Breaking Ties in School Choice: (Non-) Specialized Schools*

Preliminary and Incomplete - Please do not circulate

Lars Ehlers† and Alexander Westkamp‡

Abstract

We study school choice problems with indifferences in priority orders and identify conditions under which there exists a *strategy proof* and *student optimal stable*, or *constrained efficient*, matching mechanism. A priority structure for which strategyproofness and constrained efficiency are compatible is called *solvable*. In this paper we consider the case in which schools are either *specialized*, i.e., have a strict priority ranking of all applicants, or *non-specialized*, i.e., all applicants have equal priority. In this setting we provide a full characterization of solvable priority structures if no school can admit more than one student. For the case of general capacity vectors we derive a (weaker) sufficient condition for solvability. Our proof is constructive and uses a version of the student proposing deferred acceptance algorithm with preference-based tie-breaking.

Keywords: School Choice, Equal Priority, Strategy-Proofness, Constrained Efficiency.

1 Introduction

Recently, the school choice problem has received a lot of attention in the theoretical and applied matching literature starting with **?**. In this problem, a set of students has

*This research started while both authors were visiting Harvard University. We acknowledge the hospitality of HBS and the Department of Economics. The first author thanks the SSHRC (Canada) and the FQRSC (Québec) for financial support. The second author thanks the DAAD and the DFG for financial support. We thank Bettina Klaus for discussing an earlier version of this paper at the Coalition Theory Workshop in Maastricht, January 2009.

†Département de Sciences Économiques and CIREQ, Université de Montréal, Montréal, Québec H3C 3J7, Canada; e-mail: lars.ehlers@umontreal.ca.

‡Bonn Graduate School of Economics, University of Bonn; e-mail: awest@uni-bonn.de.

to be assigned among a set of public schools. Each school has an exogenously given priority ordering of students. A central allocative criterion in the literature is stability, which requires that no student should envy another student for a school that she has *strictly* higher priority for. If students cannot have equal priority at schools, the student proposing deferred acceptance algorithm (SDA) produces a student optimal stable matching and provides students with dominant strategy incentives to submit their preferences over schools truthfully. This is not only of theoretical interest, as school choice authorities in Boston and New York have recently decided to adopt a variant of this mechanism (**?** and **?**). A problem that has not received much attention in the theoretical literature until the recent work by **?** is that in most real-life applications students may have the same priority at a given school. In the Boston public school choice system, for example, a major determinant for the priority of a student is whether she lives in the *walk zone* of a school, that is, not further away from the school than some fixed distance. Of course, a walk zone inherits (much) more than one student in a densely populated city so that schools' priority orderings have large indifference classes. This seemingly small change in the model changes results dramatically. The major problem is that ties between equal priority students have to be broken in order to determine an assignment. This induces additional stability constraints that can lead to a substantial decrease in student welfare (**?**). We call a mechanism *constrained efficient*, if it is stable with respect to the original weak priority structure and never incurs welfare loss due to tie-breaking. Unfortunately, **?** show that there are priority structures for which a constrained efficient and strategy-proof mechanism fails to exist.

A natural question that is at the heart of the present study is whether this is an exception or the rule in school choice problems with weak priority orders. We call a priority structure *solvable*, if there exists a strategy-proof and constrained efficient mechanism. In this paper we make important initial progress in characterizing the class of solvable priority structures. We introduce a model of (non-)specialized schools in which a school is either *specialized* and has a strict priority ranking of students, or a school is *non-specialized* and all students have the same priority. While it cannot be expected that this assumption is exactly satisfied in real-life applications, we view the analysis of this model as a useful first step since it provides important insights into how we can deal with large indifference classes in the priority structure. Furthermore, this model is interesting in its own right since it unifies the school choice problem with strict priority orders of **?** and the *house allocation problem* of **?**.

For the case that no school can admit more than one student, we fully characterize solvable priority structures by two simple and intuitive conditions. These conditions ensure that most of the ties can be broken exogenously, that is, without referring to student preferences. Since the conditions required for solvability are very restrictive, our results for the unit capacity case have the flavor of an impossibility result. This is in line with **?** and **?**'s classical negative results concerning dominant strategy implementation. However, our negative results critically depend on the assumption that no school can admit more than one student. In a second step we then consider general capacity vectors and show that significantly weaker conditions are sufficient for solvability. As for the unit capacity case, our conditions connect the capacity vector with the amount of allowable variability across the priority orderings of specialized schools. Most importantly, our results show that there is some scope for breaking ties according to student preferences and we introduce a new version of the SDA with endogenous tie-breaking (SDA-ETB). For solvable priority structures the associated matching mechanism is strategy-proof for students even though tie-breaking is (partly) based on elicited preferences. Interestingly, increasing capacities substantially enlarges the scope for preference based tie-breaking.

This chapter is organized as follows: After discussing the related literature we introduce the school choice problem with weak priorities and relevant existing results in section 2. In section 3 we motivate the need for preference based tie-breaking by means of a simple example. In section 4 we introduce the (non-)specialized schools model. This section contains the main results of this chapter. In section 5 we conclude and discuss our results as well as possible extensions. All proofs are relegated to the Appendix.

Related Literature

? was the first to study the problem of indifferences in priority orders. In particular he considered a simple three student example (that does not belong to our (non-)specialized schools environment) for which no exogenous tie-breaking rule guaranteed the constrained efficiency of the SDA. Nevertheless, he showed by construction that a strategy-proof and constrained efficient matching mechanism existed. We are the first to systematically study the possibility of preference based tie-breaking in school choice problems with indifferences in priority orders.

Apart from the above paper, the literature on the school choice problem with indifferences has mainly focused on exogenous tie-breaking. Here, a central question has been whether there should be a single lottery that is used to break ties at all schools, or whether there should be a separate lottery for each school. **?** considers a random assignment problem in which all students initially have the same priority for each school. He shows that a market based approach, in which a priority structure for each school is randomly selected and students are then allowed to trade their priorities, is equivalent to the random serial dictatorship, in which a single lottery is conducted and students then choose schools in the order determined by the lottery, in the sense that both produce exactly the same lottery over outcomes.[1] In a similar vein, **?** show that for any school choice problem, any constrained efficient matching can be reached by first using a single lottery to break all ties and then running the deferred acceptance algorithm. The focus of both of these papers is to give a rationale for using a single lottery to break all ties for all schools instead of multiple lotteries. They do not discuss how one can elicit the information about student preferences that is necessary to break ties in a way that avoids additional welfare loss, which is the main focus of our study.

More related in focus is the main theoretical result in **?**, which shows that no strategy-proof matching mechanism can dominate student optimal stable matching mechanism with *any* fixed tie-breaking rule. The dominance relation they consider is very strong since it requires that *all* students weakly prefer the outcome of the dominating mechanism to the outcome of the dominate mechanism for *all* preference profiles, with at least one strict preference for at least one profile. This already suggests that the class of mechanisms that is not dominated by another strategy-proof mechanism is quite large. Our results show that it is not sufficient to restrict attention to the class of SDAs resulting from fixed tie-breaking rules if one is interested in strategy-proof and constrained efficient mechanisms. Of course, our SDA-ETB does not dominate the SDA with an arbitrary fixed tie-breaking rule for all preference profiles in the above sense. However, in contrast to the latter mechanism it guarantees that there is never additional welfare loss due to tie-breaking if the priority structure is solvable.

Another closely related paper is **?**. They show that a matching is constrained efficient if and only if there is no *stable improvement cycle*, that is, no cyclical sequence of trades that respects stability constraints and makes all students involved better off.

[1]This is an extension of a classical result by **?** who show that in the house allocation problem the random serial dictatorship is equivalent to the core from random endowments, which conducts a single lottery to determine an initial allocation of indivisible objects and then lets agents trade towards a core outcome.

This motivates a simple constrained efficient procedure: Calculate the SDA outcome using an arbitrary tie-breaking rule. If the outcome is inefficient, successively eliminate stable improvement cycles until a constrained efficient outcome is reached. We will see that there exist solvable priority structures for which the stable improvement cycles procedure is not strategy-proof no matter how ties are broken initially and no matter how stable improvement cycles are selected. In particular, it is not sufficient to restrict attention to the stable improvement cycles procedure if one is interested in strategy-proof and constrained efficient mechanisms.

For the case of strict priorities, a number of papers have studied the relation between properties of the priority structure and the existence of mechanisms with certain desirable properties. Most prominently, **?** studies the relationship between efficiency (with respect to student preferences) and stability. He introduces a simple but restrictive acyclicity condition that is shown to be necessary and sufficient for the compatibility of efficiency and stability.[2] Note that for the problem with strict priorities, the compatibility of strategy-proofness and constrained efficiency follows from the strategy-proofness of the SDA. At least for the unit capacity case we can formally show that Ergin's conditions are more restrictive than the conditions required for solvability.[3]

Finally, we mention the recent paper by **?** who study the school choice problem with equal priorities from an ex-ante cardinal welfare perspective. They introduce a "choice augmented deferred acceptance algorithm" (CADA) in which students submit an ordinal ranking of schools and also specify a target school. The auxiliary message is used as a tie-breaking device and can be interpreted as allowing a student to signal the intensity of her preference for the target school. In a model with a continuum of students, the CADA is shown to improve upon the SDA with fixed tie-breaking from an ex-ante perspective. The approach of **?** differs from ours as we concentrate on the classical ex-post welfare perspective in a model with a finite number of students.

[2] Another example is **?** who derives conditions under which the SDA coincides with the *top trading cycles* algorithm, which has been one of SDA's main competitors in applications to the school choice problem.

[3] Although **?**'s conditions are for the case of strict priorities, it is easy to see that they guarantee the compatibility of efficiency and stability when imposed on the priority structure of specialized schools in our model. If one demands that all constrained efficient matchings should be constrained efficient, stronger conditions are required (**?**).

2 The School Choice Problem with Weak Priorities

A *School Choice Problem with Weak Priorities* is given by

- a finite set of students I,
- a finite set of schools S,
- a vector of capacities $q = (q_s)_{s \in S}$,
- a profile of weak priority orders of schools $\succeq = (\succeq_s)_{s \in S}$, and
- a profile of strict student preferences $R = (R_i)_{i \in I}$.

The only difference to the school choice problem with strict priorities introduced in Chapter I.2 is that two distinct students i and i' can now have *equal priority* for a school s, denoted by $i \sim_s i'$. Remember that $i \succ_s i'$ means that i has strictly higher priority for school s than student i'. For two subsets $J, J' \subset I$, we denote by $J \succ_s J'$ that $i \succ_s i'$ for all $i \in J$ and $i' \in J'$. Note that we continue to assume students can never be indifferent between two distinct schools. As everything else is fixed, we will think of a (school choice) problem as being given by a profile R of student preferences. A *rule*, or *matching mechanism*, is a function that assigns a matching to each problem. A *correspondence* is a function that assigns a non-empty set of matchings to each problem and rule f is a *selection* from correspondence F, if $f(R) \in F(R)$ for all problems R.

Remember that a matching μ is *stable* (or *fair*) for the school choice problem R, , if it

(i) is *individually rational*, that is, $\mu(i) R_i i$ for all students $i \in I$,

(ii) *eliminates justified envy*, that is, there is no student school pair (i, s) such that $s P_i \mu(i)$ and $i \succ_s i'$ for some $i' \in \mu(s)$, and

(iii) is *non-wasteful*, that is, there is no student school pair (i, s) such that $s P_i \mu(i)$ and $|\mu(s)| < q_s$.

At this point it is important to note that stability only depends on strict rankings in the priority structure. It is known (cf example 2.15 in **?**) that in the presence of ties in the priority structure there may not exist a stable matching μ that all students weakly prefer to any other stable matching. However, given the finiteness of the problem there always exists (at least one) stable assignment which is not Pareto dominated by any other stable matching with respect to student welfare. We call a matching with this property *constrained efficient* and given some profile of strict student preferences R we denote by $OS^{\succeq}(R)$ the set of constrained efficient matchings.

If priorities are strict, $OS^{\succeq}(R)$ contains exactly one matching which can be found by applying the SDA. However, if there are ties in the priority structure the SDA cannot be applied unless we specify some rule for breaking ties. Formally, a *fixed tie-breaking rule*, or *strict transformation*, of $\succeq$ is a strict priority structure $\succ'$ that preservers the strict ranking of $\succeq$, that is, $i \succ'_s j$ whenever $i \succ_s j$. Let $ST(\succeq)$ denote the set of all strict transformations of $\succeq$. Given some $\succ' \in ST(\succeq)$, let $f^{\succ'}$ denote the matching mechanism that associates the outcome of the SDA with strict priority structure $\succ'$ to each problem. It is known (**?**, **?**) that for all $\succ' \in ST(\succeq)$, $f^{\succ'}$ is strategy-proof and stable with respect to $\succeq$. **?** note that $f^{\succ'}$ may, however, fail to be constrained efficient. **?** aim to provide a rationale for the SDA with a fixed tie-breaking rule and show that no strategy-proof mechanism can dominate $f^{\succ'}$ for any $\succ' \in ST(\succeq)$, that is, there is no strategy-proof mechanism g such that for all problems R, $g_i(R) R_i f^{\succ'}(R)$ for all $i \in I$, with at least one strict preference for at least one problem and at least one student. Of course, this dominance relation is very strong so that the set of mechanisms that are undominated in this sense is very large.

Recently, **?** introduced an algorithm that always produces a constrained efficient matching for weak priority structures. The main idea is that whenever a stable matching is not constrained efficient, then it is possible to increase student welfare via a cyclical exchange that respects stability constraints. More formally, let μ be a stable matching for some R. Then student i *desires school* s *at* μ if $sP_i\mu(i)$. For each school s, let $D_s(\mu)$ denote the set of highest $\succeq_s$-priority students among those who desire s at μ. A *stable improvement cycle (SIC)* at μ and R consists of m distinct students $i_1, \ldots, i_m$ such that for all $l = 1, \ldots, m$, $i_l \in D_{\mu(i_{l+1})}(\mu)$ (where $m + 1 := 1$). **?** show that $\mu \in OS^{\succeq}(R)$ if and only if μ admits no stable improvement cycle (SIC) at μ and P_I. This leads them to suggest the following procedure to achieve a constrained efficient outcome.

The Stable Improvement Cycles Algorithm

Select a fixed single tie-breaking rule and compute the associated SDA outcome given the submitted preferences of students.

> If the outcome is not constrained efficient, allow students involved in a SIC to realize the corresponding cyclical exchange. Continue with this procedure until we arrive at a constrained efficient matching.

As shown in **?**, this procedure is not, in general, strategy-proof. This is not necessarily a fault of the stable improvement cycles as the same authors show that there exist weak priority structures $\succeq$ which do not admit *any* strategy-proof selection from $OS^{\succeq}$.

Motivated by this result, we call a priority structure $\succeq$ *solvable*, if there exists a strategy-proof and constrained efficient selection from $OS^{\succeq}$. Our main goal is to characterize the class of solvable priority structures. In the next section we start with a motivating example.

3 Motivating Preference Based Tie-Breaking

We consider the following school choice environment with three students $1, 2, 3$: There are six schools $s_1, \ldots, s_6$ at which all three students have different priorities and one school s_7 at which all students have equal priority. All schools can admit at most one student. The priority orderings of the various schools are summarized in the following table.[4]

$\succeq_{s_1}$	$\succeq_{s_2}$	$\succeq_{s_3}$	$\succeq_{s_4}$	$\succeq_{s_5}$	$\succeq_{s_6}$	$\succeq_{s_7}$
1	1	2	2	3	3	$1, 2, 3$
2	3	1	3	1	2	
3	2	3	1	2	1	

To implement the Erdil-Ergin procedure, we have to choose a fixed tie-breaking rule. Due to the symmetries of the example, we may assume without loss of generality that the strict transformation $\succ'_{s_7}$: $1, 2, 3$ was chosen. Consider the following preference profile:[5]

R	R_1	R_2	R_3
	s_4	s_7	s_7
	s_7		

[4]This notation means that e.g. at s_1, 1 has the highest, 2 has the second highest, and 3 has the lowest priority.

[5]Remember that the above notation means that agent 1 strictly prefers school s_3 to school s_7, and that s_4 and s_7 are the only schools which agent 1 prefers to not being assigned to any school.

For this school choice problem the outcome of the SDA with the above fixed tie-breaking rule is constrained efficient: 1 obtains her first choice school s_4 and 2 obtains her first choice school s_7. Student 3 does not receive a place at some school since she did not rank enough schools. Now suppose that instead of ranking only s_7, 3 declares s_4 to be her second choice, i.e. claims that her preferences are $R'_3 : s_7, s_4$. If all other students submit the same ranking as before, the SDA outcome is not constrained efficient for the above fixed tie-breaking rule: 1 obtains s_7, since she was randomly given the highest priority for this school, and 3 obtains s_4, since she has higher priority for this school than 1 but (randomly chosen) lower priority for s_7 than 2. There is a unique stable improvement cycle in this example since 1 and 3 would prefer to trade places, which cannot be vetoed by student 2. But this means that the stable improvement cycles procedure would now assign 3 a place at s_7, her true top choice school (under R_3). Hence, 3 has a strict incentive to manipulate the procedure at the original preference profile R. Since the ordering was chosen at random, this shows that no matter which fixed tie-breaking rule is used, there does not exist a strategy proof rule for selecting stable improvement cycles. The following procedure, however, is strategy-proof and constrained efficient:

Calculate the SDA outcome assuming that s_7 has unlimited capacity and let μ^1 be the resulting temporary assignment. If $\mu^1(s_7) = \{1, 2, 3\}$, reject 3. If $\mu^1(s_7) = \{i_1, i_2\}$ with $i_1 < i_2$, reject i_1 if and only if the third student, i_3, is temporarily matched to a school s such that $i_2 \succ_s i_1$ and $i_3 \succ_s i_1$. Now continue with the SDA in which students propose down their lists starting with their most preferred school that has not rejected them yet. Should another tie-breaking become necessary, apply the same rules as above.

An exact proof of strategy-proofness and constrained efficiency of this rule is deferred to the next sections. Note that the above tie-breaking rule takes the indexing of students as a baseline, which is only modified if two students are matched to s_7 while the third student is matched to a school in $s_1, \ldots, s_6$ at which she does not have *lowest priority*. This rule ensures that a student can affect the tie-breaking decision only if she changes her own temporary assignment prior to tie-breaking. By the strategy-proofness of the SDA procedure with fixed tie-breaking, it is clear that such a manipulation can, by itself, not be profitable. But the rule for tie-breaking ensures that if a student affects the tie-breaking decision, she will be matched to the new temporary assignment. The intuition for constrained efficiency is similar. This shows - a formal proof is deferred to later sections - that if we are interested in identifying the class of solvable priority structures, it is not sufficient to restrict attention to the stable improvement cycles procedure.

4 The (Non-)Specialized Schools Model

In this paper we consider a restricted class of school choice environments with two types of schools: *Specialized schools* have a strict priority ranking, while *non-specialized* schools assign the same priority to every student. More formally, we have the following.

Definition 1. *The priority structure $\succeq$ is a* ***(non-)specialized schools environment****, if there exists a partition of S into two non-empty sets S^0 and S^1 such that*

(i) S^0 comprises the set of non-specialized schools, that is, for all $s \in S^0$ and all $i, j \in I$, $i \sim_s j$, and

(ii) S^1 comprises the set of specialized schools, that is, for all $s \in S^1$ and all $i, j \in I$ such that $i \neq j$, $i \succ_s j$ or $j \succ_s i$.

In this language schools $s_1, \ldots, s_6$ in the example of section 3 were specialized, while s_7 was the only non-specialized school. One interpretation of this model is that a specialized school's priority ordering result from subject test(s) in the discipline(s) relevant for this school, e.g. a sports oriented school makes admission contingent on sports trials. Non-specialized schools on the other hand offer general educational training and therefore do not discriminate between students. Apart from this interpretation, our motivation for studying the (non-)specialized schools model is twofold.

First, our analysis is an important initial step towards a characterization of solvable priority structures in the school choice problem with indifferences in priority orders. We consider the extreme situation where a school's priority order has either one or $|I|$ indifference classes. However, it will become clear that our results also have important implications for the general school choice problem whenever some or all schools assign equal priority to a group of students which is sufficiently large, but at the same time potentially much smaller than $|I|$.

Secondly, our model bridges the gap between two important environments, which have been studied extensively in the literature.

(i) If $S^1 = \emptyset$, then all students have equal priority at all schools. This case is known as the *house allocation problem*.[6] Among others, **?**, **?**, and **?**, are interested

[6] This problem was first studied by **?**.

in identifying rules which satisfy strategy-proofness and efficiency.[7] Since all students have equal priority at all schools, stability is vacuously satisfied by any rule and constrained efficiency is equivalent to efficiency. The class of strategy-proof and efficient rules is very large and has not been characterized in the literature.

(ii) If $S^0 = \emptyset$, then no two students have equal priority at a school and we are back to the school choice problem with strict priorities from Chapter I.2. For this problem the student optimal stable rule is the only strategy-proof and constrained efficient rule.[8]

In the presence of both specialized and non-specialized schools a strategy-proof and constrained efficient mechanism does not always exist and we derive tractable conditions under which a priority structure is solvable.

For the remainder of this paper we restrict attention to the (non-)specialized schools environment. It is important to keep in mind that stability constraints only come from the priority orders of specialized schools. In the following, we will denote the priority ordering of a specialized school $s \in S^1$ by $\succ_s$ instead of $\succeq_s$ to emphasize that no two students can have equal priority. Let $\succ^1 = (\succ_s)_{s \in S^1}$ be the priority structure of specialized schools. It is easy to see that the conditions for solvability can only concern $\succ^1$. We assume throughout that there are at least two specialized schools and at least two non-specialized schools.[9] We first consider the case of unit capacity at all schools to develop intuition for the requirements imposed by solvability.

4.1 Unit capacities - Necessary Conditions

Throughout this section we consider the case where all schools can admit at most one student, i.e. where $q_s = 1$ for all $s \in S$. Of course, schools can usually admit more than one student and the reader may prefer to think of the allocation of tasks in a society rather than the school choice problem for this section. Society has a strict preference over who takes on specialized tasks, while it is indifferent as to who takes on a non-specialized task.

[7] In fact, all these papers derive characterizations of rules that satisfy strategy-proofness, efficiency, and different sets of additional axioms.

[8] In fact, the student optimal stable rule is also the only strategy-proof and stable rule (?).

[9] If there is only one specialized school, solvability is trivial. If there is only one non-specialized school, the sufficient conditions for solvability are slightly different as we discuss below.

The example in Section 3 suggests that strategy-proofness and constrained efficiency are always compatible if there are at most three students (a formal proof can be found below). Not surprisingly, this positive result does not extend to the case of four or more students as we will shortly see. In this section we identify two related sources for the incompatibility between strategy-proofness and constrained efficiency. The first source is introduced in the following definition.

Definition 2. *Let $\succeq$ be a non-specialized schools environment with unit capacities. Then $\succ^1$ contains* ***an ambiguous 1-tie*** *if there exist two specialized schools $s_1, s_2 \in S^1$ and four distinct students $i_1, i_2, i_3, i_4 \in I$ such that both $i_1 \succ_{s_1} i_3 \succ_{s_1} i_2$ and $i_2 \succ_{s_2} i_4 \succ_{s_2} i_1$.*

To see the problems associated with ambiguous 1-ties, consider the smallest example where it can be violated: There are four students $1, \ldots, 4$, two specialized schools s_1, s_2, and one non-specialized school s_3. The priority structure $\succ^1$ is such that (the remaining rankings are irrelevant)

$$1 \succ_{s_1} 3 \succ_{s_1} 2 \text{ and } 2 \succ_{s_2} 4 \succ_{s_2} 1.$$

Now consider the preference profile

R	R_1	R_2	R_3	R_4
	s_2	s_1	s_3	s_3
	s_3	s_3		
	s_1	s_2		

For this profile, 1 and 2 would prefer to *exchange* their priorities for s_1 and s_2. Here, this is not problematic since neither 3 nor 4 are interested in these schools. However, either 3 or 4 will have to remain unassigned since s_3 cannot admit more than one student. A strategy-proof procedure has to ensure in particular that 3 and 4 cannot profit by claiming that s_1 and/or s_2 are acceptable, respectively. We show in the Appendix that this cannot be achieved by *any* constrained efficient mechanism, thus proving the following result.

Proposition 1. *Let $\succeq$ be a (non-)specialized schools environment with unit capacities. Then $\succeq$ is solvable only if $\succ^1$ does not contain an ambiguous 1-tie.*

The absence of ambiguous 1-ties is a strong restriction. Suppose for example that s_1 is a music oriented school while s_2 is a sports oriented school. Both schools assign priorities according to performance in auditions. Typically, there will be *allrounders* who do relatively well in both specializations. At the same time, there will also be *specialists* who have a musical talent but are not very sportive (and the other way around). If there are at least two allrounders and two specialists, the priority structure is not solvable since it contains an ambiguous 1-tie. However, there is still some scope for different priority orderings across specialized schools as the next example demonstrates.

Example 1. *There are four students* $1, \ldots, 4$*, six specialized schools* $s_1, \ldots, s_6$*, and one non-specialized school* s_7*. The priority structure at specialized schools is as follows.*

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$
1	2	1	1	2	2
3	3	2	2	1	1
2	1	4	3	4	3
4	4	3	4	3	4

Note that $\succ^1$ *does not contain an ambiguous 1-tie. Thus, in principle the door remains open for possibility results.*

However, it is easy to see that any fixed tie breaking rule leads to violations of constrained efficiency for some preference profiles so that some or all of the ties have to broken preference based. A natural candidate for a constrained efficient assignment procedure is the following: Set $1 \sim^0 2 \succ^0 3 \succ^0 4$ *and break ties at the non-specialized school* s_7 *according to this ordering. Thus, only the tie between 1 and 2 remains to be broken endogenously. Now if 1 and 2 apply to* s_7 *in some round of the SDA procedure, temporarily ignore the capacity constraint at* s_7*. If 3 is temporarily matched to* s_2 *by the end of the SDA procedure, 1 is rejected by* s_7*. In any other case 2 is rejected. While this certainly guarantees a constrained efficient allocation, 4 can manipulate the tie breaking decision to her benefit. To see this consider the profile*

R	R_1	R_2	R_3	R_4
	s_7	s_7	s_3	s_1 .
	s_3	s_1	s_2	

Here, 4 would be left unmatched while 3 obtains a place at s_3*. However, if she claims that* $R'_4 : s_3, s_1$*, 3 would be rejected by* s_3 *and would subsequently apply to* s_2*. But then 1 would be rejected by* s_7*, causes 4 to be rejected at* s_3*, and 4 ultimately* *obtains a place at her true top choice (under R_4) school s_1. Hence, the above procedure is not strategy-proof.*

The following definition formalizes one problematic feature of the priority structure in this example.

Definition 3. *Let $\succeq$ be a (non-)specialized schools environment with unit capacities. Then $\succ^1$ contains* ***ambiguity at the top*** *if there are four distinct students i_1, i_2, i_3, i_4 and three distinct specialized schools $s_1, s_2, s_3 \in S^1$ such that $i_1 \succ_{s_1} i_3 \succ_{s_1} i_2 \succ_{s_1} i_4$, $i_2 \succ_{s_2} i_3 \succ_{s_2} i_1 \succ_{s_2} i_4$, and $\{i_1, i_2\} \succ_{s_3} i_4 \succ_{s_3} i_3$.*

In the above example there was ambiguity at the top concerning $\succ_{s_1}, \succ_{s_2}$, and $\succ_{s_3}$. In order to avoid ambiguity at the top, at least one of the schools' priority orderings needs to be changed. For example, we could set $\tilde{\succ}_{s_2}$ equal to any of the priority orderings of the other specialized schools to obtain a priority structure that contains no ambiguous 1-ties and no ambiguity at the top. The following shows that ambiguity at the top is the second source for the incompatibility of strategy-proofness and constrained efficiency.

Proposition 2. *Let $\succeq$ be a (non-)specialized schools environment with unit capacities. Then $\succeq$ is solvable only if $\succ^1$ does not contain no ambiguity at the top.*

Above we showed that our intuitive idea for achieving a constrained efficient matching does not provide students with the right incentives. Note that the statement of Proposition 2 is much stronger since it says that *any* assignment procedure has to sacrifice either strategy-proofness or constrained efficiency.

4.2 General Capacities - Sufficient Conditions

The results in the last section support a pessimistic view about the possibilities of obtaining strategy-proof and constrained efficient mechanisms. It is important to keep in mind that we assumed that all schools could admit at most one student. In this section we turn to the case of general capacities. We derive a precise connection between the capacity vector and the *amount of variability* in school rankings allowed by a solvable priority structure. We first consider the case of identical capacities at all specialized schools.

4.2.1 Symmetric Capacities at Specialized Schools

In this subsection we concentrate on the case of identical capacities at all specialized schools. We assume for now that the set of students is *connected* in the sense that there is no strict subset $J \subset I$ such that $J \succ_s I \setminus J$ for all $s \in S^1$. We discuss below how our results translate to the case where this assumption is not satisfied.

As a first step we derive an equivalent formulation of the two necessary conditions for the unit capacity case of the last section. The idea is to then adapt these conditions to the case of general symmetric capacities, where a full characterization seems to be out of reach as we discuss in section 5. We require a bit of additional notation and terminology: For all $s \in S^1$ and $k \in \{1, \ldots, |I|\}$, $r_k(\succ_s)$ denotes the student who has kth highest priority for s, i.e. $|\{i \in I : i \succ_s r_k(\succ_s)\}| = k - 1$. For $k \in \{1, \ldots, |I|\}$, let $L_k = (\cup_{s \in S^1} \{r_k(\succ_s)\}) \setminus (L_1 \cup \cdots \cup L_{k-1})$ denote the set of students who have kth highest priority at some specialized school but never rank higher. Let K be the smallest integer such that $N = L_1 \cup \ldots \cup L_K$, so that in particular $L_k = \emptyset$ for all $k > K$. We have the following.

Proposition 3. *If $|I| > 3$ and $\succ^1$ does not contain ambiguous 1-ties or ambiguity at the top then*

- ***O1*** *$L_k \subseteq \{r_k(\succ_s), r_{k+1}(\succ_s), r_{k+2}(\succ_s)\}$ for all $s \in S^1$ and $k \in \{1, \ldots, K\}$, and*
- ***O2*** *there is exactly one student in L_1 who has third highest priority at some specialized school.*

*Conversely, if $|I| > 3$ and $\succ^1$ satisfies **O1** and **O2** then $\succ^1$ does not contain ambiguous 1-ties or ambiguity at the top.*

Propositions 1-3 imply that a student's rank in priority orderings can differ by at most two across specialized schools if all schools can admit at most one student and the priority structure is solvable. This allows us to define a global ordering $\succeq^0$ on I by setting $i_1 \succ^0 i_2$ if $i_1 \in L_k$ and $i_2 \in L_{k'}$ for some $k < k'$. The key property here is that if $i_1 \succ^0 i_2$ for two students i_1, i_2, there cannot be a third student i_3 and a specialized school $s \in S^1$ such that $i_2 \succ_s i_3 \succ_s i_1$. As we show below this implies that ties between two students who are strictly ordered according to $\succeq^0$ can be broken exogenously, i.e. without conditioning on student preferences. Thus, only ties between students in the same indifference set of $\succeq^0$ remain to be broken according to student preferences. Condition **O2** implies that if $|I| \geq 4$, we never have to consider the preferences of lower priority students in $L_2 \cup \ldots \cup L_K$ in order to break the tie between the two students in L_1. Note that since $|L_1| = 2$ this basically means that the tie between the two students at the top can be broken exogenously if $|I| \geq 4$.

Intuitively, it is clear that increasing capacities should enlarge the scope for preference based tie breaking and should increase the allowable variability in priority orderings of specialized schools. The important task here is to identify the exact form of this relationship. We now show how the conditions for solvability from the unit capacity case can be adapted to the capacity vector. For the following, we fix a capacity vector for schools with the property that all specialized schools have the same capacity. Let q^1 be the common capacity of all specialized schools, let $q^0_{(1)}$ be the lowest capacity of any non-specialized school, and $q^0_{(2)} \geq q^0_{(1)}$ be the second lowest capacity.[10]

First of all, the priority structure is solvable if the number of students is sufficiently small compared to available capacities. Here, the critical value turns out to be $p = q^1 + q^0_{(1)} + \min\{q^1, q^0_{(2)}\}$. To see that any priority structure is solvable if $|I| \leq p$ note that if tie-breaking becomes necessary, i.e. at least $q^0_{(1)} + 1$ students are interested in the same non-specialized school $s_1 \in S^0$, at most one specialized school can have filled its capacity. Furthermore, if some specialized school $s_2 \in S^1$ has filled its capacity, there cannot be a third school $s_3 \in S \setminus \{s_1, s_2\}$ that has to reject any student. We show below how the priority ordering of s_2 can be used to determine who should be rejected by s_1. Secondly, if $|I| > p$ the variability of priority orders across specialized schools has to be restricted. The next definition formally summarizes our requirements in this case.

Definition 4. *Suppose that I is connected and that $|I| > p$. Then $\succ^1$ satisfies **limited p-variability** if*

O1(p) $L_k \subset \{r_k(\succ_s), \ldots, r_p(\succ_s)\}$ *for all* $k \leq p-2$ *and all* $s \in S^1$,

O2(p) $L_k \subset \{r_k(\succ_s), r_{k+1}(\succ_s), r_{k+2}(\succ_s)\}$ *for all* $p-2 < k$ *and all* $s \in S^1$, *and*

O3(p) *there is exactly one student in* $L_1 \cup \ldots \cup L_{p-2}$ *who has pth highest priority at some specialized school.*

The idea behind this condition is that we want to assign high priority for non-specialized schools to students who have *high*, i.e. at least $(p-2)$nd highest, priority for specialized schools. Note that the amount of allowable variability declines as we move down the rankings of specialized schools. This is because the demands of students with higher priority could effectively lead to a reduction of the number of seats at some schools. Eventually, everything reduces to the unit capacity case and

[10] Remember that we assumed $|S^0| \geq 2$.

a student's priority can vary by at most two. This is illustrated by the following example.

Example 2. *There are four specialized schools $s_1, \ldots, s_4$, two non-specialized schools s_5, s_6, and six students $1, \ldots, 6$. Capacities are $q^1 = 2$ and $q_{s_5} = 1$ and $q_{s_6} = 2$. Priorities of specialized schools are given by*

$$\begin{array}{lcccccc} \succ_{s_1}: & 1 & 2 & 3 & 4 & 5 & 6 \\ \succ_{s_2}: & 4 & 3 & 2 & 1 & 5 & 6 \\ \succ_{s_3}: & 2 & 1 & 3 & 6 & 4 & 5 \\ \succ_{s_4}: & 3 & 2 & 1 & 5 & 4 & 6 \end{array}$$

Note that for this priority structure the set of all students is connected and $|I|$ exceeds the critical value of $p = 5$. We have $L_1 \cup L_2 \cup L_3 = \{1, 2, 3, 4\}$ and $L_4 = \{5, 6\}$. Since no student in $L_1 \cup L_2 \cup L_3$ is ranked lower than fifth and only 4 is ranked fifth (at schools s_3 and s_4) $\succ^1$ satisfies limited p-variability.

Consider again the interpretation of priorities at specialized schools as being determined by test scores. If all schools had unit capacity, the priority structure would not be solvable: 1 and 4 would then be specialists for schools s_1 and s_2, respectively, while 2 and 3 would be allrounders. However, given the above capacity vector 1 and 4 are not too *specialized and we will see below that the above priority structure is solvable.*

Intuitively, assigning high priority to students in the *upper segment* of students who rank at least $(p-2)$nd at some specialized school minimizes the number of rejections following a rejection at a non-specialized school. It is important to note that limited p-variability is a joint condition on $\succ^1$ and the capacity vector. In particular, the capacity vector determines the size of the upper segment. In case I exceeds the critical value of p, limited p-variability ensures that ties between two students in the upper segment can always be broken conditional only on the preferences of other upper segment students. Since the size of the upper segment cannot exceed p, this is always possible as argued above and formally proven below.[11] At this point a few remarks about limited p-variability are in order.

Remark 1:

[11] Actually, limited p-variability ensures that the upper segment contains exactly $p-1$ students if $|I| > p$. The reasoning behind restricting the upper segment of students to those ranking no lower than $(p-2)$nd (and not $(p-1)$st) is a bit subtle and will become clear in the proof of Theorem 1 in the Appendix

(i) Conditions **O1(p)** and **O2(p)** imply that $|L_1| \leq p$ and that $I = L_1$ if $|L_1| = p$. Now let K be the minimal integer such that $N = L_1 \cup \ldots \cup L_K$ so that in particular $L_k = \emptyset$ for all $k > K$. If $|I| > p$, **O1(p)** and **O2(p)** imply that

- $|L_1 \cup \ldots \cup L_{p-2}| = p - 1$,
- $|L_k| = 1$ for all $p - 2 < k \leq K - 1$, and
- $|L_K| \in \{1, 2\}$.

In particular we must have $|I| \in \{K + 1, K + 1\}$ if $|I| > p$.

(ii) A major benefit of limited p-variability is that it is tractable and very easy to verify. If $|I| > p$, we first need to check that $L_1 \subset \{r_1(\succ_s), \ldots, r_p(\succ_s)\}$ for all specialized schools $s \in S^1$. This can be implemented as follows: Take an arbitrary specialized school $s \in S^1$. Then check whether $r_1(\succ_s) \in \{r_1(\succ_{s'}), \ldots, r_p(\succ_{s'})\}$ for all $s' \in S^1 \setminus \{s\}$. This requires at most $(|S^1| - 1)p$ steps. Proceeding in this fashion, we can test whether no student in L_1 is ever ranked lower than pth in at most $|S^1|(|S^1| - 1)p < (|S^1|)^2|I|$ steps.

Now, the conditions for the remaining L_k sets can be verified completely analogously so that checking **O1(p)** and **O2(p)** requires at most $K|S^1|(|S^1| - 1)p < (|I|)^2(|S^1|)^2$ steps. Note that **O3(p)** can be tested at (almost) no additional computational cost: As soon as we find a student in $L_1 \cup \ldots \cup L_{p-2}$ who is ranked pth at some specialized school we have to check that all other students in this segment of the priority structure rank no lower than $(p - 1)$st.

We now design an assignment procedure that is strategy-proof and constrained efficient provided that limited p-variability is satisfied. For the following we fix a capacity vector as well as the priority structure $\succ^1$ of specialized schools and assume that $\succ^1$ satisfies limited p-variability. The procedure consists of two steps: In the first step we define an ordering $\succeq^0$ as in the unit capacity case. In the second step, we introduce a new version of the SDA algorithm which uses this ordering as the *common* priority ordering of all non-specialized schools. The procedure breaks ties between students in the same indifference class of $\succeq^0$ endogenously on basis of temporary assignments.

Step 1: Ordering Students

- If $|I| \leq p$, set $i \sim^0 j$ for all $i, j \in I$.

- If $|I| > p$, set
  - (i) $i \sim^0 j$ for all $i, j \in L_1 \cup \ldots \cup L_{p-2}$
  - (ii) $i \succ^0 j$ if $i \in L_k$ and $j \in L_{k'}$ with $k < k' \leq K$ and $k' \geq p$
  - (iii) $i \sim^0 j$ if $i, j \in L_K$

As in the unit capacity case this ordering has the property that if $i_1 \succ^0 i_2$ there cannot be a third student i_3 and a specialized school $s \in S^1$ such that $i_2 \succ_s i_3 \succ_s i_1$. We show below that this implies that the tie between i_1 and i_2 can be broken exogenously without violating constrained efficiency. All remaining ties can be broken endogenously. By Remark 1.(i) there are at most two non-singleton indifference sets of $\succeq^0$ if $|I| > p$ (and one indifference set if $|I| \leq p$): An *upper segment* consisting of $p-1$ students who have at least $(p-2)$nd highest priority for some specialized school and, possibly, a *lower segment* consisting of two students in L_K. For the purpose of breaking ties in these two segments endogenously we label students according to their position in $\succeq^0$. Within an indifference class, the label is arbitrary with the exception that if $|I| > p$ we assign the highest label $p-1$ (remember Remark 1.(i)) in the upper indifference set of $\succeq^0$ to the only student in $L_1 \cup \ldots \cup L_{p-2}$ who has pth highest priority at some specialized school. [12] Labels will be used as a baseline for endogenous tie breaking. This baseline is modified only if a specialized school has filled its capacity. In the following we abuse notation slightly and identify a student with her label. Thus, if $|I| > p$ we write $I = \{1, \ldots, K+1\}$ if $|L_K| = 1$ and $I = \{1, \ldots, K+2\}$ if $|L_K| = 2$, where the labeling adheres to the rules above. We are now ready to describe the SDA procedure with endogenous tie breaking (SDA-ETB).

Step 2: The SDA with Endogenous Tie Breaking

The algorithm takes as inputs the (relevant portion of the) capacity vector $(q^1, q^0_{(1)}, q^0_{(2)})$, the priority structure of specialized schools $\succ^1$, the ordering $\succeq^0$ calculated in Step 1, and a profile of student preferences.

Round 1: Each student applies to her most preferred school. Each specialized school $s \in S^1$ admits the q^1 highest priority students according to $\succ_s$. Each non-

[12] More formally, the labeling can be described as follows: If $|I| \leq p$ choose a permutation $\pi_I : I \to \{1, \ldots, |I|\}$ at random. If $|I| > p$ let $\tilde{i}$ be the unique student in $L_1 \cup \ldots \cup L_{p-2}$ who can have pth highest priority at some specialized school and set $\pi_I(\tilde{i}) = p - 1$.

(i) Choose a permutation $\pi_I : (L_1 \cup \ldots \cup L_{p-2}) \setminus \{\tilde{i}\} \to \{1, \ldots, p-2\}$ at random.

(ii) For $k \in \{p-1, \ldots, K-1\}$ and $i \in L_k$ set $\pi_I(i) = k + 1$.

(iii) If $|L_K| = 2$ randomly pick a student $i \in L_K$ and set $\pi_I(i) = K + 1$. Set $\pi_I(i') = K + 2$ for the other student i' in L_K.

specialized school $s \in S^0$ admits the q_s students with the lowest labels among those who aply to it. If necessary, it admits all students in the same indifference class of $\succeq^0$ as the q_sth highest labeled student who was admitted in addition. Let μ^1 be the resulting temporary assignment.

If one of the rejected students has not yet applied to all acceptable schools, go to **Round 2**. If all rejected students have applied to all acceptable schools and there is a non-specialized $s \in S^0$ such that $|\mu^1(s)| > q_s$, use subroutine **TB**(μ^1) to determine a rejection and go to **Round 2**. Else, stop.

⋮

Round t**:** Each student rejected in Round $t-1$ applies to her most preferred school among those that have not yet rejected one of her proposals. Each specialized school $s \in S^1$ admits the q^1 highest priority students according to $\succ_s$. Each non-specialized school $s \in S^0$ admits the q_s students with the lowest labels among those who aply to it. If necessary, it admits all students in the same indifference class of $\succeq^0$ as the q_sth highest labeled student who was admitted in addition. Let μ^t be the resulting temporary assignment.

If one of the rejected students has not yet applied to all acceptable schools, go to **Round** $t+1$. If all rejected students have applied to all acceptable schools and there is a non-specialized school $s \in S^0$ such that $|\mu^t(s)| > q_s$, use subroutine **TB**(μ^t) to determine a rejection and go to **Round** $t+1$. Else, stop.

The crucial ingredient of this algorithm is the tie-breaking subroutine which is applied to determine a rejection at non-specialized schools. The subroutine is applied only if *nothing else moves* in the sense that there is no other way for the algorithm to proceed than to break a tie within an indifference class of $\succeq^0$.

Subroutine TB(μ^t)**:** If there is a non-specialized school $s \in S^0$ such that $|\mu^t(s)| > q_s$ and $i \sim^0 j$ for all $i, j \in \mu^t(s)$, set $s_0 := s$ and go to **Step TB**(μ^t)**.1**. Else, let $s_0 \in S^0$ be the non-specialized school such that $L_K \subset \mu^t(s_0)$ and go to **Step TB**(μ^t)**.2**.

Step TB(μ^t)**.1:** If there is a specialized school $s_1 \in S^1$ s.t. $|\mu^t(s_1)| = q^1$ and $i \sim^0 j$ for all $i, j \in \mu^t(s_1) \cup \mu^t(s_0)$ let i_1 be the student with the lowest priority according to $\succ_{s_1}$ among students in $\mu^t(s_0)$. School s_0 rejects i_1 if $\mu^t(s_1) \succ_{s_1} i_1$.

In any other case s_0 rejects student with the highest label among students in $\mu^t(s_0)$.

Step TB(μ^t).2: Let $s_1 := \mu^t(K)$. If $K+2 \succ_{s_1} K+1$, s_0 rejects $K+1$.

In any other case, s_0 rejects $K+2$.

The intuition for the tie-breaking subroutine is as follows: **Step TB(μ^t).1** covers tie-breaking in the upper indifference class of $\succeq^0$. It ensures that following a tie breaking decision in the upper segment there is a further rejection of a student in this segment only if it is unavoidable. **Step TB(μ^t).2** covers tie-breaking in the lower indifference class of $\succeq^0$. It ensures that there is no further rejection following a tie-breaking decision in the lower segment. Note that this step of the tie-breaking subroutine is reached only if $|I| > p$ and $|L_K| = 2$ since otherwise there can never be a non-specialized school that temporarily admits students from different indifference classes of $\succeq^0$ and violates its capacity constraint.

Note that of the inputs required by the mechanism everything but students' preferences are assumed to be exogenously given. In the following we supress the dependency of the outcome of the SDA-ETB on the exogenous factors for notational simplicity. Given a problem R let $f^{ETB}(R)$ thus denote the associated outcome of the SDA-ETB procedure. We have the following.

Theorem 1. *Suppose that either $|I| \leq p$ or $\succ^{ss}$ satisfies limited p-variability. Then the following statements are true.*

(i) $f^{ETB}(R)$ is constrained efficient for all problems R.

(ii) f^{ETB} is strategy-proof.

In particular, $\succeq$ is solvable if either $|I| \leq p$, or $|I| > p$ and $\succ^1$ satisfies limited p-variability.

At this point it makes sense to illustrate the SDA-ETB by means of an example.

Example 3. *Consider again the environment of example 2. Note that the labels of students have been chosen in accordance with our rules since 4 is the only student in $L_1 \cup \ldots \cup L_3$ who ranks 5th (at schools s_3 and s_4). Consider the following problem:*

R	R_1	R_2	R_3	R_4	R_5	R_6
	s_6	s_6	s_3	s_6	s_5	s_5
				s_3	s_1	s_3
				s_5		

.

For this problem we obtain

$$f^{ETB}(R) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ s_6 & s_6 & s_3 & s_3 & s_1 & s_5 \end{pmatrix}.$$

This example illustrates that it is important to break ties in the upper segment $L_1 \cup L_2 \cup L_3$ *before breaking ties in the* lower segment L_4*: If we would have broken the tie at* s_5 *first (according to our rules for tie breaking), 6 would have been rejected. In subsequent rounds of SDA-ETB, student 4 would then have been rejected by* s_6 *and* s_3*. Since* $4 \succ^0 5$*, 4 would have subsequently obtained a place at* s_5*. But then there would be a stable improvement cycle consisting of* 4 *and* 6*. The main reason for breaking ties in the lower segment last is that this way we can ensure that there are no further rejections after tie-breaking. A similar example can be used to show that it is important to wait with endogenous tie-breaking until* nothing else moves*.*

4.2.2 Asymmetric Capacities

In this section we turn to the case of general capacity vectors. In the following, let $q^1_{(1)}$ be the minimal capacity of specialized schools, $q^0_{(1)}$ and $q^0_{(2)}$ be defined as in the last section, and $p = q^1_{(1)} + q^0_{(1)} + \min\{q^1_{(1)}, q^0_{(2)}\}$ be the modified critical value. It is easy to see that our previous results imply that if the set of students is connected and $\succ^1$ satisfies limited p-variability whenever $|I| > p$ then $\succeq$ is solvable.[13]

We now discuss how our results extend to the case where I is not connected. Since we are dealing with a finite problem there has to exist a minimal set J_1 such that for any $s \in S^1$, $J_1 \succ_s I \setminus J_1$. We call J_1 the *minimal top set* of I with respect to $\succ^1$. Proceeding inductively, let J_t be the minimal top set of $I \setminus (J_1 \cup \ldots \cup J_{t-1})$ with respect to $\succ^1$. We call $(J_t)_{t \geq 1}$ the *minimal top set partition* of I with respect to $\succ^1$. Suppose for the sake of clarity that $I = J_1 \cup J_2$. Let $f^{ETB}|_{J_1}$ denote the SDA-ETB mechanism when we make all places at all schools available to students in J_1. Since J_1 is connected, our previous analysis implies that $f^{ETB}|_{J_1}$ is strategy-proof and constrained efficient provided that $\succ^1|_{J_1}$ satisfies limited p-variability. In principle, there are two ways to guarantee that there is a strategy-proof and constrained efficient procedure for students in J_2 which we now discuss. In both cases, we assign all students in J_1 with higher priority for non-specialized schools than all students in J_2. This ensures in particular that a student in J_1 can never envy a student in J_2.

[13]One just needs to replace q^1 with the actual capacities of specialized schools in the formulation of the SDA-ETB. Everything else remains exactly the same.

(i) In some instances it might be feasible to elicit reports from students in J_2 after assignments for students in J_1 have been determined. In this case given a profile R_{J_1} elicited from students in J_1, we can reduce capacities at schools according to $f^{ETB}|_{J_1}(R_{J_1})$. Let p^1 be the resulting modified critical value and let $f^{ETB}|_{J_2}$ denote the SDA-ETB that allocates remaining places among students in J_2. Again, our analysis from the connected case implies that $f^{ETB}|_{J_2}$ is strategy-proof and constrained efficient provided that $\succ^1|_{J_2}$ satisfies limited p^1-variability.

(ii) If we restrict attention to assignment procedures that simultaneously elicit a report from all students, the restrictions for solvability in J_2 become more restrictive. We have to require solvability for the *lowest* possible critical value that could be induced by the demands of students in J_1. For example, consider the case $q^1_{(1)} = 4$, $q^0_{(1)} = q^0_{(2)} = 2$, and $|J_1| = 3$. Here, the worst case would be if all students in J_1 were interested in the minimal capacity specialized school leading to a new critical value of 4.

From the above discussion it is clear that even when all specialized schools initially have identical capacities, we have to consider the case of asymmetric capacities if I is not connected since the demands of students in J_1 may lead to a problem with asymmetric capacities for the remaining student population.

However, note that in the unit capacity case the critical value is always 3. This implies that the same conditions guaranteeing solvability for the connected case also guarantee solvability for the general case. Hence, we obtain the following theorem as a corollary to Propositions 1-3 and Theorem 1.

Theorem 2. *Suppose $\succeq$ is a (non-)specialized schools environment with unit capacities. Then $\succeq$ is solvable if and only if $\succ^1$ does not contain ambiguous 1-ties or ambiguity at the top.*

To conclude this section note that it could be the case that even though $\succeq$ is not solvable, there is a strategy-proof and constrained efficient procedure for a subpopulation of students. To see this consider again the case of unit capacities and suppose that $I = J_1 \cup J_2$. If $\succ^1|_{J_1}$ satisfies limited 3-variability, but $\succ^1|_{J_2}$ does not, there is a strategy-proof and constrained efficient mechanism for students in J_1 but not for students in J_2.

5 Conclusion and Discussion

This chapter derived a full characterization of solvable priority structures in (non-)specialized schools environments with unit capacity. Significantly weaker sufficient conditions were introduced for the case of general capacity vectors. Our conditions show precisely how much variability in priority orderings across specialized schools can be allowed in order to guarantee existence of a constrained efficient and strategy-proof mechanism. The proof of sufficiency was constructive and used a modified deferred acceptance procedure with (potentially) preference based tie-breaking. The results show that it is not sufficient to concentrate on fixed tie-breaking rules if one is interested in strategy-proof and constrained efficient school choice systems. Furthermore, the scope for preference based tie-breaking increases in the number of slots available at schools. We now discuss several important open questions.

5.1 Uniqueness of the Tie-Breaking Rule

In this chapter we introduced tractable conditions that guarantee solvability of a priority structure. Given that these conditions are satisfied we introduced the strategy-proof and constrained efficient SDA-ETB procedure. One important idea of this mechanism was to assign those students who have high priority at specialized school also high priority for all non-specialized schools. This could be considered problematic from an equity perspective and school choice authorities might be interested in knowing whether there are other strategy-proof and constrained efficient mechanisms. In the following we discuss whether there could be other ways to break ties. In this section we concentrate on the case of unit capacities at all schools, assume that I is connected, and fix a solvable environment $\succeq$.

First, suppose we have set $i_1 \succ^0 i_2$ for two students $i_1, i_2 \in I$ so that i_2 can never obtain a non-specialized school desired by i_1. Can there be strategy-proof and constrained efficient procedure that exogenously breaks ties at non-specialized schools in favor of i_2? To see that this is impossible, note that the construction of $\succeq^0$ ensures that there exists a specialized school $s \in S^1$ and a third student i_3 such that $i_1 \succ_s i_3 \succ_s i_2$. Let $\tilde{s} \in S^0$ be one of the non-specialized schools.[14] Now let f be a strategy-proof and constrained efficient mechanism such that i_2 always has higher priority for $\tilde{s}$ than i_1. Consider first the problem

[14] Note that we must have $|I| \geq 4$ if we exogenously break any tie. For the following we assume that no student in $I \setminus \{i_1, i_2, i_3\}$ is interested in s or $\tilde{s}$.

R^1	$R^1_{i_1}$	$R^1_{i_2}$	$R^1_{i_3}$
	$\tilde{s}$	s	$\tilde{s}$
		$\tilde{s}$	s

We must have $f_{i_1}(R^1) = 1$, $f_{i_2}(R^1) = s$, and $f_{i_3} = \tilde{s}$. Otherwise there would be a stable improvement cycle given that i_2 can never envy i_1 for $\tilde{s}$. Now suppose that $R^2_{i_3} : \tilde{s}$ and consider $R^2 = (R^1_{i_1}, R^1_{i_2}, R^2_{i_3})$. By strategy-proofness we must have $f(R^2) = f(R^1)$. Next, let $R^2_{i_1} : \tilde{s}, s$ and $R^3 = (R^2_{i_1}, R^1_{i_2}, R^2_{i_3})$. By strategy-proofness and stability, we must have $f_{i_1}(R^3) = s$. Constrained efficiency then implies $f_{i_2}(R^3) = i_2$ and $f_{i_3}(R^3) = \tilde{s}$. Finally, let $R^3_{i_3} = s, \tilde{s}$ and consider $R^4 = (R^2_{i_1}, R^1_{i_2}, R^3_{i_3})$. Since i_2 cannot envy i_1 for a place at $\tilde{s}$ and i_2 cannot obtain a place at s given $R^3_{i_3}$, it is not possible that $f_{i_3}(R^4) = s$. But if $f_{i_3}(R^4) = \tilde{s}$, we must have $f_{i_1}(R^4) = s$ so that i_1 and i_3 form a stable improvement cycle. Hence, we must have $f_{i_3}(R^4) = i_3$. But then i_3 has an incentive to submit $R^2_{i_3}$ when the other students submit $R^2_{i_1}$ and $R^1_{i_2}$!

More generally, we would like to know whether strategy-proofness and constrained efficiency require us to *always* follow the ordering $\succeq^0$ for the case of solvable priority structures. That is, if f is a strategy-proof and constrained efficient mechanism can it be the case that for some problem R we have $\tilde{s} P_i f_i(R)$, $i \succ^0 j$, and $f_j(R) = \tilde{s}$ for some non-specialized school $\tilde{s} \in S^0$? To see that this is possible let $\hat{\mathcal{R}} = \{R : |\{i \in I : A(R_i) \neq \emptyset\}| \leq 2\}$ denote the set of profiles where at most two students $i \in I$ have a non-empty set of acceptable schools $A(R_i)$. Let $\succ' \in ST(\succeq)$ be an arbitrary strict transformation. Now we can modify the rule f^{ETB} as follows: for any profile R, (i) if $R \notin \hat{\mathcal{R}}$, then $\hat{f}(R) = f^{ETB}(R)$; and (ii) if $R \in \hat{\mathcal{R}}$, then $\hat{f}(R) = f^{\succ'}(R)$. It is easy to see that $\hat{f}$ is strategy-proof and constrained efficient. An important open question is whether we can allow such *violations* of $\succeq^0$ on more interesting domains of preferences.

5.2 Full Characterization for General Capacities

Beyond the case of unit capacities we have only derived sufficient conditions for solvability. An important question is whether these conditions can be weakened further. We first illustrate the additional problems for designing strategy-proof and constrained efficient mechanisms when our conditions are not satisfied using a simple example.

There are two specialized schools s_1, s_2 and two non-specialized schools s_3, s_4. Both specialized schools can admit three students while the two non-specialized schools can only admit one student. There are six students $1, \dots, 6$ and the priority ordering is given by

$$\succ_{s_1}: \ 1 \ 2 \ 3 \ 4 \ 5 \ 6$$
$$\succ_{s_2}: \ 6 \ 5 \ 4 \ 3 \ 2 \ 1$$

Note that in this example the critical value is $p = 5$ and that the priority structure does not satisfy limited p-variability. Now consider the preference profile

R	R_1	R_2	R_3	R_4	R_5	R_6
	s_3	s_2	s_4	s_4	s_2	s_3
	s_7	s_3		s_2		s_2

.

Now suppose we were to use the SDA-ETB with the tie-breaking procedure we defined above for this example assuming that $i \sim^0 j$ for all $i, j \in \{1, \dots, 6\}$. Then $\mu^1(s_1) = \emptyset$, $\mu^1(s_2) = \{2, 5\}$, $\mu^1(s_3) = \{1, 6\}$, and $\mu^1(s_4) = \{3, 4\}$. Now 6 would be rejected in the first round and 4 would be rejected by s_4 in the second round. In the third round, 4 applies to s_2 so that 2 would be rejected and applies to s_3. Since s_2 has filled its capacity and 1 is the lowest priority student at s_2, SDA-ETB would break the resulting tie in favor of 2. But then 2 and 6 form a stable improvement cycle. The problem in this example is that there are two non-specialized schools (s_3 and s_4) that have to reject students. If $\succ^1$ had satisfied limited p-variability, it would have been irrelevant which student is rejected by s_3 or s_4 since there could not have been a subsequent rejection at some specialized school. Here, in contrast it is important to condition tie-breaking on the priority ranking of school s_2 in the first place even though this school has not filled its capacity in round 1 of SDA-ETB. Nevertheless, we do not have a counterexample showing that the above priority structure is not solvable so that the door remains open for further possibility results despite the just mentioned complications.[15]

Secondly, consider the case of identical capacity $q \geq 2$ at all schools so that $p = 3q$ (and the above issue does not arise). Is limited p-variability necessary for solvability here? While we do believe that this is true, potential counterexamples needed to show necessity quickly become intractable. The main problem here is that it is very

[15] We do expect the tie-breaking rule to be somewhat more complicated to describe. In general, we conjecture that a critical value of $2q^1 + q^0_{(1)}$ could work in case of identical capacities at specialized schools. This is easily seen to be true when there is just one non-specialized school. However, we have not (yet) been able to prove that this critical value works in the general case. A similar remark applies to the case of asymmetric capacities, where there also seems to be some additional room for solvability.

hard to pin down assignments in case non-specialized schools can admit more than one student. We view the weakening of sufficient conditions for solvability to be substantially more important than extending our impossibility results and have thus not worked towards obtaining a full characterization for the case of general capacities at non-specialized schools.

5.3 Beyond Non-specialized schools environments

In this paper we have concentrated on the (non-)specialized schools environment. To see that there is room for positive results outside this environment, we now consider an easy example.

There are three schools s_1, s_2, s_3 and three students $1, 2, 3$. All schools have a capacity of one and the priority structure is as follows

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
1	2	3
$\{2,3\}$	$\{1,3\}$	$\{1,2\}$

To see that this priority structure is solvable, note that the above environment is isomorphic to a *house allocation with existing tenants* problem as introduced by **?**: Student i is an existing tenant for school s_i. Their version of the top-trading cycles algorithm is strategy-proof and constrained efficient for such problems.[16]

In general, it will not be possible to rely on the top-trading cycles algorithm since it is known that it may lead to unstable allocations. Furthermore, the above approach is not applicable, for example, when the set of students with top priority for some school is larger than the school's capacity (as in the (non-)specialized schools environment). However, the above shows that the door in principal remains open for possibility results and a (partial) characterization of solvable priority structures in the general case is an important question for future research.

Proof of Proposition 1

Suppose to the contrary that $\succ^1$ contains an ambiguous 1-tie but that f is a strategy-proof and constrained efficient selection from the stable correspondence. W.l.o.g. we

[16] This is a special case of the hierarchical exchange rules introduced by **?**. She shows that this class of rules exhaust the class of rules that are group strategy-proof, efficient, and satisfy a notion of reallocation-proofness. Recently, **?** have characterized the (slightly) larger class of group strategy-proof and efficient rules.

can assume that there are exactly four students $1, 2, 3, 4$ and two specialized schools s_1, s_2 such that

$$1 \succ_{s_1} 3 \succ_{s_1} 2 \text{ and } 2 \succ_{s_2} 4 \succ_{s_2} 1. \tag{1}$$

Let s_3 be one of the non-specialized schools. To derive a contradiction, we consider six preference profiles which define a cycle in the space of preference profiles. The following diagram summarizes the preference profiles used in our proof. Arrows indicate how we move between the profiles.

R^2	R_1^1	R_2^1	R_3^2	R_4^1
	s_2	s_1	s_1	s_3
	s_3	s_3	s_3	
	s_1	s_2		

$\leftarrow$

R^1	R_1^1	R_2^1	R_3^1	R_4^1
	s_2	s_1	s_3	s_3
	s_3	s_3		
	s_1	s_2		

$\leftarrow$

R^6	R_1^1	R_2^1	R_3^1	R_4^2
	s_2	s_1	s_3	s_3
	s_3	s_3		s_2
	s_1	s_2		

$\downarrow$ (from R^2) $\qquad \uparrow$ (to R^6)

R^3	R_1^1	R_2^3	R_3^2	R_4^1
	s_2	s_1	s_1	s_3
	s_3	s_3	s_3	
	s_1			

$\rightarrow$

R^4	R_1^1	R_2^3	R_3^2	R_4^2
	s_2	s_1	s_1	s_3
	s_3	s_3	s_3	s_2
	s_1			

$\rightarrow$

R^5	R_1^1	R_2^1	R_3^2	R_4^2
	s_2	s_1	s_1	s_3
	s_3	s_3	s_3	s_2
	s_1	s_2		

We start with the profile R^1. Let

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_2 & s_1 & s_3 & 4 \end{pmatrix} \text{ and } \bar{\mu} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_2 & s_1 & 3 & s_3 \end{pmatrix}.$$

It is straightforward that these are the only constrained efficient matchings for the profile R^1. Thus, we must have $f(R^1) = \mu$ or $f(R^1) = \bar{\mu}$. By the symmetries of the example, we can assume $f(R^1) = \mu$ without loss of generality.

Now let $R_3^2 : s_1, s_3$ and $R^2 = (R_3^2, R_{-3}^1)$. By strategy-proofness, $f_3(R^2) \neq 3$. Note that for R^2 there is no constrained efficient matching that assigns 3 to s_3. Hence, we must have $f_3(R^2) = s_1$. It is easy to see that this in conjunction with constrained efficiency implies

$$f(R^2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_2 & s_3 & s_1 & 4 \end{pmatrix}. \tag{2}$$

Next, suppose 2 declares s_2 unacceptable, that is, consider $R_2^3 : s_1, s_3$ and the profile $R^3 = (R_1^1, R_2^3, R_3^2, R_4^1)$. By strategy-proofness we must have $f_2(R^3) = s_3$ so that constrained efficiency implies $f(R^3) = f(R^2)$.

Now consider the profile $R^4 = (R_1^1, R_2^3, R_3^2, R_4^2)$. By strategy-proofness, $f_4(R^4) \neq s_3$. Since $4 \succ_{s_2} 1$ and 1 and 4 are the only students who would like to be assigned to s_2, we have $f_1(R^4) \neq s_2 = f_4(R^4)$. If $f_1(R^4) = s_3$, 1 and 4 would form a SIC under $f(R^4)$ - a contradiction to constrained efficiency. Thus, $f_1(R^4) = s_1$. But then $f_3(R^4) \neq s_3$ since otherwise by $3 \succ_{s_1} 2$, 1 and 3 would form a SIC. Thus, $f_3(R^4) = 3$ so that

$$f(R^4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_1 & s_3 & 3 & s_2 \end{pmatrix}. \tag{3}$$

Now consider the profile $R^5 = (R_1^1, R_2^1, R_3^2, R_4^2)$. By strategy-proofness, $f_2(R^5) = s_3$ and similarly to above, $f(R^5) = f(R^4)$.

Finally, consider the profile $R^6 = (R_1^1, R_2^1, R_3^1, R_4^2)$. By strategy-proofness, $f_3(R^6) = 3$. Since the first choices of the other three agents are compatible, constrained efficiency implies $f(R^6) = f(R^5)$.

This yields the desired contradiction since $s_3 = f_4(R_4^2, R_{-4}^1) P_4^1 f_4(R^1) = 4$. Hence, there is no strategy-proof and constrained efficient mechanism. □

Proof of Proposition 2

Consider a $\succ^1$ that contains ambiguity at the top. Let s_1, s_2, s_3 be three specialized schools and $1, 2, 3, 4 \in I$ be four distinct students such that[17]

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
1	2	1
3	3	2
2	1	4
4	4	3

Let s_4 be one of the non-specialized schools. Suppose to the contrary that there exists a strategy-proof and constrained efficient rule f.

The main part of the proof considers four preference profiles, which are summarized in the following diagram. As in the proof of Proposition 5, arrows indicate how we move between preference profiles.

[17]Note that due to the symmetries of the definition it is without loss of generality to assume that $1 \succ_{s_3} 2$.

R^1	R^1_1	R^1_2	R^1_3	R^1_4
	s_4	s_1	s_4	s_3
		s_4	s_3	
			s_1	

R^2	R^2_1	R^2_2	R^2_3	R^1_4
	s_2	s_4	s_4	s_3
	s_4		s_3	
			s_2	

$\downarrow$ (from R^1 to R^3; from R^2 to R^4)

R^3	R^1_1	R^3_2	R^1_3	R^2_4
	s_4	s_4	s_4	s_1
		s_3	s_3	s_3
			s_1	s_2

$\leftrightarrow$

R^4	R^1_1	R^3_2	R^2_3	R^2_4
	s_4	s_4	s_4	s_1
		s_3	s_3	s_3
			s_2	s_2

Let $R^1_1 : s_4; R^1_2 : s_1, s_4; R^1_3 : s_4, s_3, s_1; R^1_4 : s_3$, and R^1 be the resulting preference profile. It is easy to see that there are exactly two constrained efficient matchings at R^1,

$$\mu^1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & 2 & s_1 & s_3 \end{pmatrix} \text{ and } \bar{\mu}^1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & s_1 & s_4 & s_3 \end{pmatrix}.$$

Claim 1: $f(R^1) = \mu^1$.

Proof of Claim 1. Suppose to the contrary that $f(R^1) = \bar{\mu}^1$. Starting from profile R^1 we will consider the following preference profiles for these students:

R^1	R^1_1	R^1_2	R^1_3	R^1_4
	s_4	s_1	s_4	s_3
		s_4	s_3	
			s_1	

$\downarrow$

$R^{1,1}$	R^1_1	R^1_2	$R^{1,1}_3$	R^1_4
	s_4	s_1	s_4	s_3
		s_4	s_3	

$\downarrow$

$R^{1,2}$	$R^1_{1,1}$	R^1_2	$R^{1,1}_3$	R^1_4
	s_4	s_1	s_4	s_3
	s_1	s_4	s_3	

$\rightarrow$

$R^{1,3}$	$R^{1,1}_1$	$R^{1,1}_2$	$R^{1,1}_3$	R^1_4
	s_4	s_1	s_4	s_3
	s_1	s_4	s_3	
		s_3		

$\rightarrow$

$R^{1,4}$	$R^{1,1}_1$	$R^{1,1}_2$	$R^{1,2}_3$	R^1_4
	s_4	s_1	s_3	s_3
	s_1	s_4	s_4	
		s_3		

$\uparrow$ (to $R^{1,5}$)

$R^{1,5}$	$R^{1,1}_1$	$R^{1,1}_2$	$R^{1,2}_3$	$R^{1,1}_4$
	s_4	s_1	s_3	s_4
	s_1	s_4	s_4	s_3
		s_3		

$\leftarrow$

$R^{1,6}$	$R^{1,1}_1$	$R^{1,1}_2$	$R^{1,2}_3$	$R^{1,3}_4$
	s_4	s_1	s_3	s_4
	s_1	s_4	s_4	
		s_3		

$\uparrow$ (to $R^{1,7}$)

$R^{1,7}$	$R^{1,1}_1$	R^1_2	$R^{1,2}_3$	$R^{1,3}_4$
	s_4	s_1	s_3	s_4
	s_1	s_4	s_4	

$\rightarrow$

$R^{1,8}$	$R^{1,1}_1$	R^1_2	$R^{1,2}_3$	$R^{1,1}_4$
	s_4	s_1	s_3	s_4
	s_1	s_4	s_4	s_3

$\downarrow$ (to $R^{1,5}$)

Let $R^{1,1}_3 : s_4, s_3$ and $R^{1,1} = (R^1_1, R^1_2, R^{1,1}_3, R^1_4)$. By strategy-proofness (for student 3) and constrained efficiency we must have that $f(R^2) = f(R^1)$. Let $R^{1,1}_1 : s_4, s_1$ and $R^{1,2} = (R^{1,1}_1, R^1_2, R^{1,1}_3, R^1_4)$. By strategy-proofness (for student 1) and $1 \succ_{s_1} 3 \succ_{s_1} 2 \succ_{s_1} 4$, we must have $f_1(R^{1,2}) = s_1$. Now if $f_2(R^{1,2}) = s_4$, then 1 and 2 form a SIC, a contradiction. Thus, by constrained efficiency, we must have that $f(R^{1,2}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_1 & 2 & s_4 & s_3 \end{pmatrix}$.

Let $R_2^{1,1}: s_1, s_4, s_3$ and $R^{1,3} = (R_1^{1,1}, R_2^{1,1}, R_3^{1,1}, R_4^1)$. By strategy-proofness (for student 2) and constrained efficiency we must have that $f(R^{1,3}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_1 & s_3 & s_4 & 4 \end{pmatrix}$.

Let $R_3^{1,2}: s_3, s_4$ and $R^{1,4} = (R_1^{1,1}, R_2^{1,1}, R_3^{1,2}, R_4^1)$. By strategy-proofness (for student 3) and constrained efficiency we must have that $f(R^{1,4}) = f(R^{1,3})$.

Let $R_4^{1,1}: s_4, s_3$ and $R^{1,5} = (R_1^{1,1}, R_2^{1,1}, R_3^{1,2}, R_4^{1,1})$. By strategy-proofness (for student 4) we must have $f_4(R^{1,5}) \in \{s_4, 4\}$. Suppose that $f_4(R^{1,5}) = s_4$. Let $\tilde{R}_4 : s_3, s_4$ and $\tilde{R} = (R_1^{1,1}, R_2^{1,1}, R_3^{1,2}, \tilde{R}_4)$. If $f_4(\tilde{R}) = s_3$, 4 could manipulate f at the profile $R^{1,4}$ by submitting $\tilde{R}_4$. Given that $f_4(R^{1,5}) = s_4$, strategy-proofness thus implies $f_4(\tilde{R}) = s_4$ as well. But this contradicts the constrained efficiency of f since 4 and 2 would then form a SIC at $f(\tilde{R})$ and $\tilde{R}$. This contradiction shows that we must have $f_4(R^{1,5}) = 4$ and $f(R^{1,5}) = f(R^{1,3})$ as well.

Let $R_4^{1,3}: s_4$ and $R^{1,6} = (R_1^{1,1}, R_2^{1,1}, R_3^{1,2}, R_4^{1,3})$. By strategy-proofness we must have $f_4(R^{1,6}) = 4$. Since the top choices of the other students are compatible, constrained efficiency implies that $f(R^{1,6}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & s_1 & s_3 & 4 \end{pmatrix}$.

Let $R^{1,7} = (R_1^{1,1}, R_2^1, R_3^{1,2}, R_4^{1,3})$. By strategy-proofness we must have that $f_2(R^{1,7}) = s_1$. Stability implies that $f_1(R^{1,7}) = s_4$ and hence $f(R^{1,7}) = f(R^{1,6})$.

Let $R^{1,8} = (R_1^{1,1}, R_2^1, R_3^{1,2}, R_4^{1,1})$. By strategy-proofness and constrained efficiency we must have $f_4(R^{1,8}) = s_3$ and $f_3(R^{1,8}) = 3$. Thus $f(R^{1,8}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & s_1 & 3 & s_3 \end{pmatrix}$.

Since $f_2(R^{1,5}) = s_3$ and $f_2(R^{1,8}) = s_1$, this implies that 2 is strictly better off reporting R_2^1 rather than her true preference $R_2^{1,1}$ when the other students submit $R_1^{1,1}, R_3^{1,2}$, and $R_4^{1,1}$. This contradicts strategy-proofness and completes the proof of Claim 1. □

Now let $R_1^2: s_2, s_4$, $R_2^2: s_4$, $R_3^2: s_4, s_3, s_2$, and $R^2 = (R_1^2, R_2^2, R_3^2, R_4^1)$. Similar to above, there are exactly two constrained efficient matchings at R^2,

$$\mu^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_2 & 2 & s_4 & s_3 \end{pmatrix} \text{ and } \bar{\mu}^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & s_4 & s_2 & s_3 \end{pmatrix}.$$

Claim 2: $f(R^2) = \bar{\mu}^2$

Proof of Claim 2. The proof is analogous to the proof of Claim 1. One just needs to switch the roles of 1 and 2 as well as s_1 and s_2 and note that at any profile in the proof of Claim 1, school s_3 is never acceptable for both 1 and 2 (and the proof only uses the fact $\{1, 2\} \succ_{s_3} 4 \succ_{s_3} 3$ and not how students 1 and 2 are ranked under $\succ_{s_3}$).□

Now let $R_2^3 : s_4, s_3$, $R_4^2 : s_1, s_3, s_2$, and $R^3 = (R_1^1, R_2^3, R_3^1, R_4^2)$. Let $\mu^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & s_3 & s_1 & s_2 \end{pmatrix}$.

Claim 3: $f(R^3) = \mu^3$.

Proof of Claim 3. By Claim 1 we have that $f(R^1) = \mu^1$. Now let $R_2^{3,1} : s_1, s_4, s_3$ and $R^{3,1} = (R_1^1, R_2^{3,1}, R_3^1, R_4^1)$. Since $f_2(R^1) = 2$, strategy-proofness and stability imply $f_2(R^{3,1}) = s_3$. Constrained efficiency then implies $f_3(R^{3,1}) = s_1$ so that $f(R^{3,1}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & s_3 & s_1 & 4 \end{pmatrix}$.

Now let $R_4^{3,1} : s_3, s_1$ and $R^{3,2} = (R_1^1, R_2^{3,1}, R_3^1, R_4^{3,1})$. By strategy-proofness, $f_4(R^{3,2}) \neq s_3$ so that constrained efficiency implies $f(R^{3,2}) = f(R^{3,1})$.

Let $R_4^{3,2} : s_1, s_3, s_2$ and $R^{3,3} = (R_1^1, R_2^{3,1}, R_3^1, R_4^{3,2})$. By strategy-proofness and $f_4(R^{3,2}) = 4$, $f_4(R^{3,3}) \notin \{s_1, s_3\}$. Thus, by constrained efficiency, $f(R^{3,3}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_4 & s_3 & s_1 & s_2 \end{pmatrix}$.

Let $R_1^{3,1} : s_4, s_3$ and $R^{3,4} = (R_1^{3,1}, R_2^{3,1}, R_3^1, R_4^{3,2})$. By strategy-proofness, $f_1(R^{3,4}) = s_4$ so that constrained efficiency implies $f(R^{3,4}) = f(R^{3,3})$.

Let $R_2^{3,2} : s_4, s_3$ and $R^{3,5} = (R_1^{3,1}, R_2^{3,2}, R_3^1, R_4^{3,2})$. By strategy-proofness we must have $f_2(R^{3,5}) = s_3$. Since $1 \succ_{s_3} 2$, stability implies $f_1(R^{3,5}) = s_4$ and thus $f(R^{3,5}) = f(R^{3,3})$.

Let $R_1^{3,2} : s_4$ and $R^{3,6} = (R_1^{3,2}, R_2^{3,2}, R_3^1, R_4^{3,2})$. By strategy-proofness, $f_1(R^{3,6}) = s_4$ and hence $f(R^{3,6}) = f(R^{3,3})$. Since $R^{3,6} = R^3$ this proves Claim 3. □

Let $R^4 = (R_1^1, R_2^3, R_3^2, R_4^2)$.

Claim 4: $f_3(R^4) \in \{s_3, s_4\}$.

Proof. By Claim 2 we have that $f(R^2) = \bar{\mu}^2$. Starting from profile R^2 we will consider the following preference profiles for these students:

$R^{4,1}$	R_1^2	$R_2^{4,1}$	R_3^2	R_4^2
	s_2	s_4	s_4	s_3
	s_4	s_3	s_3	
			s_2	

←

R^2	R_1^2	R_2^2	R_3^2	R_4^2
	s_2	s_4	s_4	s_3
	s_4		s_3	
			s_2	

$R^{4,10}$	$R_1^{4,2}$	$R_2^{4,1}$	R_3^2	$R_4^{4,3}$
	s_4	s_4	s_4	s_1
		s_3	s_3	s_3
			s_2	s_2

↓

$R^{4,2}$	R_1^2	$R_2^{4,1}$	R_3^2	$R_4^{4,1}$
	s_2	s_4	s_4	s_3
	s_4	s_3	s_3	s_1
			s_2	

↑

$R^{4,9}$	$R_1^{4,1}$	$R_2^{4,1}$	R_3^2	$R_4^{4,3}$
	s_4	s_4	s_3	s_1
	s_3	s_3	s_4	s_3
			s_2	s_2

↓

$R^{4,3}$	$R_1^{4,1}$	$R_2^{4,1}$	R_3^2	$R_4^{4,1}$
	s_4	s_4	s_4	s_3
	s_3	s_3	s_3	s_1
			s_2	

→

$R^{4,7}$	$R_1^{4,1}$	$R_2^{4,2}$	R_3^2	$R_4^{4,1}$
	s_4	s_4	s_4	s_3
	s_3	s_1	s_3	s_1
			s_2	

→

↑

$R^{4,8}$	$R_1^{4,1}$	$R_2^{4,2}$	R_3^2	$R_4^{4,3}$
	s_4	s_4	s_4	s_1
	s_3	s_1	s_3	s_3
			s_2	s_2

↓

$R^{4,4}$	$R_1^{4,1}$	$R_2^{4,1}$	$R_3^{4,1}$	$R_4^{4,1}$
	s_4	s_4	s_4	s_3
	s_3	s_3	s_3	s_1
			s_1	

→

$R^{4,5}$	$R_1^{4,1}$	$R_2^{4,1}$	$R_3^{4,1}$	$R_4^{4,2}$
	s_4	s_4	s_4	s_1
	s_3	s_3	s_3	s_3
			s_1	s_2

→

$R^{4,6}$	$R_1^{4,2}$	$R_2^{4,1}$	$R_3^{4,1}$	$R_4^{4,2}$
	s_4	s_4	s_4	s_1
		s_3	s_3	s_3
			s_1	s_2

Let $R_2^{4,1} : s_4, s_3$ and $R^{4,1} = (R_1^2, R_2^{4,1}, R_3^2, R_4^2)$. By strategy-proofness, $f_2(R^{4,1}) = s_4$ so that constrained efficiency implies $f(R^{4,1}) = \bar{\mu}^2$

Let $R_4^{4,1} : s_3, s_1$ and $R^{4,2} = (R_1^2, R_2^{4,1}, R_3^2, R_4^{4,1})$. By strategy-proofness, $f_4(R^{4,2}) = s_3$ so that, by stability, $f_2(R^{4,2}) = s_4$ and $f(R^{4,2}) = \bar{\mu}^2$.

Let $R_1^{4,1} : s_4, s_3$ and $R^{4,3} = (R_1^{4,1}, R_2^{4,1}, R_3^2, R_4^{4,1})$. By strategy-proofness, stability, and $1 \succ_{s_3} 2 \succ_{s_3} 4 \succ_{s_3} 3$, we must have $f_1(R^{4,3}) = s_3$. Constrained efficiency then implies that either $f_2(R^{4,3}) = s_4$ or $f_3(R^{4,3}) = s_4$. We show by contradiction that the second case is impossible. Suppose $f_3(R^{4,3}) = s_4$. Let $R_3^{4,1} : s_4, s_3, s_1$ and $R^{4,4} = (R_1^{4,1}, R_2^{4,1}, R_3^{4,1}, R_4^{4,1})$. By strategy-proofness $f_3(R^{4,4}) = s_4$ so that in particular $f_1(R^{4,4}) = s_3$ and $f_4(R^{4,4}) = s_1$. Let $R_4^{4,2} : s_1, s_3, s_2$ and $R^{4,5} = (R_1^{4,1}, R_2^{4,1}, R_3^{4,1}, R_4^{4,2})$. By strategy-proofness we must have $f_4(R^{4,5}) = s_1$. This is compatible with stability only if $f_3(R^{4,5}) = s_4$ and $f_1(R^{4,5}) = s_3$ so that $f(R^{4,5}) = f(R^{4,4})$. Let $R_1^{4,2} : s_4$ and $R^{4,6} = (R_1^{4,2}, R_2^{4,1}, R_3^{4,1}, R_4^{4,2})$. By strategy-proofness $f_1(R^{4,6}) = 1$. But note that $R^{4,6} = R^3$ and, by Claim 3 above, $f_1(R^3) = s_4$. This is a contradiction and hence we must have that $f_2(R^{4,3}) = s_4$. Constrained efficiency then implies $f(R^{4,3}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ s_3 & s_4 & s_2 & s_1 \end{pmatrix}$.

Let $R_2^{4,2} : s_4, s_1$ and $R^{4,7} = (R_1^{4,1}, R_2^{4,2}, R_3^2, R_4^{4,1})$. By strategy-proofness, $f_2(R^{4,7}) = s_4$ so that by stability $f(R^{4,7}) = f(R^{4,3})$.

Let $R_4^{4,3} : s_4, s_3, s_2$ and $R^{4,8} = (R_1^{4,1}, R_2^{4,2}, R_3^2, R_4^{4,3})$. By strategy-proofness, $f_4(R^{4,8}) = s_1$. Stability then implies $f_2(R^{4,8}) = s_4$ and $f(R^{4,8}) = f(R^{4,3})$.

Let $R^{4,9} = (R_1^{4,1}, R_2^{4,1}, R_3^2, R_4^{4,3})$. Strategy-proofness implies $f_2(R^{4,9}) = s_4$ so that, by stability, $f(R^{4,9}) = f(R^{4,3})$.

Let $R^{4,10} = (R_1^{4,2}, R_2^{4,1}, R_3^2, R_4^{4,3})$ and note that $R^{4,10} = R^4$. By strategy-proofness $f_1(R^4) = 1$. But then constrained efficiency implies $f_3(R^4) \in \{s_4, s_3\}$ since either $f_2(R^4) = s_4$ or $f_3(R^4) = s_4$, and if $f_2(R^4) = s_4$ then $f_3(R^4) = s_3$. □

Combining Claim 3 and Claim 4 we see that student 3 has an incentive to submit R_3^2 when other students submit their preferences from the profile R^3 since $f_3(R^4) P_3^1 f_3(R^3) = s_1$ given that $f_3(R^3) = s_1$ by Claim 3 and $f_3(R^4) \in \{s_3, s_4\}$ by Claim 4. This contradicts strategy-proofness of f and completes the proof. □

Proof of Proposition 3

We show first that no ambiguos 1-ties and no ambiguity at the top imply that **O1** and **O2** are satisfied. Fix some $k \leq K$ and $i \in L_k$. Suppose to the contrary that there exists a specialized school $s_1 \in S^1$ such that $r_{k+l}(\succ_{s_1}) = i$ for some $l \geq 3$. On the other hand since $i \in L_k$ there exists a specialized school $s_2 \in S^1 \setminus \{s_1\}$ such that $r_k(\succ_{s_2}) = i$. For any $l' \in \{1, \ldots, l\}$ let $i_{l'} = r_{k+l'}(\succ_{s_2})$. Now if there is a student j such that $i_l \succ_{s_1} j \succ_{s_1} i$ we have found an ambiguous 1-tie since $l \geq 3$ so that at least two distinct students rank between i and i_l with respect to $\succ_{s_2}$. If $i_l = r_{k+l-1}(\succ_{s_1})$ and $i_{l-1} \succ_{s_1} i_l$ we obtain a contradiction since $i \succ_{s_2} i_{l-2} \succ_{s_2} i_{l-1}$. If $i_l \succ_{s_1} i \succ_{s_1} i_{l-1}$, there has to be a student j such that $i_l \succ_{s_2} j$ and $j \succ_{s_1} i_l$ yielding another contradiction. Hence, we must have $i \succ_{s_1} i_l$. But then there has to exist a student j such that $i_l \succ_{s_2} j$ and $j \succ_{s_1} i$. No matter whether $j \succ_{s_1} i_{l-1}$ or $i_{l-1} \succ_{s_1} j$ we obtain an ambiguous 1-tie. This shows that **O1** has to be satisfied.

To see that **O2** must be satisfied note that if **O1** is satisfied we must have $|L_1| = 2$ if $|I| > 3$: If $|L_1| = 1$ we must have $|I| = 1$ and if $|L_1| = 3$ we must have $I = L_1$ since otherwise one of the students in L_1 would have to rank fourth at some specialized school. By similar arguments we must have $|L_2| \in \{1, 2\}$ and $K = 2$ if $|L_2| = 2$. Let $L_1 = \{1, 2\}$. If $L_2 = \{3, 4\}$ but there exist two specialized schools $s_1, s_2 \in S^1$ such that $r_3(\succ_{s_1}) = 1$ and $r_3(\succ_{s_2}) = 2$ we obtain an ambiguous 1-tie. If $L_2 = \{3\}$, let 4 be one of the students in L_3. Suppose there exist two specialized schools $s_1, s_2 \in S^1$ such that $r_3(\succ_{s_1}) = 1$ and $r_3(\succ_{s_1}) = 2$. By **O1** we must have $2 \succ_{s_1} 3 \succ_{s_1} 1 \succ_{s_1} 4$ and $1 \succ_{s_2} 3 \succ_{s_2} 2 \succ_{s_2} 4$. Now since $4 \in L_3$ and $|L_1 \cup L_2| = 3$, at least one agent in $\{1, 2, 3\}$ must have lower priority than 4 for some specialized school. By **O1** this agent cannot be 1 or 2. This implies that there exists a third specialized school s_3 such that $\{1, 2\} \succ_{s_3} 4 \succ_{s_3} 3$. Hence, $\succ^1$ contains ambiguity at the top (with respect to schools s_1, s_2, s_3).

Now suppose that $\succ^1$ satisfies **O1** and **O2**. Note that if $|I| > 3$, we must have $|L_1| = 2$, $|L_k| = 1$, for all $k \in \{1, \ldots, K-1\}$, and $|L_K| \in \{1, 2\}$. Now by **O2** there cannot be an ambiguous 1-tie involving the two students in L_1 since at most one of them can rank third. By **O1** there cannot be an ambiguous 1-tie between an agent $i \in L_k$ and an agent $j \in L_{k'}$ for $k < k' \leq K$ since i always has at least $(k+2)$nd highest priority at specialized schools. Lastly, there cannot be an ambiguous 1-tie between two students in L_K (if $|L_K| = 2$) since only one student in L_{K-1} can rank in between these two agents by **O1** and **O2**.

Now suppose that there exist four distinct students i_1, i_2, i_3, i_4 and three specialized schools s_1, s_2, s_3 such that $i_1 \succ_{s_1} i_3 \succ_{s_1} i_2 \succ_{s_1} i_4$, $i_2 \succ_{s_2} i_3 \succ_{s_2} i_1 \succ_{s_2} i_4$, and $\{i_1, i_2\} \succ_{s_3} i_4 \succ_{s_3} i_3$. Clearly, we cannot have $i_1, i_2 \in L_K$ since this would imply $i_4 \in L_1 \cup \ldots \cup L_{K-1}$ while $i_4 \notin \{r_1(\succ_{s_1}), \ldots, r_{K+2}(\succ_{s_1})\}$, contradicting **O1**. By **O1**, we can also not have that $i_1, i_2 \in L_1$ and it is easy to see that i_1 and i_2 cannot belong to different L_ks. This completes the proof.

□

Proof of Theorem 1

(i) Fix an arbitrary school choice problem R and let $\mu := f^{ETB}(R)$ be the matching produced by the SDA-ETB algorithm. Let $(\mu^t)_{t \geq 1}$ be the sequence of temporary assignments in the SDA-ETB. We show that there are no stable improvement cycles (SICs) at μ and R by contradiction.

Suppose that $i_1, \ldots, i_m$ is a SIC at μ and R, and let $s_l := \mu(i_l)$ for all $l \leq m$. We assume in the following that the cycle is minimal in the sense that no strict subset of students $i_1, \ldots, i_m$ forms a SIC. Note that since $s_{l+1} P_{i_l} s_l$, i_l must have applied to s_{l+1} before applying to s_l in the SDA-ETB. We start with a few preliminary observations about SICs that are summarized in the following Lemma.

Lemma 1. *(i) $s_l \neq s_{l'}$ for all $l \neq l'$.*
(ii) $s_l \in S^0$ for at least one $l \leq m$.
(iii) $s_l \in S^1$ for at least one $l \leq m$.

Proof.

(i) It is clear that no specialized school can appear more than once on a SIC since $D_s(\mu)$ contains at most one student if $s \in S^1$. If $s_l = s_{l'} = s \in S^0$, $i_1, \ldots, i_{l-1}, i_{l'}, \ldots, i_m$ is a SIC of smaller size which contradicts the assumed minimality of the cycle.

(ii) Suppose to the contrary that $\{s_1, \ldots, s_m\} \subseteq S^1$ and let t_1 be the first round of the SDA-ETB (under R)[18] in which a student i_l is rejected by s_{l+1}. But then, there must be a student $j \in \mu^{t_1}(s_{l+1}) \setminus \mu(s_{l+1})$ such that $j \succ_{s_{l+1}} i_l$. Since $s_{l+1} \in S^1$, this implies that $i_l \notin D_{s_{l+1}}(\mu)$; contradiction.

[18]This qualifying statement will henceforth be omitted and we will speak of the SDA-ETB. There is no ambiguity involved here since the problem R is fixed throughout.

(iii) Suppose to the contrary that $\{s_1, \ldots, s_m\} \subseteq S^0$.

Consider first the case $|I| \leq p$ and note that $m \geq 2$. Again, let t_1 be the first round of the SDA-ETB in which a student i_l is rejected by s_{l+1}. If there was a specialized school $s \in S^1$ such that $|\mu^{t_1}(s)| = q^1$, there could not have been a round $t' > t_1$ in which a student $i_{l'} \neq i_l$ was rejected by $s_{l'+1}$ given $|I| \leq p$; contradiction. Now let $t'' > t_1$ be some round of the SDA-ETB in which a student $i_{l'} \neq i_l$ is rejected by $s_{l'+1}$. Since $|\mu^{t''}(s_{l+1})| \geq q_{s_{l+1}}$ and $|\mu^{t''}(s_{l'+1})| > q_{s_{l'}+1}$, there could not have been a specialized school $s \in S^1$ such that $|\mu^{t''}(s)| = q_s$ given that $|I| \leq p$. This implies that all rejections by non-specialized schools on the SIC were based on the labeling of students so that $i_1 > i_2 > \ldots > i_m > i_1$[19]; contradiction.

If $|I| > p$ note that the construction of $\succeq^0$ and the just completed argument imply that no student on the SIC belongs to the upper segment $L_1 \cup \ldots \cup L_{p-2}$. Furthermore, the cycle cannot consist exclusively of students who are strictly ordered according to $\succeq^0$. This implies that the only remaining possibility for a SIC consisting only of non-specialized schools is $m = 2$ and $i_1, i_2 \in L_K$. The proof is completed by noting that, by construction of $\succeq^0$, no student in L_K can cause a student in L_k to be rejected by a non-specialized school for all $k < K$.

□

Now consider the case of $|I| \leq p$. By Lemma 1 we can assume w.l.o.g. that $s_1 \in S^0$ and $s_2 \in S^1$. Since $|I| \leq p$, we must have $m \leq 3$. Suppose first that $m = 3$ and that $s_3 \in S^1$ so that s_1 is the only non-specialized school on the SIC. Note that i_3 must have been rejected by s_1 before i_2 was rejected by s_3: Otherwise $i_2 \notin D_{s_3}(\mu)$ since at least one higher priority student must have been rejected by s_3 in the course of SDA-ETB. Similarly, i_2 must have been rejected by s_3 before i_1 was rejected by s_2. But this implies that at the point where i_1 was supposedly rejected by s_2, at least $q^0_{(1)}$ students were temporarily matched to s_1 and q^1 students were temporarily matched to s_3. Since $|I| \leq p$, i_1 could not have been rejected by s_2; contradiction. Now suppose that $s_3 \in S^0$. As in the previous case, i_2 must have been rejected by s_3 before i_1 was rejected by s_2. This implies that there is a round t of SDA-ETB such that $i_2 \in \mu^t(s_3)$, $\mu^t(i_1) R_{i_1} s_2$, and $s_3 P_{i_2} \mu^{t+1}(i_2)$. Now in some round $t' > t$, i_1 must have been

[19]Remember that we identify students with their labels.

rejected by s_2. If $s_1 P_{i_3} \mu^{t'}(i_3)$, we obtain an immediate contradiction since at the point were i_1 was supposedly rejected by s_2, at least $q^0_{(1)} + q^0_{(2)}$ students must have been matched to s_1 and s_3 so that there could not have been q^1 students apart from i_1 applying to s_2 in round t'. If $\mu^{t'}(i_3) R_{i_3} s_1$, we similarly obtain a contradiction to the assumption that i_3 was rejected by s_1 in some later round of SDA-ETB. Hence, we must have $m = 2$. As in the previous cases, i_2 must have been rejected by s_1 before i_1 was rejected by s_2. This implies that there is a round t of the SDA-ETB such that $i_2 \in \mu^t(s_1)$, $\mu^t(i_1) R_{i_1} s_2$, and $s_1 P_{i_2} \mu^{t+1}(i_2)$. If $\mu^t(i_1) P_{i_1} s_2$, i_1 could not have been rejected by $\mu^t(i_1)$ and s_2 in subsequent rounds of SDA-ETB given that $|I| \leq p$. Hence, we must have $\mu^t(i_1) = s_2$ by strict preferences. If $i_1 \notin \mu^{t+1}(s_2)$, it has to be the case that $\mu^t(s_1) \succ_{s_2} i_1$ since i_2 would not have been rejected in $TB(\mu^t)$ otherwise. But then i_1 could not have subsequently obtained a place at s_1; contradiction. If $i_1 \in \mu^{t+1}(s_2)$, there must be a student $j \in \mu(s_2)$ such that $\mu^t(j) P_j s_2$. If $\mu^t(j) \neq s_1$, let t' be the round where j was rejected by $\mu^t(j)$. Given $|I| \leq p$, there cannot be a round $t'' > t'$ in which i_1 was rejected by s_2. But then i_2 must have been rejected by s_2 before round t' so that in particular $i_2 \notin D_{s_2}(\mu)$; contradiction. Hence, we must have $\mu^t(j) = s_1$ for any $j \in \mu(s_2)$ such that $\mu^t(j) P_j s_2$. Iterating this argument it is easy to see that there must be a round $t' > t$ of SDA-ETB such that $|\mu^{t'}(s_2)| = q^1$, $i_1 \in \mu^{t'}(s_2)$, $|\mu^{t'}(s_1)| \geq q_{s_1} + 1$, and $\mu^{t'}(s_1) \succ_{s_2} i_1$. But then i_1 could not have obtained a place at s_1 subsequently to being rejected by s_2; contradiction.

Now we consider the case that $|I| > p$. Note that we can assume w.l.o.g. that i_1 is the student with the lowest label among all students on the SIC. We distinguish two subcases.

Case 1: $i_1 \geq p$.

We assume for now that $|L_K| = 2$. It will become clear from our arguments that there cannot be SIC in case $|L_K| = 1$ either.

If $s_2 \in S^0$, we must have $i_1, i_2 \in L_K$ by the construction of $\succeq^0$ and the assumption that $p \leq i_1 < i_2$. This implies in particular that $L = 2$, $i_1 = K + 1$, $i_2 = K + 2$, and $s_1 \in S^1$. By exogenous tie-breaking in the SDA-ETB at most $q_{s_2} - 1$ students with lower labels than $K + 1$ could have applied to s_2. If more than $q^1 - 1$ students with lower labels than $K + 1$ applied to s_1, stability of μ and limited p-variability imply that K must have been rejected by s_1 in the SDA-ETB and $K+1 \succ_{s_1} K \succ_{s_1} K+2$. But then we cannot have that $K+2 \in D_{s_1}(\mu)$. Hence, at most $q^1 - 1$ students with lower labels than $K+1$ could have applied to s_1. But then SDA-ETB would have assigned $K+1$ to s_2 and $K+2$ to s_1; contradiction. Hence, we must have $s_2 \in S^1$.

Now suppose that $s_2 \in S^1$ and $i_2 = i_1 + 2$. Since $s_2 \in S^1$, the stability of μ w.r.t. $\succ^1$ and limited p-variability imply that $i_1 = K$, $i_2 = K+2$, and $K+2 \succ_{s_2} K$. If $m = 2$ we must have $s_1 \in S^0$ by Lemma 1. Given exogenous tie-breaking in the SDA-ETB and $\mu(K) = s_1$, at most $q_{s_1} - 1$ students with labels lower than K could have applied to s_1. As above, $K+2$ must have been rejected by s_1 before K was rejected by s_2. It is easy to see that we obtain a contradiction unless $K+1$ applied to s_1 in the SDA-ETB. Since ties in the lower segment are broken last, this implies that there is a round t of the SDA-ETB procedure such that $\{K+1, K+2\} \subset \mu^t(s_1)$ and $i_1 \in \mu^t(s_2)$. Since $K+2 \succ_{s_2} K$ by the stability of μ, we must have $K \succ_{s_2} K+1$ by limited p-variability. But then $K+1$, and not $K+2$, would have been rejected by s_1. Since there are no further rejections after tie-breaking in the lower segment this is a contradiction. If $m = 3$, $i_3 = K+1$ (this is the only possibility given the definition of i_1), and $s_3 \in S^0$, i_2 must have been rejected by s_3 before i_1 was rejected by s_2. Suppose first that $s_1 \in S^0$. By exogenous tie-breaking in the SDA-ETB and limited p-variability at most $q_{s_1} - 1$ students with lower labels than K could have applied to s_1. Similarly, at most $q_{s_3} - 1$ lower labeled students could have applied to s_3. Furthermore, it cannot be the case that $s_1 P_{K+2} s_3$ since there would be a SIC of size 2 otherwise. But then neither i_3 nor i_2 would have been rejected by s_1 and s_3, respectively; contradiction. If $s_1 \in S^1$, it is easy to see that there must have been a round t of the SDA-ETB such that $\{i_2, i_3\} \subset \mu^t(s_3)$ and $i_1 \in \mu^t(s_2)$. But then, i_2 would not have been rejected by s_3 since $i_2 \succ_{s_2} i_1 \succ_{s_2} i_3$ given the stability of μ and limited p-variability. An analogous argument can be used to show that $m = 3$, $i_3 = K+1$, and $s_3 \in S^1$ is also impossible. Hence, we must have $i_2 = i_1 + 1$.

Now suppose we have shown that, for some $l \leq m$, $i_{l'} = i_{l'-1} + 1$ and $s_{l'} \in S^1$, for all $l' \in \{2, \ldots, l\}$. We now establish that $m > l$, $i_{l+1} = i_l + 1$, and $s_{l+1} \in S^1$. This inductive argument completes the proof since it contradicts the finiteness of the set of students.

Suppose first that $m = l$. Since there has to be at least one non-specialized school on the SIC by Lemma 1, we must have $s_1 \in S^0$. Note that it has to be the case that $i_m \in \{K+1, K+2\}$. Otherwise we could use exactly the same argument used to establish that a SIC cannot contain only specialized schools to derive a contradiction since we assumed $i_1 \geq p-1$. Suppose first that $i_m = K+1$ so that $i_{m-1} = K$. By minimality of the cycle, we must have $s_{l+1} P_{i_l} s_1$ for all $l < m$. Furthermore, at most $q_{s_1} - 1$ students indexed lower than i_1 could have applied to s_1 by exogenous tie-breaking in the SDA-ETB. Since all schools except s_1 are specialized, i_m must have been rejected by s_1 before i_l was rejected by s_{l+1} *for all* $l \leq m-1$. Similar to above this implies that there must have been a round t of SDA-ETB such that $\{K+1, K+2\} \subset \mu^t(s_1)$ and $K \in \mu^t(s_m)$. By stability of μ and limited p-variability, we must have $K+1 \succ_{s_m} K \succ_{s_m} K+2$. This implies again that $K+2$, and not $K+1$, would have been rejected by s_1. Since there are no rejections after tie-breaking in the lower segment this is a contradiction. If $i_m = K+2$, we obtain an immediate contradiction since it is easy to see that the minimality of the cycle implies that at most $q_{s_1} - 1$ other students could have applied to s_1 in the SDA-ETB. Hence, i_m could not have been rejected by s_1 in the SDA-ETB; contradiction.

The proofs that $s_{l+1} \in S^1$ and $i_{l+1} = i_l + 1$ are virtually identical to the proofs that $s_2 \in S^1$ and $i_2 = i_1 + 1$. The details are omitted.

Case 2: $i_1 < p$.

Note first that there has to exist an $l \leq m-1$ such that $i_l = p-1$ and $i_{l+1} \in L_{p-1} \cup \ldots \cup L_K$. Otherwise, the SIC would consist entirely of students in $L_1 \cup \ldots \cup L_{p-2}$. The proof that this is impossible is completely analogous to the proof for the case of $|I| \leq p$.

Hence, there has to exist an index l with the above mentioned properties. Note that in particular $s_{l+1} \in S^1$ since i_l cannot envy a student in L_{p-1} for a non-specialized school. Furthermore, by limited p-variability i_{l+1} is the only student in $L_{p-1} \cup \ldots \cup L_K$ who can rank higher at s_{l+1} than i_l. This implies $|\mu(s_{l+1}) \cap (L_1 \cup \ldots \cup L_{p-2})| = q^1 - 1$. For all $l' \in \{2, \ldots, l\}$ we must have $\mu(s_l) \subset L_1 \cup \ldots \cup L_{p-1}$ and $|\mu(s_l)| = q_{s_l}$. Now note that $|I| > p$ implies $|L_1 \cup \ldots \cup L_{p-2}| = p-1$ so that we must have $l \leq 2$.

Now let $l \leq 2$, $i_l = p-1$ and $i_{l+1} \in L_{p-1}$. Note that there cannot be an $l < l' \leq m$ such that $i_{l'} \in L_1 \cup \ldots \cup L_{p-2}$: Otherwise, we would have that $\{i_{l'}, \ldots, i_m\} \subset L_1 \cup \ldots \cup L_{p-2}$. If $l = 1$, this yields a contradiction to the assumption that i_1 was the lowest labeled agent on the SIC. If $l = 2$, we would have that $\mu(s_{l'+1}) \cup \ldots \cup \mu(s_1) \subset L_1 \cup \ldots \cup L_{p-2}$. Since $|\mu(s_{l+1}) \cap (L_1 \cup \ldots \cup L_{p-2})| = q^1 - 1$, this contradicts $|L_1 \cup \ldots \cup L_{p-2}| = p-1$. By construction of $\succeq^0$ and limited p-variability, we must thus have $i_{l'+1} = i_{l'} + 1$ for all $l' \in \{l, \ldots, m-3\}$, and $\{s_{l+1}, \ldots, s_{m-1}\} \subset S^1$. If $s_m \in S^0$, it has to be the case that $\{i_{m-1}, i_m\} = L_K$. As above, i_{m-1} must have been rejected by s_m before $i_{m-2} = K$ was rejected by $s_{m-1} \in S^1$. Since we break ties in the lower segment last and $i_m \in \mu(s_m)$, there must have been a round t of the SDA-ETB such that $\{i_{m-1}, i_m\} \subset \mu^t(s_m)$, $|\mu^t(s_m)| = q_{s_m} + 1$, and $i_{m-2} \in \mu^t(s_{m-1})$. But then i_{m-1} could not have been rejected by s_m; contradiction. Hence, we must have $\{s_{l+1}, \ldots, s_m\} \subset S^1$.

Now suppose that $l = 2$, $s_1 \in S^1$, and $s_2 \in S^0$. It has to be the case that i_1 was rejected by s_2 before i_l was rejected by s_{l+1} for all $l \in \{2, \ldots, L\}$. Let t be the round of SDA-ETB in which i_1 was rejected by s_2 and let $t' > t$ be the round of SDA-ETB in which i_2 applied to s_2. By limited p-variability we must have $|\mu^{t'}(s_3) \cap (L_1 \cup \ldots \cup L_{p-2})| \geq q^1 - 1$ and since $t' > t$ it has to be the case that $\mu^{t'}(s_2) \setminus \{i_2\} \subset L_1 \cup \ldots \cup L_{p-2}$, $|\mu^{t'}(s_2)| \geq q^0_{(1)} + 1$, and $i_1 \notin \mu^{t'}(s_3) \cup \mu^{t'}(s_2)$. But since $|L_1 \cup \ldots \cup L_{p-2}| = p-1$, there cannot be a specialized school $s \in S^1 \setminus \{s_3\}$ such that $\mu^{t'}(s) \subset L_1 \cup \ldots \cup L_{p-2}$ and $|\mu^{t'}(s)| = q^1$. Since i_2 is the highest indexed student in the upper segment, she could not have obtained a place at s_2; contradiction. Next, consider the case $s_1 \in S^0$, $s_2 \in S^1$. As above, i_m must have been rejected by s_1 before i_l was rejected by s_{l+1} for all $l \leq m-1$. Let t be the round of the SDA-ETB in which i_m was rejected by s_1. If $\mu^t(s_1) \subset L_1 \cup \ldots \cup L_{p-2}$, it is easy to see that i_1 could not have been rejected by s_2 given that $|L_1 \cup \ldots \cup L_{p-2}| = p-1$ since i_2 must have been rejected by s_3 in some earlier round. By minimality of the cycle, we must have $s_l P_{i_l} s_1$ for all $l \neq m$. This implies that there is an agent $j \in I \setminus (L_1 \cup \ldots \cup L_{p-2} \cup \{i_1, \ldots, i_m\})$ such that $\mu^{t+1}(j) = \mu^t(j) = s_1$. Given the above the only possibility is that $\{i_m, j\} = L_K$. Since ties in the lower segment are broken last, we must have $\mu^t(i_{m-1}) = s_m$. As usual, stability and limited p-variability imply that i_m would not have been rejected by s_1; contradiction. The only remaining case to consider for $l = 2$ is $s_1, s_2 \in S^0$. Here, i_m must have been rejected by s_1 in particular before i_2 was rejected by s_3. If i_1 was rejected by s_2

before i_2 was rejected by s_3, i_2 could not have been rejected by s_3 in a subsequent round. If i_2 was rejected by s_3 before i_1 was rejected by s_2, i_1 could not have been rejected by s_2 in a subsequent round. The details are similar to the previous cases and omitted.

Thus, the only remaining case is $l = 1$ and $s_1 \in S^0$ (by Lemma 1). As usual, i_m must have been rejected by s_1 before i_l was rejected by s_{l+1} for all $l \leq m-1$. Similar to the proof that $l = 2, s_1 \in S^0, s_2 \in S^1$, we can show that if t is the round in which i_m was rejected by s_1 we must have $\mu^t(s_1) \setminus \{i_m\} \subset L_1 \cup \ldots \cup L_{p-2}$. If $t' > t$ is the round in which i_1 is rejected by s_2, we must have $|\mu^{t'}(s_2) \cap (L_1 \cup \ldots \cup L_{p-2})| = q^1 - 1$. Since $|I| \leq p-1$, this implies that if $t'' > t'$ denotes the round of SDA-ETB in which i_2 applies to s_1, we must have $\mu^{t''}(s_1) \subset L_1 \cup \ldots \cup L_{p-2}$ and there could not have been a specialized school $s \neq s_2$ such that $\mu^{t''}(s_2) \subset L_1 \cup \ldots \cup L_{p-2}$ and $|\mu^{t''}(s_2)| = q^1$. But then i_2 could not have obtained a place at s_2; contradiction.

(ii) We first consider the case $|I| \leq p$. Let $i \in I$ be a student and R_i be an arbitrary strict preference relation for this student. Let $top_j(R_i)$ be the jth most preferred school according to R_i if R_i contains at least j acceptable schools, and $top_j(R_i) = i$ if R_i contains less than j acceptable schools.

Note that if $f_i^{ETB}(R) \notin \{top(R_i), top_2(R_i), top_3(R_i)\}$ for some $i \in I$, student i cannot manipulate at the profile R:[20] Let $s_1 = top(R_i), s_2 = top_2(R_i)$, and $s_3 = top_3(R_i)$. Given that $|I| \leq p$, we must have $s_1, s_2, s_3 \in S^0$ and any specialized school s must have at least $q_{s_1} + q_{s_2} + q_{s_3}$ free places in the matching $f^{ETB}(R)$. But then, no matter which preference relation i submits, no specialized school can ever fill its capacity. By the rules of the tie-breaking subroutine, tie-breaking decisions will thus always be based on the (fixed) labels of students. By strategy-proofness of the SDA for fixed tie-breaking rules, i cannot manipulate the SDA-ETB.

Now consider a profile R and a student i such that $f_i^{ETB}(R) \in \{top_2(R_i), top_3(R_i)\}$. Suppose there was an alternative report R_i' for i such that $f_i^{ETB}(R_i', R_{-i}) = top(R_i) =: s_1$. Let $s_2 := top(R_i')$ and note that we must have $s_1 \neq s_2$. Denote by $(\mu^t)_{t\geq 1}$ and $(\tilde{\mu}^t)_{t\geq 1}$ the sequences of temporary assignments of the SDA-ETB

[20] It is obvious that the SDA-ETB never assigns a student to an unacceptable school. This implies in particular that $\{top(R_i), top_2(R_i), top_3(R_i)\} \subset S$ in the above situation.

under R and R', respectively. Let t_1 be the round of the SDA-ETB under R in which i is rejected by s_1 and let t_2 be the round of the SDA-ETB under R' in which i is rejected by s_2. Consider first the case $s_1 \in S^0$ and $s_2 \in S^1$. We must have $|\mu^{t_1}(s_1)| = q_{s_1} + l$ for some $l \geq 1$. If there was a specialized school s such that $|\mu^{t_1}(s)| = q_s$, it would have to be the case that $|\mu^t(s_2)| \leq q_{s_2} - l$. But this would imply that in the SDA-ETB under R no student is ever rejected by s_2, so that $f_i^{ETB}(R') = s_2$. Continuing this line of reasoning it is easy to see that all tie-breaking decisions in the SDA-ETB under R must have been made conditional on the fixed labeling of students and no specialized school could have rejected any student in the course of this algorithm. But the same statements must hold for the SDA-ETB under R' since i was rejected by s_1 in the SDA-ETB under R. Now there must be at least q_{s_1} students with lower labels than i who applied to s_1 in the SDA-ETB under R. But all of these students will apply to s_1 in the SDA-ETB under R' given the above so that i cannot obtain a place at s_1. Next, consider the case $s_1 \in S^0$ and $s_2 \in S^1$. Note that if $t_2 = 1$, i cannot end up matched to s_1 in the SDA-ETB under R' if $top_2(R_i') = s_1$ since there must be a round of this procedure in which the temporary assignment is exactly the same as in round t_1 of the SDA-ETB under R. If $top_2(R_i') \neq s_1$, we must have $f_i^{ETB}(R') = top_2(R_i')$ given $|I| \leq p$. Thus, $t_2 > 1$ and there has to be a school $s \neq s_2$ that has to reject a student in some round $t < t_2$ of the SDA-ETB under R'. If $s \neq s_1$, s_1 could not have rejected any student in the SDA-ETB under R and R' given that $|I| \leq p$ which contradicts $f_i^{ETB}(R) \neq s_1$. Hence, $s = s_1$ and we must have $|\tilde{\mu}^{t_2}(s_1)| > q_{s_1}$. But then subroutine $\mathbf{TB}(\tilde{\mu}^{t_2})$ ensures that all students in $\tilde{\mu}^{t_2}(s_1)$ have higher priority for s_2 than i and that i could not have obtained a place at s_1 in one of the subsequent rounds. Now suppose $s_1 \in S^1$ and note that i could not obtain s_1 by any misrepresentation if $\mu^1(i) \neq s_1$. Hence, there must be school $s \neq s_1$ that had to reject at least one student prior to t_1. Since $|I| \leq p$, i is matched to $top(R_i')$ if $top(R_i') \in S \setminus \{s_1, s\}$. Hence, we must have $s = s_2$ and s_1, s_2 are the only schools who had to reject a student in the SDA-ETB under R'. If $s_2 \in S^1$ and $f_i^{ETB}(R') = s_1$, i could also manipulate if s_1 and s_2 were the only schools available. This contradicts strategy-proofness of the SDA when $S^0 = \emptyset$. So suppose that $s_2 \in S^0$. Since i was rejected subsequently to a rejection at s_2 in the SDA-ETB under R, all students in $\mu^{t_1}(s_2)$ must have had higher priority for s_1 than i. This implies that in the SDA-ETB under R' ultimately the same set of students will be rejected by s_2 as in the SDA-ETB under R. Since all students who have applied to s_1 prior to t_1 in the SDA-ETB under R also apply to s_1 in some round of the SDA-ETB under R', we cannot have $f_i^{ETB}(R') = s_1$.

It remains to be shown that i cannot obtain $top_2(R_i)$ if $f_i^{ETB}(R) = top_3(R_i)$. Given $|I| \leq p$, it is easy to see that the only potentially profitable manipulation is for i to rank $top_2(R_i)$ first. The proof can be completed using a similar case distinction as above and the details are omitted.

Next, we prove the statement for the case of $|I| \geq p+1$. Note first that students outside the upper segment $L_1 \cup \ldots \cup L_{p-2}$ cannot influence tie breaking in the upper segment. To see this note that the only possible effect such a student could have on this tie-breaking decision is to initiate a rejection chain leading to the rejection of the student indexed $p-1$ by some specialized school s. Since there cannot be more than one student from L_{p-1} who has higher priority for s than $p-1$, this implies that there are $q^1 - 1$ students from the upper segment temporarily matched to s. Now remember that we only use temporary assignments for tie-breaking in the upper segment if some specialized school has filled all of its places with students from the upper segment. But if $p-1$ is temporarily matched to some specialized school $s' \in S^1 \setminus \{s\}$ together with q^1-1 other students from the upper segment, no student from the upper segment is rejected by a non-specialized school given that $|I| \leq p$.[21] The proof that no student in the upper segment can manipulate tie-breaking in the upper segment to her benefit is completely analogous to the proof for the case of $|I| \leq p$ and the details are omitted.

It remains to be shown that no student can profitable manipulate the tie-breaking decision in the lower segment. In particular, the proof is complete unless $|L_K| = 2$. Now note that the endogenous tie-breaking of the SDA-ETB in the lower segment ensures that there are no additional rejections after the tie-breaking stage since (i) only student K can ever rank lower than one of $K+1$ and $K+2$, and (ii) K cannot rank below both of these students at some specialized school. Hence, a student can profitably manipulate tie-breaking in the lower segment only if she obtains a better school prior to tie-breaking. Now suppose that contrary to what we want to show some student i can profitably

[21]It is precisely at this point where we need that the upper segment contains $p-1$ students and not p students if $|I| \geq p+1$.

manipulate the tie-breaking procedure when the profile of (true) preferences is R by submitting R'_i. Note that it has to be the case that the tie between $K+1$ and $K+2$ needs to be broken endogenously under R and $R' = (R'_i, R_{-i})$: Otherwise we could use the same strict priority structure under R and R' so that we obtain a contradiction to the strategy-proofness of SDA for a fixed strict priority structure. This already implies that neither $K+1$ nor $K+2$ can manipulate the SDA-ETB procedure to their benefit. Let s and s' be the schools to which $K+1$ and $K+2$ are temporarily matched before tie-breaking in the lower segment under R and R', respectively. By the exogenous tie-breaking of the SDA-ETB exactly $q_s - 1$ students in $I \setminus \{K+1, K+2\}$ apply to s in the course of SDA-ETB under R and exactly $q_{s'} - 1$ students in $I \setminus \{K+1, K+2\}$ apply to s' in the course of SDA-ETB under R'. Note that $s = s'$ unless i applies to s' (prior to tie-breaking) under R'_i but not under R_i. Since there are no rejections after tie-breaking in the lower segment, $s \neq s'$ would imply that $s' = f_i^{ETB}(R')$ and thus $s = f_i^{ETB}(R) R_i f_i^{ETB}(R') = s'$. So we may assume that $s = s'$. But then we would obtain the same final matching for students outside the lower segment (under both R and R'), if we assumed (contrary to fact) that s could admit $q_s + 1$ students and has a strict priority ranking of students (by arbitrarily breaking the remaining ties in $\succeq^0$). This is again a contradiction to strategy-proofness of the SDA for a fixed strict priority structure and completes the proof.

□