

[39] However, the results obtained from choosing a specific point-game matrix can depend greatly on the exact entries chosen. For example, the point-game matrix used by Walker and Wooders (2001) to estimate the power function for the test of equal winning percentages is [0.58 0.79; 0.73 0.49] and corresponds to the server serving to the left 53.3% of the time and winning 65.0% of the points in their aggregate data.

The differences between entries in the same row but different columns (that is, the differences (0.79-0.58)=0.21 and (0.73-0.49)=0.24) drive the power of the test. If these difference values were lower (higher), the power of the test would decrease (increase). For example, I can construct a matrix where these differences are 2/3 as great as before but the server's chance to serve left and the server's chance to win the point in equilibrium are unchanged. So the matrix looks like [0.58, 0.58+ 2/3*0.21; 0.49 + 2/3*0.24]—I then add a constant term to each entry in order to make the server's chance to win each point the same as before.

The resulting matrix is [0.6173, 0.7573; 0.6873 0.5273] (this gives the same 53.3% chance for the server to serve to the left and 65.0% chance for the server to win the point). Since I decreased the row differences by 2/3, this matrix has the property that if Walker and Wooders had used it instead of the matrix they chose, the power function would rise 2/3 as quickly as the returner moved away from playing his equilibrium mixed strategy.

[40] In about half of point games, the server's win percentage is greater than 65% so any matrix chosen according to the conditions $0<(\pi_{LR} - \pi_{LL})< 0.3$ and $0<(\pi_{RL} - \pi_{RR})< 0.3$ will have $(\pi_{LR} + \pi_{LL})> 1$ and $(\pi_{RL} + \pi_{RR}) >1$ hold (see "Further Assumptions About Point-Game Matrices" in the Theoretical Framework section). To give a general idea of the impact of the timing variable, in the 44 matrices chosen the estimate T=0.299 causes $S_{L,R,}$- $S_{L,L}$ to range from 0.1% to 3.1%.

L	0.61	0.73
R	0.789	0.56

I performed the same procedure on the other point games in the 11 matches in the sample (the 43 matrices estimated for the other point games are not shown for brevity).[41] Using these matrices, I found the likelihood function based on the sequence of serves from the points in the actual match as well as the matrices I had estimated. The likelihood function measures the chance of obtaining the sequence of points from the game by multiplying the probabilities of obtaining each serve (conditional on previous serves). For example, the Safin-Federer deuce point game begins as follows. The type of point in the left hand column is overridingly $S_{L,0}$ if the point begins a new game, is $S_{L,L}$ if the point follows a serve to the left and $S_{L,R}$ if the point follows a serve to the right.

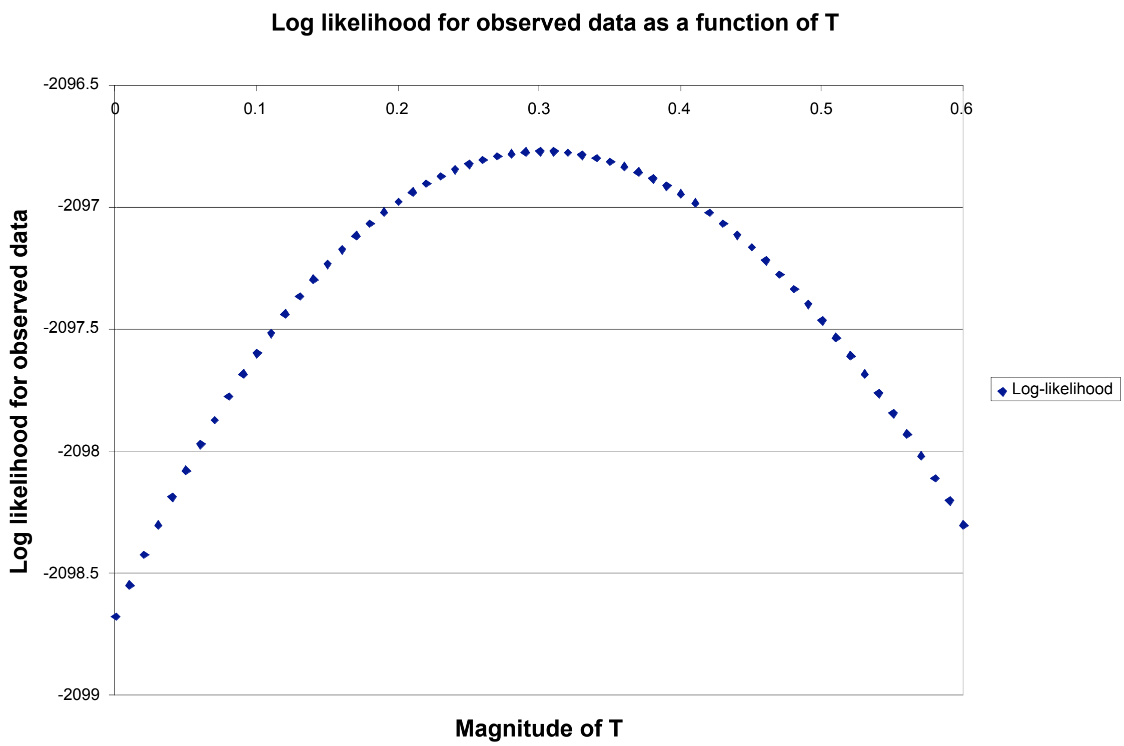
Type of point?	Service location
S(L,0)	RIGHT
S(L,R)	LEFT
S(L,L)	LEFT
S(L,0)	LEFT
S(L,L)	LEFT

The value for T determines the values of $S_{L,L}$ and $S_{L,R}$—$S_{L,0}$ is fixed no matter what T is. Thus, the likelihood is a function of T and if the model presented here is correct there should be a single positive value of T that maximizes the likelihood while

[41] I did not generate the matrices according to some fixed rule because any rule would cause some of the matrices to be invalid (either because the matrix would contain entries that are not between 0 and 1 or because dominant strategies would exist). This is probably inevitable in a varied data set with good reason: if all of the matrices were of a particular form, it might have implications (such as the server hitting most of his serves to the returner's weaker side) that do not always hold.

leaving the equilibrium conditions intact. I maximize the (natural) log of the likelihood function to simplify calculations.

The graph of the log-likelihood as a function of the timing variable T is shown on the following page. The value of T that maximizes the log-likelihood is 0.299. This value is consistent with the assumptions I have been making so far. As shown earlier, $T>1$ would not be consistent with the mixed strategy equilibrium conditions in the model in general. Furthermore, based on the specific matrices that I chose there is often some value of T for each point game (usually close to but less than 1) that would cause dominant strategies to emerge in that point game. The result of $T=0.299$ is significantly less than this "fall to dominance" value for each of the matrices chosen. At the same time, I would not expect T to be very close to 0 (less than 0.1, for example). If T was very small, it would have almost no effect on a server's choices and would consequently be able to explain almost none of the negative serial correlation observed in Walker and Wooders (2001). A T-value of 0.299 should have neither of these problems.



As a quick estimate of the power of this test, I generated data for 20 trials on the assumption that the underlying matrices that I have been using are correct but that T=0 (so that points are actually independent). I then estimated the value for T using the procedure outlined so far. Out of the 20 trials, 18 of the trials had T<= 0.12, one trial had T=0.22 and one trial had T=0.34 (which is greater than the result observed from the data). This suggests that if this model is correct, the estimated T-value of 0.299 is a good indication that T is actually non-zero.[42]

[42] It seems that any sensitivity analysis of this test that involves changing the matrices chosen would be exceedingly difficult. If I came up with 2 possibilities for what the matrix of each point game looks like then there would be 2^44 possible combinations of these matrices (each requiring a separate maximum likelihood calculation for the value of T). It may be possible to estimate a new set of matrices based on the old matrices with some uniform change applied to each. The results using the new matrices could then be compared to the results using the old matrices. Even if this result would not be as powerful, calculating T twice is a lot easier than doing it 20 trillion times!

The Run Test

I use the "run test" that was used by Walker and Wooders (2001) to test for serial independence of service choices. A run is defined as a maximal string of consecutive identical serves, either all to the left or all to the right.[43] In the case where serves are chosen independently, the formula for the probability of **r** runs in a string of $\mathbf{n_L}$ serves to the left and $\mathbf{n_R}$ serves to the right is known (see Ross, 1992).[44]

Here are the results of the run test as applied to the tennis data—the variable r^i measures the number of runs for a given point game. $F(r^i-1)$ is the chance to get less than or equal to (r^i-1) runs, given the number of serves to the left and right in this point game. $F(r^i)$ is the chance to get less than or equal to (r^i) runs.[45] Since the run test is not a continuous distribution I follow the method used in Walker and Wooders (2001) and construct a uniform draw on $[F(r^i-1), F(r^i)]$ for each point game. The U $[F(r^i-1, F(r^i)]$, column is a *sample* draw from this distribution. When serves are chosen independently these draws should have the uniform (0,1) distribution.[46]

		Total Points	Left Serves	Right Serves	Runs (r^i)	$F(r^i-1)$,	$F(r^i)$	U[$F(r^i-1),F(r^i)$]
2005	Safin , deuce	70	49	21	35	0.876	0.936	0.888

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[43] For example, if there are four points with serves (Left, Left, Right, Left), there is a run of two lefts, a run of one right and a run of one left for a total of 3 runs.

[44] The chance to get a certain distribution of runs under independence is known. By comparing the number of runs in the data to the number of runs predicted under independence, you can see if the data is consistent with the assumption that players' service choices are serially independent. If service choices are negatively serially correlated, there should be more runs than if service choices are independent. If service choices are positively serially correlated, there should be fewer runs than if service choices are independent.

[45] For example, if in a given point game $[F(r^i-1), F(r^i)]= [0.98, 1]$ then if serves were being chosen independently you would have a 100% chance to have this many or fewer runs and a 98% chance to have fewer runs, which means that you would have a 2% chance to have exactly this many runs and a 98% chance to have fewer runs. In this case, when you have many more runs than would be typical under independence, it means that serves are probably being chosen with negative serial correlation (as Walker and Wooders (2001) find is the case in the tennis data).

Consider a different example, where $[F(r^i-1), F(r^i)]= [0.48, 0.51]$ in a given point game. If serves were being chosen independently you would have a 48% chance to have fewer than this many runs, a 3% chance to have exactly this many runs, and a 49% chance to have more than this many runs. In this case you have a number of runs that would be typical if serves were chosen independently.

[46] See footnote 24 in Walker and Wooders (1999) for a proof of this assertion.

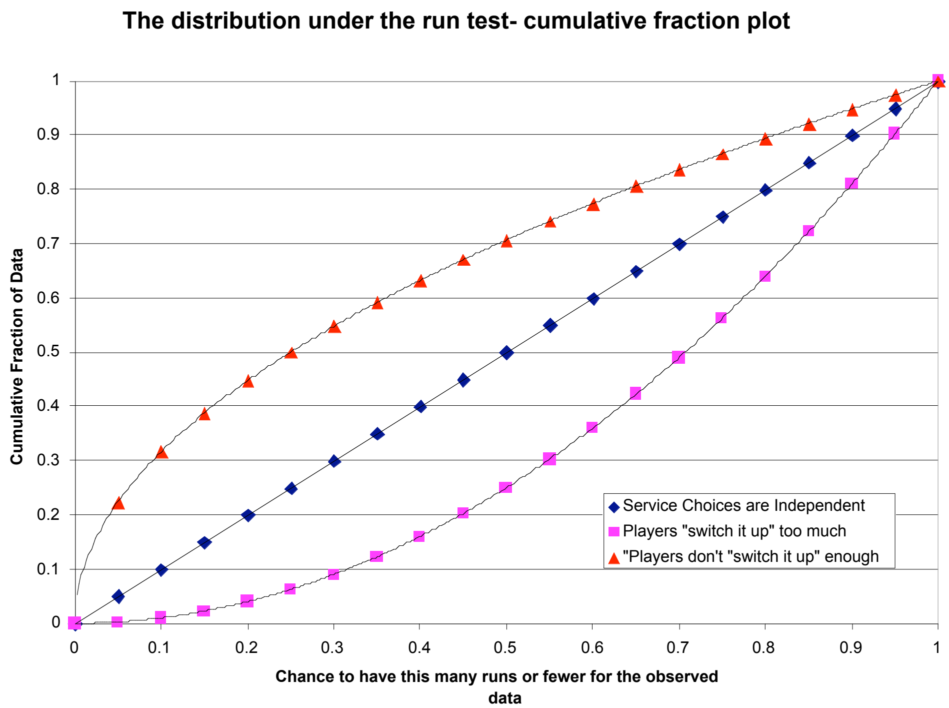
Australian								
	Safin, ad	74	51	23	39	0.943	0.974	0.948
	Fed, deuce	72	27	45	33	0.282	0.376	0.335
	Fed, ad	64	46	18	27	0.436	0.580	0.516
1974 Wimbledon	Rosewall, deuce	75	70	5	11	0.349	1.000	0.926
	Rosewall, ad	74	37	37	43	0.854	0.901	0.870
	Smith, deuce	82	53	29	43	0.832	0.892	0.869
	Smith, ad	76	66	10	21	0.812	1.000	0.825
1980 Wimbledon	Mc, Deuce	89	45	44	49	0.739	0.803	0.799
	Mc, Ad	85	45	40	44	0.512	0.599	0.514
	Borg, deuce	99	37	62	52	0.817	0.866	0.820
	Borg, ad	92	19	73	33	0.633	0.788	0.647
1980 US Open	Mc, Deuce	84	52	32	36	0.118	0.168	0.128
	Mc, Ad	79	37	42	38	0.259	0.338	0.322
	Borg, deuce	80	29	51	43	0.863	0.914	0.907
	Borg, ad	76	29	47	42	0.873	0.916	0.906
1982 Wimbledon	Mc, Deuce	79	35	44	36	0.152	0.212	0.186
	Mc, Ad	71	32	39	36	0.437	0.533	0.467
	Connors, deuce	91	76	15	31	0.958	1.000	0.987
	Connors, ad	78	33	45	48	0.976	0.987	0.978
1984 French Open	Mc, Deuce	72	42	30	45	0.982	0.991	0.991
	Mc, Ad	67	38	29	40	0.921	0.952	0.933
	Lendl, deuce	71	26	45	41	0.955	0.976	0.976
	Lendl, ad	67	33	34	41	0.931	0.958	0.933
1987 Australian	Edberg, D	75	19	56	29	0.374	0.519	0.433
	Edberg, Ad	69	47	22	40	0.994	0.997	0.996
	Cash, D	68	39	29	37	0.711	0.791	0.711
	Cash, Ad	65	38	27	40	0.964	0.980	0.980
1988 Australian	Wilander, deuce	76	20	56	29	0.265	0.389	0.309
	Wilander, ad	68	32	36	38	0.739	0.813	0.787
	Cash, deuce	74	37	37	28	0.007	0.013	0.009
	Cash, ad	63	40	23	29	0.316	0.424	0.322
1988 Masters	Becker, Deuce	84	53	31	45	0.847	0.900	0.849
	Becker, Ad	76	50	26	38	0.724	0.796	0.786
	Lendl, Deuce	84	46	38	43	0.489	0.577	0.524
	Lendl, Ad	76	55	21	32	0.515	0.607	0.521
1995 US Open	Samp, Deuce	59	33	26	22	0.011	0.021	0.013
	Sampras, ad	57	20	37	25	0.231	0.335	0.247
	Ag, deuce	59	30	29	24	0.032	0.058	0.042
	Ag, ad	55	39	16	29	0.943	0.980	0.969
1997 US Open	Samp, Deuce	94	50	44	39	0.026	0.041	0.039
	Sampras, Ad	83	33	50	37	0.162	0.227	0.184
	Korda, deuce	82	52	30	43	0.793	0.859	0.802
	Korda, ad	74	55	19	28	0.301	0.389	0.389

The run test is used in order to test my null hypothesis that the model that includes the timing variable is sufficient to explain the deviation from serial independence observed in Walker and Wooders. I determined the distribution of the run test under my null hypothesis (where points are no longer independent because the timing variable T=0.3) to do this. Since the distribution of the run test under my null hypothesis is not known I simulated it instead. I used the sequence of "first points" for the entire data sample as well as the 44 underlying matrices (chosen to represent the point games) and my estimate that T=0.3. The simulation filled in all of the service choices made during the tennis matches according to the optimal strategies calculated from the matrices, "first points" and previous serves chosen in the course of the simulation. I then calculated the "run test" results for these simulated point games and constructed draws on the appropriate $U[F(r^i-1),F(r^i)]$ distribution for each simulated point game.

I use the Kolmogorov-Smirnov test ("KS test") used in Walker and Wooders (2001) to test for the equality of the run test distributions for the actual data compared to the simulated data.[47] If the distribution of the run test in the actual data falls "too far" from the simulated distribution the KS test will reject the null hypothesis that they are from the same distribution (and that my model can explain the observed serial correlation of serves). If I reject the null hypothesis, it should be because optimal play under my model only explains part of the excess "switching it up" found by Walker and Wooders (2001).[48] Graphically, the run test should give you three general results that would look

[47] The KS test is a non-parametric test for the equality of two data sets—it makes no assumptions about the distribution of the data.
[48] Since I have added only one variable to the model in Walker and Wooders (2001), it seems extremely unlikely that I would reject the null hypothesis that my model account for the actual data observed because

something like the following:



The 45° line is what the cumulative distribution should look like if players were making service choices completely independently. Intuitively, if serves are independent the run test should have the uniform distribution (as it is designed to test for independence) and thus the cumulative distribution should be the 45° line.

The parabola (at the bottom) is a distribution where the number of runs observed in the data is higher than is consistent with serially independent service choices. Note that for any set cumulative fraction of the data (the y-axis), the number of runs observed in the sample (the x-axis) is greater than it would be in the case of serial independence. Since there are too many runs players are "switching it up" more often than is consistent

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the model predicted significantly *more* negative serial correlation of service choices than is found in the actual data.

with appropriately random play—this is what Walker and Wooders (2001) find for tennis serves.

The hyperbola (at the top) is a distribution where the number of runs observed in the data is lower than is consistent with appropriately random play. This means that players don't "switch it up" enough.

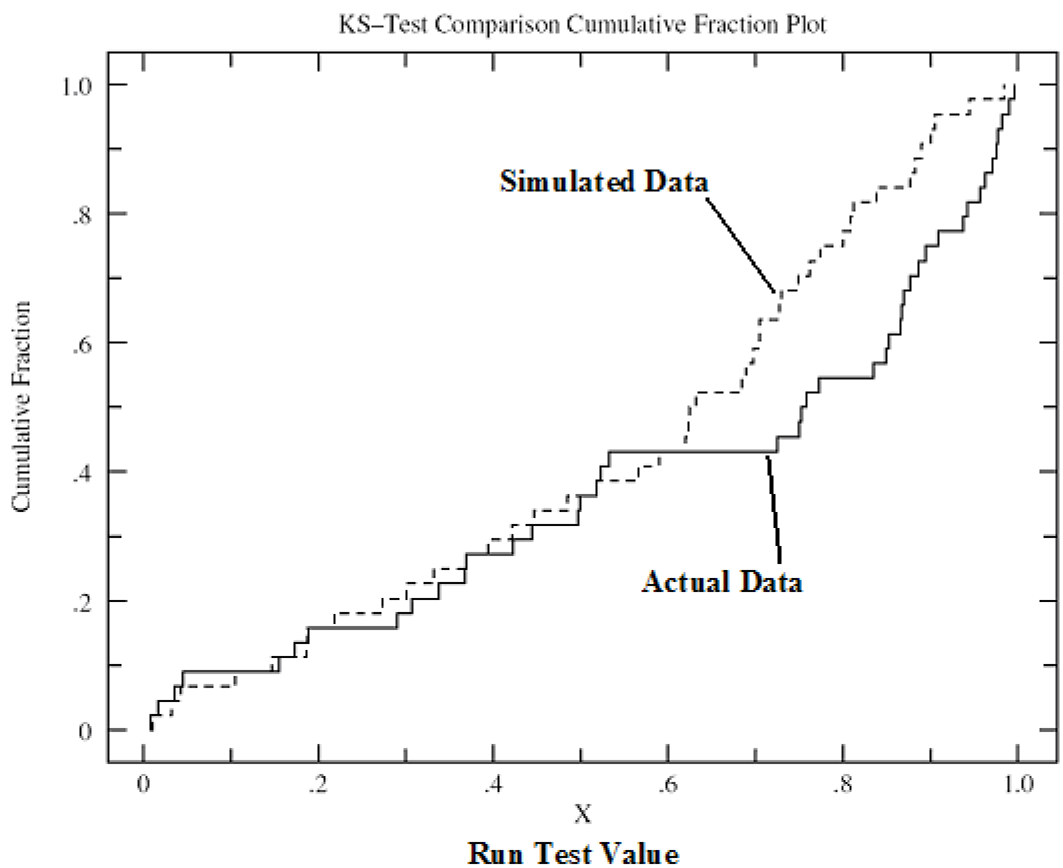
The question my simulation answers is: if the model presented so far is correct, how would players choose serves optimally? I then determine if this optimal strategy under my null hypothesis is consistent with the choices observed by in the tennis data. That is, can the combination of the entries in each payoff matrix and the constant timing variable T (calculated earlier to be 0.299) account for the serial dependence of serves observed in Walker and Wooders (2001)?

On the following page there is a sample cumulative fraction plot for the one of the trials in the data where the KS test was run to compare the actual and simulated data (associated p-value=0.062).[49] The actual data tends to have more runs for each cumulative fraction than the simulated data (that is, the graph of the actual data is to the right of the graph of the simulated data).[50] This is a general result—my model does not account for all of the serial correlation in tennis player's serves. I ran 1500 comparisons of the actual and simulated data, obtaining a p-value for each—if the model was true, it should only be rejected roughly 5% of the time at the 5% level. This did not happen. At the 5% level, the null hypothesis was rejected in 25.2% of the trials; it was rejected at the

[49] The simulated data is generated entirely anew in each trial while the actual data in each trial differs only based on the difference in the $U[F(r^i-1), F(r^i)]$ draws each time (in the actual tennis data, $F(r^i-1)$ and $F(r^i)$ are fixed for each point game based on the data from that point game).

[50] Another way to see this it is the cumulative fraction of point-games with run test values less than a particular value. For example, the cumulative fraction of point games with run test values less than 0.8 is about 0.75 in the simulated data and 0.55 in the actual data. There are more runs in the actual data—45% of the point games in the actual data have run test values greater than 0.8 while only 25% of the point games in the simulated data have run test values greater than 0.8.

1% level in 7.1% of the trials and at the 10% level in 38.0% of the trials. These results indicate that my model is not sufficient to explain why tennis players "switch up" their service choices more than is consistent with purely random play.



How much of the negative serial correlation of serves in the data is explained by this model? The Kolmogorov-Smirnov test evaluates the equality of distributions but it does not provide a direct measure of how far apart the distributions are. As the KS test is nonparametric, there is no direct way to compute an R^2 value. In order to evaluate how much of the negative serial correlation of serves observed in the data the model explains, I constructed two simple statistics. The first was to compare the averages of the values for the run test on the simulated data with the average value in the actual data.[51] The average value for the actual data is 0.63. If service choices were independent, the run test would have the uniform distribution and the average would be 0.5. Thus tennis players in the data have an average value under the run test 0.13 higher than would be expected if service choices were independent. The simulated data, on the other hand, had an average run test value of 0.56, 0.06 higher than the value associated with independence. Thus, going by this measure, the model explained 46% of the data (0.06/0.13).

Another method that I used was to determine how often the simulated data for the run test accounted for the actual data. To do this I determined, for each point game, whether the value for the U $[F(r^i-1, F(r^i)]$ sample draw from the run test in the simulated data was greater than the value for the run test in the actual data. In the simulated data set, this happened with probability 0.42. To see what this means, consider some alternatives.

If the actual data had the same distribution as the simulated data (i.e., my null hypothesis explained exactly the variation in service choices), this should happen with probability _. If the simulated data did not account for any of the excess "switching it up" in service choices (so service choices in the simulation were independent, as would have been the case if T=0), then it would be expected to happen with probability (1-the average value of the sample draws from the run test in the actual data)= (1- 0.63, the value calculated above)= 0.37. A method that did not explain any serial correlation would fail to work 0.13 more often than a method that explained all of it. Since the simulated data worked (0.42-0.37)=0.05 more often than this, it can be said to account for

[51] These are the U $[F(r^i-1, F(r^i)]$ random values chosen to make the run test distribution continuous. While any individual value can vary considerably, so many values were generated that the average across all of the random values has a very small variance.

0.05/0.13= 39% of the additional switching up in the data.[52] Thus, these two measures indicate that this model accounts for slightly less than half of the amount that servers "switch up" their choices beyond that which would be predicted by serial independence.

[52] These measures could give results greater than 1. This would indicate that the simulated data had more negative serial correlation than the actual data.

VI. Conclusion:

Although game theory predicts that people will play according to mixed-strategy equilibrium, most empirical research has not been consistent with this prediction. Walker and Wooders (2001) examined mixed-strategy play professional tennis matches and found that professional tennis players satisfied the equal winning percentages condition of mixed-strategy equilibrium, but did not satisfy the condition of serial independence of service choices.

This paper changed the model of tennis used in Walker and Wooders (2001) by assuming that there was a short-term timing effect where a serve that you have just hit is less effective the next time. I estimated the magnitude of this effect, determined what optimal play was under this model, and simulated how professional tennis players would play under this model. I then used the "run test" to compare the distribution of serves under my model with the distribution observed in the data from professional tennis matches. My conclusion was that the model explained a little under half of the deviation from optimal play found in the data from professional tennis matches. Professional tennis players play closer to game theory's predictions when tested using a model that reflects more of the complexities of tennis, but they still do not play perfectly.

These results suggest that professional tennis players are still not fully rational actors—their strategic abilities are very good but still limited in some way.[53] It is possible that players have imperfect recall, which means that the returner cannot take advantage of serial correlation by the server—it is also possible that the server tries to choose serves in a relatively random fashion, but does not succeed. There are many

[53] Since the players involved are top professional tennis players, it may also be reasonable to suppose that these players *do* play optimally and deviations from optimal play are primarily due to the fact that the model does not fully reflect the game of tennis.

explanations for why players may not play optimally in strategic settings and there is a growing literature devoted to explaining strategic interaction when players are not the highly rational players that game theory typically assumes.[54]

One final explanation for deviations from mixed-strategy equilibrium play may be that there is an inherent cost to making good strategic decisions, especially while you are trying to remain focused on the physical aspects of a tennis match. To use the memorable words of Jim Courier, a former #1 in the world tennis player, “I don't have that much mental energy, so I have to kind of guard it with my life."

[54] For example, Erev and Roth (1998) study reinforcement learning in experimental games with unique mixed strategy equilibria.

Appendix:

Example 1: In a tennis game where service choices affect the payoff matrix of the successive point, the strategy used by the server to maximize his chance to win the game is not generally equivalent to trying to win each point as if it was the only point.

Consider a "game" of tennis played under the following simplified scoring rules (all points are played on the same side of the court). If the server wins the first point, he wins the game (and the game is stopped). If the server loses the first point, then a second point is played and the winner of the second point wins the game.

Each player tries to maximize his chance to win the *game*. Designate the chance for the server to win the first point as V_1 and the chance for the server to win the second point as V_2. The server's chance to win the game= $V_1 + (1-V_1)V_2$

By backward induction, on the second point each player will try to maximize his chance to win the point as if this were the only point (since there are no future points).

On a point when each player maximizes his chance to win the point as if it was the only point, designate the server's resulting chance to win this point as $\mathbf{V_L}$ if his previous serve was to the left and as $\mathbf{V_R}$ if his previous serve was to the right.

From the perspective of the players deciding what to do on the first point of the game, the matrix now looks like this (entries represent the server's chance to win the game):

		Returner Left	Returner Right
Server	Left	$\pi_{LL}+ (1- \pi_{LL})V_L$	$\pi_{LR}+ (1- \pi_{LR})V_L$
	Right	$\pi_{RL}+ (1- \pi_{RL})V_R$	$\pi_{RR}+ (1- \pi_{RR})V_R$

The same formula for the chance for the server to serve left as before applies and gives (after some algebra): $S^*_L= (\pi_{RL} - \pi_{RR})/[(\pi_{RL} -\pi_{RR}) + (\pi_{LR}- \pi_{LL})\mathbf{*(1- V_L)/(1- V_R)]}$

The chance to serve left if the server treated this point as the only point is $S_L= (\pi_{RL} - \pi_{RR})/[(\pi_{RL} -\pi_{RR}) + (\pi_{LR}- \pi_{LL})]$

These chances are equal if and only if $V_L=V_R$. That is, in this "game" of tennis, the chance for the server to serve in each direction is unchanged by the state of the game (so he plays each point as if it was the only point) if and only if the server's chance to win the next point, given that it will be played by both players as if it is the only point, is independent of his choice of where to serve on this point.

Call this condition—that the server's chance to win the next point, given that it will be played by both players as if it is the only point, is independent of his choice of where to serve on this point—the Equal Subsequent Point Winning Percentages Condition (EWC).

It can also be shown by simple algebra that when V_L=V_R, the returner's chance to anticipate to the left is the same as when he treats this point as the only point.

Proposition 1: If the Equal Subsequent Point Winning Percentages Condition holds, in a game of finite maximum length where service choices only affect the payoff matrix of the subsequent point, it is the optimal strategy for each player to maximize his chance to win the game by playing each point as if it were the only point.[55]

Proof: By backwards induction,
1. On the last possible point of the service game, there are no future points so the players trivially play the current point as if it was the only point.

2. Assume that for a game with a maximum of n points left, Proposition 1 holds.

3. Prove that if Proposition 1 is true for a game with n points left, it is true for a game with n+1 points left. The matrix for the current point is as follows:

		Returner	
		Left	Right
Server	Left	$\pi_{LL}H_L + (1-\pi_{LL})L_L$	$\pi_{LR}H_L + (1-\pi_{LR})L_L$
	Right	$\pi_{RL}H_R + (1-\pi_{RL})L_R$	$\pi_{RR}H_R + (1-\pi_{RR})L_R$

π_{LL}, π_{LR}, π_{RL}, and π_{RR} are the server's chance to win the current point, based on the serve and anticipation chosen. H_L is the server's chance to win the game, conditional on hitting the current serve to the *left* and *winning* the current point. L_L is the server's chance to win the game, conditional on hitting the current serve to the *left* and *losing* the current point. H_R is the server's chance to win the game, conditional on hitting the current serve to the *right* and *winning* the current point. L_R is the server's chance to win the game, conditional on hitting the current serve to the *right* and *losing* the current point.

On the next point in the game after the current point, we know by assumption that the players will use strategies as if it were the only point. By EWC, when the server plays the point after this point as if it were the only point, he must have the same winning percentage regardless of where his current serve is hit. Thus, the chance for the server to win the next point is equal regardless of the current service location.[56] This logic can be applied to every point in the game after the current point, so the chance for the server to win each subsequent point cannot depend on his current serve's location.

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[55] A game with a fixed maximum length does not have to reach its maximum length each time it is played. For example, a game that is won by the first player to reach four points (at which point the game is stopped) has a maximum length of seven points, even though it could stop after as few as four points.
In the model in this paper (using the actual scoring system of tennis), each point affects the payoffs only of the subsequent point on *the same side of the court*, which is the point two points from now in the game.
[56] Note that due to the effects of timing, the matrices governing the next point will be different based on the server's current action (so the server's chance to serve left on the next point may depend on the choice for the current serve). However, the equilibrium win percentage for the server will be the same by EWC.

Since the server's chance to win each future point is independent of his service choice on this point, it must be the case that $H_L = H_R$ and $L_L = L_R$. Use this to calculate S_L and A_L, the server and returner's equilibrium probability to serve or anticipate left.
$S_L = (\pi_{RL} - \pi_{RR})/[(\pi_{RL} - \pi_{RR}) + (\pi_{LR} - \pi_{LL})]$
$A_L = (\pi_{LR} - \pi_{RR})/[(\pi_{RL} - \pi_{RR}) + (\pi_{LR} - \pi_{LL})]$

These chances for the server to serve to the left and for the returner to anticipate to the left are the same as the chances that when each player maximizes his chance to win the current point as if it was the only point (since π_{LL}, π_{LR}, π_{RL}, and π_{RR} are the server's chance to win the current point, based on the serve and anticipation chosen).

Thus, since this holds for a game with a maximum of one point remaining and holds for a game with a maximum of n+1 points remaining if it holds for a game with n points, this result holds for any game with a maximum number of points. **QED**

In the model in this paper, each game in a tennis match does not affect the other games. Thus, players should maximize their chance to win each game as if it was the only game.[57]

When maximizing the chance to win each game as if it was the only game means maximizing the chance to win each point as if it was the only point, players should play each point in the match as if it was the only point.

It is an intuitive result that equal subsequent winning percentages cause players to make choices independent of the current game state. Intuitively it seems likely that this result would hold under the normal scoring system of tennis as well, (where players alternate sides of the court and games are played first to four points, with a lead of two points), but a proof of that is beyond this paper.

In the model that I use in the paper with the parabolic timing variable, the condition $V_L = V_R$ will only occur (for all values of T) when specific conditions are placed on the matrix in question. In particular, I will now show that $V_L = V_R$ occurs (for all values of T) if and only if when the matrix is symmetric ($\pi_{LR} = \pi_{RL}$ and $\pi_{LL} = \pi_{RR}$).

However, it is clear that symmetry of point game matrices is a bad assumption to make in general. By simple calculation, any symmetric point game matrix results in S_L= _ (that the server serves to the left with probability _), which is definitely not the case. Thus, the optimal strategy for each player will in general be different from maximizing his chance to win each point as if it were the only point.

[57] This result is intuitive and seems like it follows from Walker and Wooders (2000).

Proposition 2: In the model of the timing variable used in this paper, $V_L=V_R$ occurs if and only if the underlying point game matrix is symmetric ($\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$).

Proof that if ($\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$), then $V_L=V_R$: When $\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$ it is trivial to show algebraically that $V_L=V_R$ (note that the point game matrix that results after a serve to the left is the same as the point game matrix that results after the serve to the right, except with the bottom and top rows reversed). **QED**

Proof that if $V_L=V_R$, then ($\pi_{LR}=\pi_{RL}$ and $\pi_{LL}= \pi_{RR}$): For ease of notation, let $\pi_{LL}= a$; $\pi_{LR}=b$; $\pi_{RL}=c$; and $\pi_{RR}= d$. This means that the point game matrix can be written as [a b; c d] and the mixed strategy equilibrium conditions are b>a, b>d, c>a, c>d.

After a serve to the left, the equilibrium winning percentage for the server on the next point can be calculated to be:
$V_L= [(bc- ad) + \mathbf{T}*(ad(1-a) - bc(1-b))] /[(b+ c -a -d)+ \mathbf{T}(a(1-a) -b(1-b))]$

Likewise, after a serve to the right, the equilibrium winning percentage for the server on the next point can be calculated to be: [58]

$V_R= [(bc- ad) + T*(ad(1-d) - bc(1-c))] /[(b+ c- a- d)+ T(d(1-d) -c(1-c))]$

For simplicity of notation, let (bc-ad)= K and let (b+c-a-d)= R. Then let (ad(1-a)- bc(1-b))= N_L, a(1-a)-b(1-b)= D_L, (ad(1-d) - bc(1-c))= N_R, and (d(1-d) –c(1-c))= D_R

This reduces the equations to:
$V_L= (K + T*N_L)/(R + T*D_L)$
$V_R= (K + T*N_R)/(R + T*D_R)$

We are looking for cases when $V_L= V_R$ for all T ($0\leq T<1$), so equating them and cross multiplying (T<1 guarantees that the denominators will be non-zero) gives:

$(K + T*N_L)*(R + T*D_R) = (K + T*N_R)*(R + T*D_L)$
Multiplying out and subtracting K*R from both sides gives:

$T(N_LR + D_RK) + T^2N_LD_R = T(RN_R + KD_L) + T^2N_RD_L$ (for all T)
Since this equation must hold for all T, we can equate the components multiplying the same powers of T to get the two equations:

$N_LR + D_RK= RN_R + KD_L$ (1)
$N_LD_R= N_RD_L$ (2)

First, solve for N_R in terms of the other variables in (1).

$N_R = (N_LR + D_RK - KD_L)/R$

[58] These calculations assume that the mixed strategy equilibrium conditions still hold on these points.

Then, substitute N_R into (2) and solve for D_L in terms of N_L and D_R.
$N_L D_R = D_L(N_L R + D_R K - KD_L)/R$

$RN_L D_R = (N_L R + D_R K)D_L – K(D_L)^2$

$K(\mathbf{D_L})^2 - (N_L R + D_R K)\mathbf{D_L} + RN_L D_R = 0$

Use the quadratic equation to solve for D_L in terms of the other variables.

$D_L = [(N_L R + D_R K) \pm \sqrt{((N_L R + D_R K)^2 – 4KRN_L D_R)}] / (2K)$
$D_L = [(N_L R + D_R K) \pm \sqrt{((N_L R)^2 + 2N_L RD_R K + (D_R K)^2 – 4KRN_L D_R) }] / (2K)$
$D_L = [(N_L R + D_R K) \pm \sqrt{((N_L R)^2 - 2N_L RD_R K + (D_R K)^2)}]/ (2K)$
$D_L = [(N_L R + D_R K) \pm \sqrt{((N_L R - D_R K)^2)}] /(2K)$

$D_L = [(N_L R + D_R K) \pm (N_L R - D_R K)] /(2K)$

Two roots:
Case 1: $D_L = N_L R/K$, or
Case 2: $D_L = D_R$

In Case 1, $D_L = N_L R/K$, so by (2) $KD_R = RN_R$, and $N_R = KD_R/R$

Substituting back into V_R, $V_R = (K + TKD_R/R)/ (R + TD_R)$
$V_R = K(1 + TD_R/R) / [R(1 + TD_R/R)]$
$V_R = K/R$. (by similar algebra, $V_L = K/L$ as well)

In solutions under case 1, V_R is independent of T. However, we know (see Lemma 1 in the proof of Proposition 3) that when the mixed strategy equilibrium conditions are satisfied, V is a strictly increasing function of the entries of the matrix and we also know that as T increases, two of the entries in the matrix must decrease. As T increases and the entries in the matrix decrease, V must decrease. Thus we cannot have any solutions of the form of Case 1.

Consider case 2 from here on: In case 2, $D_L = D_R$ so by (2) $N_L = N_R$

Substitute back in to these two equations to get the following two equations:

$a(1-a)-b(1-b)= d(1-d) –c(1-c)$ (3)

$ad(1-a)- bc(1-b)= ad(1-d) - bc(1-c)$ (4)

Equation 3 reduces to:

$b^2 – c^2 – (b-c)= a^2 – d^2 –(a-d)$ and then reduces further to

(b+c-1)(b-c)= (a+d-1) (a-d) **(3^*)**

Equation (4) reduces to:

$b^2c – a^2d = -ad^2 + bc^2$ and then reduces further to

bc(b-c)=ad(a-d) **(4^*)**

It is immediately apparent that if b=c, then a=d; also, if a=d, then b=c. Thus (b=c, a=d) is one solution to the equations 3^* and 4^* and is the unique solution to these equations when at least one of ((b=c) or (a=d)) holds.

I now show that there are no other solutions. Assume there is a solution to this equation such that b≠c and a≠d. Solve for the value for **a** in this solution, given **b**, **c**, and **d**.

Divide (3^*) by (4^*) to get
(b+c-1)/bc= (a+d-1)/(ad) Cross multiply to get
abd + acd – ad = abc + bcd – bc Group terms with **a** on the left side
a(bd + cd – d – bc)= bc(d-1)
a= bc(1-d)/ (bc+d – bc –cd)

I now show that this value of **a** violates the mixed strategy equilibrium condition b>a (it violates c>a as well):

Assume that there exists a value of **a** that satisfies the mixed strategy equilibrium condition b>a. That means that (substituting in for a):

b> bc(1-d)/(bc + d- bd- cd) Divide both sides by b and cross multiply:
bc + d –bd – cd > c(1-d)
bc + d – bd – cd > c-cd
bc + d – bd > c
bc-bd > (c-d)
b(c-d) > (c-d) c-d >0, so divide both sides by (c-d)
b>1

→But this contradicts our original assumption that b<1. So there cannot be a solution to these equations for b≠c and a≠d that satisfies the mixed strategy equilibrium conditions. As a result, solutions to these equations must satisfy at least one of ((b=c) and (a=d)).

Thus, the only solution to these equations is (b=c, a=d), which means that the only payoff matrix for which $V_L= V_R$ (the server's chance to win the next point, given that players use their equilibrium mixed strategies on that point as if it was the only point, is the same if he hits the serve on the current point to the left or the right) is a symmetric payoff matrix. **QED**

Proposition 3: Whenever ($\pi_{LL} + \pi_{LR} \geq 1$) and ($\pi_{RL} + \pi_{RR} \geq 1$) both hold and at least one does not hold with equality, the server's equilibrium chance to win the point must be greater than _.

Proof:

Lemma 1: The server's equilibrium chance to win the point is strictly increasing with each of π_{LL}, π_{LR}, π_{RL}, and π_{RR}.

Proof of Lemma 1: The server's chance to win the point in equilibrium, V, is $V = (\pi_{LR}\pi_{RL} - \pi_{LL}\pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})$.

By symmetry, it is sufficient to show that the server's chance to win the point is strictly increasing in π_{LR} and π_{LL}.

By calculation, $\partial V/\partial\pi_{LR} = (\pi_{RL} - \pi_{LL})(\pi_{RL} - \pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})^2$.
Since by assumption $\pi_{RL} > \pi_{LL}$, $\pi_{RL} > \pi_{RR}$, $\pi_{LR} > \pi_{LL}$, and $\pi_{LR} > \pi_{RR}$, each of these terms is positive so $\partial V/\partial\pi_{LR} > 0$ and the chance to win the point is increasing in π_{LR}.

Likewise, by calculation $\partial V/\partial\pi_{LL} = (\pi_{LR} - \pi_{RR})(\pi_{RL} - \pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})^2$
By the same assumptions as before, each of these three terms is positive so $\partial V/\partial\pi_{LL} > 0$ and the chance to win the point is increasing in π_{LL}. **QED**

Lemma 2: If ($\pi_{LL} + \pi_{LR} = 1$) and ($\pi_{RL} + \pi_{RR} = 1$), then the server's equilibrium chance to win the point must equal _.

Proof of Lemma 2: $V = (\pi_{LR}\pi_{RL} - \pi_{LL}\pi_{RR}) / (\pi_{RL} + \pi_{LR} - \pi_{LL} - \pi_{RR})$. By substitution for π_{LL} and π_{RR},
$V = (\pi_{RL} + \pi_{LR} - 1)/[2*(\pi_{RL} + \pi_{LR} - 1)] = 1/2$. **QED**

Without loss of generality, assume that ($\pi_{LL} + \pi_{LR} > 1$) and ($\pi_{RL} + \pi_{RR} = 1$). Let $\pi_{LL} + \pi_{LR} = 1 + 2\_{}_1$, where $\_{}_1$ is some positive constant.

Consider the case when you have $\pi^*_{LL} = \pi_{LL} - \_{}_1$, $\pi^*_{LR} = \pi_{LR} - \_{}_1$, $\pi^*_{RL} = \pi_{RL}$, and $\pi^*_{RR} = \pi_{LL}$. Then since $\pi^*_{LL} + \pi^*_{LR} = 1$ and $\pi^*_{RL} + \pi^*_{RR} = 1$, then the server's chance to win the point is _.

In the case of π_{LL}, π_{LR}, π_{RL}, and π_{RR}, two of these values (π_{LL} and π_{LR}) are greater than the values π^*_{LL}, π^*_{LR}, π^*_{RL}, and π^*_{RR}. By Lemma 1 the chance to win is a strictly increasing function of each of these values, so the overall chance to win the point in this case must be greater than the chance to win in the previous (π^*_{LL}, π^*_{LR}, π^*_{RL}, π^*_{RR}) case. The chance to win the point before was _, so the chance to win the point now must be greater than _.

Because of symmetry, an equivalent argument can be used to show that when ($\pi_{LL} + \pi_{LR} = 1$) and ($\pi_{RL} + \pi_{RR} > 1$), the server's chance to win the point must be greater than _.

The server's chance to win the point is a strictly increasing function of the point game matrix entries, so if ($\pi_{LL} + \pi_{LR} > 1$) and ($\pi_{RL} + \pi_{RR} > 1$), the server's chance to win the point must be strictly higher than it would be for the point game matrix ($\pi_{LL} -$ _, $\pi_{LR}-$ _, π_{RL}, π_{RR}) if _>0. Choose _ such that $\pi_{LL} + \pi_{LR} = 1 + 2$_. We know that the server's chance to win a point will be greater than _ for the point game matrix (π_{LL} - _, π_{LR}- _, π_{RL}, π_{RR}), so it must also be greater than _ for the point game matrix (π_{LL}, π_{LR}, π_{RL}, π_{RR}).

Combining the preceding three cases gives the result that whenever ($\pi_{LL} + \pi_{LR} \geq 1$) and ($\pi_{RL} + \pi_{RR} \geq 1$) both hold and at least one does not hold with equality, the server's equilibrium chance to win the point must be greater than _. **QED**

As was shown in part of the proof for Proposition 3, when ($\pi_{LL} + \pi_{LR} > 1$) and ($\pi_{RL} + \pi_{RR} > 1$) both hold, the server's equilibrium chance to win the point must be greater than _.

By a similar argument to Proposition 3, when ($\pi_{LL} + \pi_{LR} < 1$) and ($\pi_{RL} + \pi_{RR} < 1$) both hold, the server's equilibrium chance to win the point must be less than _.

When ($\pi_{LL} + \pi_{LR} \geq 1$) and ($\pi_{RL} + \pi_{RR} \geq 1$) and at least one of these inequalities does not hold with equality, we know by Proposition 3 that the server's equilibrium chance to win the point must be > _. When both of these inequalities hold with equality, the server's equilibrium chance to win the point (Lemma 2, above)= _. Thus, when ($\pi_{LL} + \pi_{LR} \geq 1$) and ($\pi_{RL} + \pi_{RR} \geq 1$), the server's chance to win the point must be ≥ _.

By a similar argument, when ($\pi_{LL} + \pi_{LR} \leq 1$) and ($\pi_{RL} + \pi_{RR} \leq 1$), the server's chance to win the point must be ≤ _. The contrapositive to this statement is that when the server's chance to win the point is >1/2, then ($\pi_{LL} + \pi_{LR} \leq 1$) and ($\pi_{RL} + \pi_{RR} \leq 1$) cannot both hold, so at least one of ($\pi_{LL} + \pi_{LR} > 1$) and ($\pi_{RL} + \pi_{RR} > 1$) must hold.

Proposition 4: In a point game matrix, if $0 < (\pi_{LR} - \pi_{LL}) \leq$ **M** and $0 < (\pi_{RL} - \pi_{RR}) \leq$ **M**, then if the server's chance to win the point is greater than (1+**M**)/2 this is sufficient to establish that both $\pi_{LL} + \pi_{LR} > 1$ and $\pi_{RL} + \pi_{RR} > 1$ hold.

Here is a heuristic proof:[59]

For ease of notation, let π_{LL}= a ; π_{LR}=b; π_{RL}=c ; and π_{RR}= d. This means that the point game matrix can be written as [a b; c d]. Also for ease of notation assume that the mixed strategy equilibrium conditions can now hold with equality—that is, b ≥ a, b≥ d, c≥ a, c≥ d.[60] The server's chance to win the point in equilibrium, V, is **V=(bc- ad)/(b+ c– a– d)**

With some loss of generality, assume that **b= a+M**. That is, assume that the difference in the server's chance to win the point, given that he serves to the left, when the returner

[59] In addition to this heuristic proof, I generated tens of thousands of point-game matrices randomly and none of those matrices disproved this proposition.
[60] When one or more of these conditions hold with equality the equations derived under the mixed strategy equilibrium conditions still predict equilibrium play, but equilibrium play now corresponds to one player having a dominant strategy.

anticipates correctly versus when the returner anticipates incorrectly is at its maximum permitted value, **M**. The condition **$a+b>1$** is thus equivalent to **$2a + M > 1$**.

The point game matrix can thus be written as $[a \; a+M; \; c \; d]$

From here, there are different cases concerning the values of **c** and **d** (in particular concerning the value of **$c-d$**). I will show that Proposition 4 holds in the two extreme cases, **$c-d= 0$** and **$c-d=M$**, on the assumption that if this property fails to hold fails to hold in general cases, then it fails to hold at extremes (so if it holds at extremes, it will hold in general cases as well).

Case 1: Assume that $c=d \rightarrow$
$c=c$; **$d=c$**; (**$a \leq c \leq a+M$**) (determined by the mixed strategy equilibrium conditions)

Show that if $V> (1+M)/2$, then this implies $(a+b)>1$ and $(c+d)>1$. By substitution, this is equivalent to showing that $V> (1+M)/2$ implies both $(2a+M) >1$ and $(2c)>1$.

$V= [c(a+M)- ca] /M = cM/M \rightarrow$ **$V=c$**

V $> (1+M)/2 \rightarrow$ **c**$> (1+M)/2 \rightarrow 2c> 1+M \rightarrow 2c-M > 1$
If $(2c-M > 1)$, then clearly $(2c > 1)$ holds as well.

We know $a \leq c \leq a+M$, so $2a \leq 2c \leq 2a+2M \rightarrow 2a-M \leq 2c-M \leq 2a+M$
Since $2c-M> 1$ and $2a + M \geq 2c-M$, $2a+M > 1$ holds as well.

Thus by direct calculation, we have shown that $V>(1+M)/2$ implies both $(a+b)>1$ and $(c+d)>1$.

Case 2: Assume that $c= d+M \rightarrow$
$c=c$; **$d=c-M$ ($a \leq c \leq a+2M$)** (determined by the mixed strategy equilibrium conditions)

Show that if $V> (1+M)/2$, then this implies $(a+b)>1$ and $(c+d)>1$. By substitution, this is equivalent to showing that $V> (1+M)/2$ implies both $(2a+M) >1$ and $(2c-M)>1$.

$V= [c(a+M)- (c-M)a]/(2M) = [ca +cM -ca +aM]/(2M) \rightarrow V= (c+a)M/ (2M) \rightarrow$ **$V= (a+c)/2$**

V $> (1+M)/2 \rightarrow$ **$(a+c)/2$**$> (1+M)/2 \rightarrow a+c> 1+M \rightarrow a+ c-M >1$

Note that the condition $(2c-M)> 1$ can be rewritten as $c + (c-M)> 1$.
Since **$c \geq a$**, and **a**$+ (c-M)> 1$, we know that **c**$+(c-M)> 1$ holds as well.

Note that since $a \leq c \leq a+2M \rightarrow 2a \leq c+a \leq 2a+2M \rightarrow 2a-M \leq c+a-M \leq 2a-M$

Since $2a-M \geq c+a-M$, and $c+a-M>1$, $2a-M >1$ holds as well. Since $2a-M >1$, it clearly follows that $2a+ M > 1$ holds as well. Thus by direct calculation, we have shown that $V>(1+M)/2$ implies both $(a+b)>1$ and $(c+d)>1$.

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