# A Theory of School-Choice Lotteries*

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**Abstract**

A new centralized mechanism was introduced in New York City and Boston to assign students to public schools in district school-choice programs. This mechanism was advocated for its superior fairness property, besides others, over the mechanisms it replaced. In this paper, we introduce a new framework for investigating school-choice matching problems and two ex-ante notions of fairness in lottery design, *strong ex-ante stability* and *ex-ante stability*. This framework generalizes known one-to-many two-sided and one-sided matching models. We first show that the new NYC/Boston mechanism fails to satisfy these fairness properties. We then propose two new mechanisms, the *fractional deferred-acceptance mechanism*, which is ordinally Pareto dominant within the class of strongly ex-ante stable mechanisms, and the *fractional deferred-acceptance and trading mechanism*, which satisfies equal treatment of equals and constrained ordinal Pareto efficiency within the class of ex-ante stable mechanisms.

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# 1 Introduction

Following the 1987 decision of the U.S. Court of Appeals, the Boston school district introduced a possibility of "choice" for public schools by relaxing the mandatory zoning policy. For this purpose the district introduced priority classes for students for each school, based on how far away a student lives and whether a sibling of the student attends the school. In 1989, a centralized clearinghouse, now commonly referred to as the *Boston mechanism* (Abdulkadiroğlu and Sönmez, 2003) was adopted by the district. Each year since then this clearinghouse collects preference rankings of students over schools, and determines a matching of students to schools based on students' priorities. Since there are typically several students tied for priority at schools, *random tie-breaking* has been the common practice for obtaining a strict priority ranking among students within equal priority classes. Today many U.S. school districts employ clearinghouses that operate on the random tie-breaking practice.

Abdulkadiroğlu and Sönmez (2003) pointed out that the Boston mechanism is flawed in many ways (also see Chen and Sönmez (2006); Ergin and Sönmez (2006); Pais and Pinter (2007); Pathak and Sönmez (2008)). They showed, for example, that student priorities are not necessarily respected by this mechanism. Moreover, the Boston mechanism is susceptible to strategic manipulation in a very obvious manner.[1] Abdulkadiroğlu and Sönmez proposed two alternatives to this mechanism from the mechanism design literature on indivisible good allocation and two-sided matching.[2] Eventually one of these mechanisms, the Gale-Shapley student-optimal stable mechanism, replaced the Boston mechanism in 2005 due to the collaborative efforts of Abdulkadiroğlu and Sönmez with economists Pathak and Roth. A version of the same mechanism was also adopted by the New York City public school system in 2004 (Abdulkadiroğlu, Pathak, and Roth, 2005, 2009; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006; Pathak and Sönmez, 2008).

This new mechanism relies on the idea of producing a "stable matching" first introduced by Gale and Shapley (1962). This approach has been widely and successfully used in several two-sided matching applications. Probably the most well-known of these applications is the National Resident Matching Program (Roth, 1984, 1991; Roth and Peranson, 1999) that was designed to match hospital residency programs with graduating medical doctors. However, there are two important differences between school-choice and two-sided matching problems.

The first difference is about the definition of the mechanism design problem. Both hospital residency programs and doctors are active "agents" in a two-sided market, and thus, both state preferences over agents on the other side of the market. On the other hand, in school choice, schools

[1]See Haeringer and Klijn (2009) and Calsamiglia, Haeringer, and Klijn (2009) for strategic issues regarding both mechanisms when the number of choices to be listed is limited for each student.

[2]See for example Gale and Shapley (1962); Shapley and Scarf (1974); Roth (1984, 1991); Balinski and Sönmez (1999); Abdulkadiroğlu and Sönmez (1999); Papai (2000); Ergin (2002); Kesten (2006, 2009b); Pycia and Ünver (2009).

are passive in most cases and viewed only as indivisible objects to be consumed. In other words, a student's priority is irrelevant to schools' preferences. This difference has been emphasized by several studies starting with Balinski and Sönmez (1999).[3]

The second major difference, however, has not been explicitly studied until recently. In two-sided matching markets, hospital and doctor choices are elicited as strict preferences. Therefore, there are no indifferences within preferences, and thus, tie-breaking is not needed. In school choice, however, only students' choices are elicited as strict preferences, whereas students' priorities are coarse. That is, there are often many students who belong to the same priority class. For example, there are only four priority classes in Boston, although there are thousands of students applying each year. Consequently, to adapt the deterministic two-sided matching approach, ties among equal-priority students need to be explicitly broken in an endogenous or exogenous fashion. All previous mechanism design efforts emphasizing this difference relied on the deterministic two-sided matching approach (which builds on the assumption of strict preferences and strict priorities) using random tie-breaking (Ehlers, 2006; Pathak, 2006; Abdulkadiroğlu, Pathak, and Roth, 2009; Erdil and Ergin, 2008; Abdulkadiroğlu, Che, and Yasuda, 2008). Our approach in this paper does not rely on any form of tie-breaking.

Another related problem to school choice is the so-called random assignment problem. A random assignment problem can be viewed as a special school-choice problem where each school has unit quota and all students have equal priority for all schools. In this context a mechanism chooses a probability distribution over schools for each student. The seminal work of Bogomolnaia and Moulin (2001) has revealed that such a richer setup can allow one to define a much stronger welfare criterion than ex-post efficiency, and in particular, improve upon a well-known and commonly used real-life mechanism (the so-called *random priority*) that relies on randomly breaking the ties among equal-priority students. Our approach, inspired in part by Bogomolnaia and Moulin (2001), also rests on the idea of avoiding welfare losses due to random tie-breaking in the context of school choice.

Although the $14^{\text{th}}$ Amendment of the U.S. Constitution does not directly apply to education, it clearly states that some sort of fairness should exist in the treatment of students. Different interpretations of the amendment by school districts have resulted in different fairness concepts. There are two important fairness properties of the recently adopted NYC/Boston mechanism. The first one is *equal treatment of equals*, which says that two students with exactly the same preferences and equal priorities at all schools should be guaranteed the same chance of enrollment in all schools. Since school choice is a discrete allocation problem, this property has been achieved simply in previous approaches by using uniform randomization in tie-breaking. The second fairness property is "ex-post stability."At a *stable* matching, there does not exist any student $i$ who prefers a seat at a different

[3]Also see Ergin (2002); Abdulkadiroğlu and Sönmez (2003); Sönmez and Ünver (2010); Abdulkadiroğlu, Che, and Yasuda (2008).

school $c$ than the one he is assigned to such that either (1) school $c$ has not filled its quota, or (2) school $c$ has an enrolled student who has strictly lower priority than $i$. A mechanism is *ex-post stable* if it induces a lottery over stable matchings (i.e., an ex-post stable lottery).[4] One of the reasons that ex-post stability was chosen to interpret priorities is that when an assignment is unstable, local school districts are prone to lawsuits that can be filed regarding a 14th Amendment violation.[5]

Although ex-post stability is a meaningful interpretation of fairness for deterministic outcomes, for lottery mechanisms, such as the ones used for school choice, it is not clear that it is the "right" fairness property. A plausible argument is that ex-post stability may not be a sufficiently strong fairness condition to eliminate the possibility of legal action. To begin with, ex-post stability is not defined over random matchings (as opposed to equal treatment of equals), but it is defined over deterministic matchings. Hence it has no implications when contrasting the enrollment chances of different-priority students. Nonetheless, it is not unreasonable for students' families to seek fairness from an ex-ante perspective as well. From such a perspective, in line with the rationale of stability, ex ante fairness can then be interpreted as giving "the chance of enrollment to the higher-priority student at a preferred school." To capture this intuition we introduce the following analogous stability requirement on students' enrollment probabilities: if student $i$ has strictly higher priority for school $c$ than student $j$, then student $i$ should not suffer any risk of being assigned to a less desirable school than $c$ while student $j$ enjoys a positive chance of being assigned to school $c$. Clearly, this requirement implies that any ex-post realization from such a *random matching* (which prescribes the assignment probability of each student to each school) is necessarily stable.

To avoid welfare losses due to random tie-breaking and to address ex-ante fairness concerns, we present a general model of school choice in which (1) school priorities can be coarse as in real life, and (2) matchings can be random. Over random matchings, we propose two powerful notions of fairness that are stronger than ex-post stability. We say that a random matching causes *ex-ante justified envy* if there are two students $i$ and $j$ and a school $c$ such that student $i$ has strictly higher priority than $j$ for school $c$ but student $j$ can be assigned to school $c$ with positive probability while $i$ can be assigned to a less desirable school for him than $c$ with positive probability (i.e., $i$ has ex-ante justified envy toward $j$). We refer to a random matching as *ex-ante stable* if it eliminates ex-ante justified envy. The following simple example shows that the current NYC/Boston mechanism, when viewed from an ex-ante perspective, may cause ex-ante justified envy. Interestingly, one can find an alternative lottery that, despite being equivalent to the one used by the NYC/Boston mechanism, is ex-post unstable.

**Example 1** Consider the following problem with five students $\{1, 2, 3, 4, 5\}$ and four schools $\{a, b, c, d\}$ where each of schools $a$, $b$, and $c$ has one seat, and $d$ has two seats. The priority orders and student

[4] Abdulkadiroğlu and Sönmez (2003) discuss different interpretations of priorities in the context of school choice.

[5] Personal communication with Atila Abdulkadiroğlu, Parag Pathak, Alvin Roth, and Tayfun Sönmez.

preferences are as follows:

| $\succsim_a$ | $\succsim_b$ | $\succsim_c$ | $\succsim_d$ |
|---|---|---|---|
| 5 | 4, 5 | 1, 3 | $\vdots$ |
| 1 | $\vdots$ | $\vdots$ | $\vdots$ |
| 2 | $\vdots$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ |
|---|---|---|---|---|
| $c$ | $a$ | $c$ | $b$ | $b$ |
| $a$ | $d$ | $d$ | $d$ | $a$ |
| $d$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Consider the new NYC/Boston mechanism, which uniformly randomly chooses a single tie-breaking order for equal-priority students at each school and then employs the student-proposing deferred-acceptance algorithm using the modified priority structure. It is straightforward to compute that this mechanism implements the following lottery:

$$\lambda = \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ d & d & c & b & a\end{pmatrix}}_{\mu_1} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ a & d & c & d & b\end{pmatrix}}_{\mu_2} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ c & d & d & b & a\end{pmatrix}}_{\mu_3} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ c & a & d & d & b\end{pmatrix}}_{\mu_4}$$

The above four deterministic matchings in the support of $\lambda$ are stable since they are obtained by the student-proposing deferred acceptance algorithm for tie-breakers $3 \succ_c 1$ and $4 \succ_b 5$; $3 \succ_c 1$ and $5 \succ_b 4$; $1 \succ_c 3$ and $4 \succ_b 5$; $1 \succ_c 3$ and $5 \succ_b 4$, respectively. Thus $\lambda$ is ex-post stable. However, the random matching that lottery $\lambda$ induces is not ex-ante stable because student 1 has ex-ante justified envy toward student 2 for school $a$: Matching $\mu_1$ implies that student 1 suffers from a positive probability of being assigned to school $d$, while matching $\mu_4$ implies that student 2 enjoys a positive probability of being assigned to school $a$, for which he has strictly lower priority than 1.

Now consider the following lottery which is equivalent to $\lambda$.

$$\lambda' = \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ d & a & c & d & b\end{pmatrix}}_{\mu'_1} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ c & d & d & b & a\end{pmatrix}}_{\mu'_2} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ c & d & d & b & a\end{pmatrix}}_{\mu'_3} + \frac{1}{4}\underbrace{\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ a & d & c & d & b\end{pmatrix}}_{\mu'_4}$$

The support of $\lambda'$ contains an unstable matching, namely $\mu'_1$, since student 1 has justified envy toward student 2 at this matching. Lottery $\lambda'$ exacerbates the justified envy situation under $\lambda$ by transforming it from ex-ante to ex-post. $\Diamond$

Given coarse priorities, an even stronger fairness consideration can be based on also imposing a similar requirement on the enrollment chances of students who belong to the same priority group for some school. We say that a random matching causes *ex-ante discrimination (among equal-priority students)* if there are two students $i$ and $j$ with equal priority for a school $c$ such that $j$ enjoys a higher probability of being assigned to school $c$ than student $i$ even though $i$ suffers from a positive probability of being assigned to a less desirable school for him than school $c$.[6] The following simple example shows that the new NYC/Boston mechanism also induces ex-ante discrimination between equal-priority students.

**Example 2** Consider the following problem with three students $\{1, 2, 3\}$ and three schools $\{a, b, c\}$ each with a quota of one. The priority orders and student preferences are as follows:

| $\succsim_a$ | $\succsim_b$ | $\succsim_c$ |
|---|---|---|
| 3 | 2 | 2 |
| 1, 2 | 1 | 1 |
| | 3 | 3 |

| $P_1$ | $P_2$ | $P_3$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $c$ |

The tie-breaking lottery assigns the second priority at school $a$ to equal-priority students 1 and 2 with equal chances. Then the new NYC/Boston mechanism (which operates on the student-proposing deferred-acceptance algorithm coupled with either strict priority structure) implements the following lottery:

$$\lambda = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix}}_{\mu_1} + \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}}_{\mu_2}$$

Observe that random matching $\rho^\lambda$ induces ex-ante discrimination between students 1 and 2 for school $a$ since matching $\mu_1$ implies that student 1 is given a positive probability of being assigned to school $a$ while student 2 who, despite having equal priority for $a$, always ends up at school $c$ which she finds worse than $a$. In particular, the ex-post observation that student 2 has been assigned to school $c$ by this mechanism cannot be attributed to an unlucky lottery draw to determine the priority order at school $a$. $\diamondsuit$

We refer to a random matching as *strongly ex-ante stable* if it eliminates both ex-ante justified envy and ex-ante discrimination. Both of these concepts imply ex-post stability. Strong ex-ante stability also implies equal treatment of equals. We propose two new mechanisms that select "special" ex-ante stable and strongly ex-ante stable random matchings.

[6]Ozek (2008) introduces a particular ex-ante fairness criterion for school-choice problems, which is in a similar vein to our *ex-ante discrimination of equal-priority students* property, and investigates whether the prominent mechanisms studied in previous literature satisfy this property.

Our first proposal, the *fractional deferred-acceptance (FDA) mechanism,* selects *the* strongly ex-ante stable random matching that is ordinally Pareto dominant among all strongly ex-ante stable random matchings (Theorems 2 and 3). The algorithm it employs is in the spirit of the deferred-acceptance algorithm of Gale and Shapley (1962), with students applying to schools in an order of decreasing preference and schools tentatively holding students based on priority. Unlike previous mechanisms, however, the FDA mechanism does not rely on tie-breaking. Loosely, schools always reject lower-priority students in favor of higher ones (if the need arises) as in the deferred-acceptance algorithm. However, whenever there are multiple equal-priority students being considered for assignment to a school, for which there is insufficient quota, the procedure tentatively assigns an equal fraction of these students and rejects the rest of the fractions. These rejected "fractions of students" continue to apply to their next-preferred schools in the usual deferred-acceptance fashion as if they were individual students. The procedure iteratively continues to make tentative assignments, until one full fraction of each student is assigned to some school. We interpret the assigned fractions of a student at the end of the procedure as his assignment probability to each corresponding school by the FDA mechanism. In contrast with the deferred-acceptance algorithm of Gale and Shapley (1962), the above described procedure may involve rejection cycles that prevent the procedure from terminating in a finite number of steps. Therefore, to obtain a convergent algorithm we also couple this procedure with a "cycle resolution phase" that solves a linear programming problem.

Our second proposal, the *fractional deferred-acceptance and trading (FDAT) mechanism* selects an ex-ante stable random matching that (1) treats equals equally and (2) is ordinally Pareto undominated within the set of ex-ante stable random matchings (Theorems 4 and 5). It employs a two-stage algorithm that stochastically improves upon the FDA matching. The FDAT mechanism starts from the random-matching outcome of the FDA algorithm and creates a trading market for school-assignment probabilities. In this market, the assignment probability of a student to a school can be traded for an equal amount of probabilities at better schools for the student so long as the trade does not result in ex-ante justified envy of some other student. Such trading opportunities are represented by *stochastic ex-ante stable improvement cycles*, i.e., the list of students who can trade fractions of schools among each other without violating any ex-ante stability constraints. We show that a random matching is constrained ordinally efficient among ex-ante stable random matchings if and only if there is no stochastic ex-ante stable improvement cycle (Proposition 5). Under the FDAT mechanism stochastic ex-ante stable improvement cycles are iteratively executed in a manner that preserves equal treatment of equals.

There are major difficulties in both approaches for finding feasible computational solutions. To overcome these difficulties we develop some new methods and use tools from the related literatures on other problems. We also show that the implementation of either mechanism is of polynomial time complexity (Propositions 3 and 6).

## 2 Related Literature

In addition to the papers mentioned in the Introduction, there are several strands of literature related to our paper. In the two-sided matching literature, a version of the random matching problem with strict school preferences was analyzed by Roth, Rothblum, and Vande Vate (1993). Our ex-ante stability and strong ex-ante stability concepts are equivalent when school priorities are strict, and they coincide with Roth, Rothblum, and Vande Vate's strong stability concept. They also characterize ex-post stability in this random framework using linear programming methods. However, their characterization is not valid with weak priorities. Thus, our analysis and results are independent from theirs.[7]

Alkan and Gale (2003) consider a deterministic two-sided matching model in which the two sides are referred to as firms and workers. In their model, a worker can work for one hour in total, but he can share his time between different firms. A firm can hire fractions of workers that sum up to a certain quota of hours. Both firms and workers have preferences over these fractions. The solution concept Alkan and Gale propose is Gale-Shapley stability with respect to these preferences. They prove that the set of stable matchings is non-empty and forms a partial-order lattice under certain assumptions over the firm and worker preferences. One can interpret a fractional deterministic matching as a random matching. Thus, this similarity creates some overlap between their and our setups. However, instead of complicated preference structures over deterministic outcomes, we use (1) responsive (Roth, 1985) school priorities and stochastic dominance to induce a partial order over probabilistic outcomes, and (2) a weaker definition than their stability notion, which we refer to as "ex-ante stability. "Moreover, Alkan and Gale do not propose any well-defined algorithm.

Recently, researchers have started to think about ex-ante efficiency in school-choice mechanisms. Specifically, to capture preference intensities, these approaches assume that each student is endowed with cardinal preferences over schools (as opposed to our assumption of ordinal preferences). Abdulkadiroğlu, Che, and Yasuda (2009) show that under a reasonable incomplete information setting the Boston mechanism's Bayesian equilibria Pareto dominate the dominant strategy equilibrium of the student-optimal stable Gale-Shapley mechanism. Based on this motivation, Abdulkadiroğlu, Che, and Yasuda (2008) propose a new hybrid mechanism that gives each student the choice of a target school, for which the student is given priority in case tie-breaking is needed, while using the Gale-Shapley student-optimal stable mechanism. The authors show that this leads to an ex-ante welfare improvement over random tie-breaking in certain instances. Featherstone and Niederle (2008) show that the Boston mechanism would result in ex-ante efficient random matchings when the students priorities are equal in an incomplete information equilibrium, and they support their finding through experiments.

[7] Moreover, they do not propose this model for school choice.

Another strand of literature deals with the probabilistic assignment of indivisible goods without assuming a priority structure. At least since Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001), it is well known that this approach is superior in terms of efficiency to randomization over priority-based deterministic methods. The latter paper and Katta and Sethuraman (2006) propose ordinally efficient procedures, treating equals equally for the strict and weak preference domains, respectively. Yilmaz (2006, 2009) generalize these methods to an indivisible good assignment problem, where some agents have initial property rights of some of the goods for the strict and weak preference domains, respectively. Finally, Athanassoglou and Sethuraman (2007) extend these models to a framework where the initial property rights could be over fractions of goods. We also embed one version of their algorithm into the second stage of our fractional deferred-acceptance and trading procedure as a way to achieve "fair" probability trading.

Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak, and Roth (2009) have pointed out that the new NYC/Boston mechanism may be subject to welfare losses when ties in priorities are broken randomly. Erdil and Ergin (2007, 2008) propose methods for improved efficiency without violating exogenous stability constraints for school-choice and two-sided matching problems, respectively.[8] All these papers emphasize that random tie-breaking may entail an ex-post efficiency loss. We, on the other hand, argue that it may also entail an ex-ante stability loss both among students with different priorities (ex-ante justified envy) and among students with equal priorities (ex-ante discrimination).

The current paper generalizes the unrelated approaches summarized in the previous two paragraphs and obtains a unified framework in dealing with school-choice problems in a probabilistic setting with ordinal preferences. Although we do not focus on ex-post stable mechanisms per se, our analysis also establishes some important infrastructure for studying ex-post stable mechanisms in a probabilistic setting where much stronger welfare criteria than ex-post efficiency can be conceived. Since the mechanism recently adopted in Boston and New York was chosen instead of a Pareto-efficient alternative[9] due to its superior fairness/stability features, we believe that the stability consideration plays a key role for the practicality of a mechanism in the context of school choice distinguishing this problem from most other allocation problems. Consequently, our study focuses not only on constrained ordinal efficiency but also on ex-ante stability, which has not been addressed before as well.[10]

The rest of the paper is organized as follows. Section 3 formally introduces a general model of

[8]Kesten (2009a) provides a new mechanism that aims to eliminate the efficiency loss under the Gale-Shapley mechanism by allowing students to give up certain priorities whenever it does not hurt them to do so.

[9]This is the so-called *top trading cycles mechanism* which has been also advocated by Abdulkadiroğlu and Sönmez (2003) as an attractive replacement. This mechanism is strategy-proof just like the new NYC/Boston mechanism but not ex-post stable.

[10]Erdil and Kojima (2007) independently develop a school-choice framework and concepts similar to ours. They do not pursue mechanisms satisfying their proposed properties.

school choice. Section 4 discusses desirable properties of mechanisms and introduces the new ex-ante stability criteria. Section 5 presents our first proposal, the fractional deferred-acceptance mechanism, and the related results. Section 6 presents our second proposal, the fractional deferred-acceptance and trading mechanism, and the related results. Section 7 concludes. The proofs of our main results are relegated to the Appendices.

# 3 The Model

We start by introducing a general model for school choice. A *school-choice problem* is a five-tuple $[I, C, q, P, \succsim]$ where:

- $I$ is a finite set of *students* each of whom is seeking a seat at a school.
- $C$ is a finite set of *schools*.
- $q = (q_c)_{c \in C}$ is a *quota vector* of schools such that $q_c \in \mathbb{Z}_{++}$ is the maximum number of students who can be assigned to school $c$. We assume that there is enough quota for all students, that is $\sum\limits_{c \in C} q_c = |I|$.[11]
- $P = (P_i)_{i \in I}$ is a *strict preference profile* for students such that $P_i$ is the *strict preference relation of student $i$ over the schools*.[12] Let $R_i$ refer to the associated weak preference relation with $P_i$. Formally, we assume that $R_i$ is a linear order, i.e. a complete, transitive, and antisymmetric binary relation. That is, for any $c, a \in C$, $cR_i a$ if and only if $c = a$ or $cP_i a$.
- $\succsim = (\succsim_c)_{c \in C}$ is a *weak priority structure for schools* such that $\succsim_c$ is the *weak priority order of school $c$ over the students*. That is, $\succsim_c$ is a reflexive, complete, and transitive binary relation on $I$. Let $\succ_c$ be the acyclic portion and $\sim_c$ be the cyclic portion of $\succsim_c$. That is, $i \succsim_c j$ means that student $i$ has at least as high priority as student $j$ at school $c$, $i \succ_c j$ means that $i$ has strictly higher priority than $j$ at $c$, and $i \sim_c j$ means that $i$ and $j$ have equal priority at $c$.

[11] If originally $\sum\limits_{c \in C} q_c > |I|$, then we introduce $|I| - \sum\limits_{c \in C} q_c$ additional virtual students, who have the lowest priorities at each school (say, a uniform priority ranking is available among virtual students for all schools and all virtual students have common strict preferences over schools). If originally $\sum\limits_{c \in C} q_c < |I|$, then we introduce a virtual school with a quota $|I|$-$\sum\limits_{c \in C} q_c$, which is the worst choice of each student, such that all students have equal priority for this school.

[12] For simplicity of exposition, we assume that all schools are acceptable for all students. All of our results are easy to generalize to the setting with unacceptable schools using a null school with quota $\infty$ and substochastic matrices instead of bistochastic matrices.

Occasionally, we will fix $I, C, q$ and refer to a problem by the strict preference profile of the students and weak priorities of schools, $[P, \succsim]$.

We are seeking matchings such that each student is assigned a seat at a single school and the quota of no school is exceeded. We also allow random (or probabilistic) matchings.

A *random matching* $\rho = [\rho_{i,c}]_{i\in I,\ c\in C}$ is a real stochastic matrix, i.e., it satisfies (1) $0 \leq \rho_{i,c} \leq 1$ for all $i \in I$ and $c \in C$; (2) $\sum_{c\in C} \rho_{i,c} = 1$ for all $i \in I$ ; and (3) $\sum_{i\in I} \rho_{i,c} = q_c$ for all $c \in C$. Here $\rho_{i,c}$ represents the probability that student $i$ is being matched with school $c$. Moreover, the stochastic row vector $\rho_i = (\rho_{i,c})_{c\in C}$ denotes the *random matching (vector) of student $i$ at $\rho$*, and the stochastic column vector $\rho_c = (\rho_{i,c})_{i\in I}$ denotes the *random matching (vector) of school $c$ at $\rho$*. A random matching $\rho$ is a *(deterministic) matching* if $\rho_{i,c} \in \{0, 1\}$ for all $i \in I$ and $c \in C$. Let $\mathcal{X}$ be the set of random matchings and $\mathcal{M} \subseteq \mathcal{X}$ be the set of matchings. We also represent a matching $\mu \in \mathcal{M}$ as the unique non-zero diagonal vector of matrix $\mu$, i.e., as a list $\mu = \binom{i_1\ i_2\ \dots\ i_{|I|}}{c_1\ c_2\ \dots\ c_{|I|}}$ such that for each $\ell$, $\mu_{i_\ell,c_\ell} = 1$. We interpret each student $i_\ell$ as matched with school $c_\ell$ in this list and, with a slight abuse of notation, use $\mu_{i_\ell}$ to denote the match of student $i_\ell$.

A *lottery* $\lambda$ is a probability distribution over matchings. That is, $\lambda = (\lambda_\mu)_{\mu\in\mathcal{M}}$ such that for all $\mu \in \mathcal{M}$, $0 \leq \lambda_\mu \leq 1$ and $\sum_{\mu\in\mathcal{M}} \lambda_\mu = 1$. Let $\Delta\mathcal{M}$ denote the set of lotteries. For any $\lambda \in \Delta\mathcal{M}$, let $\rho^\lambda$ be the *random matching of lottery* $\lambda$. That is, $\rho^\lambda = \left[\rho^\lambda_{i,c}\right]_{i\in I, c\in C} \in \mathcal{X}$ is such that $\rho^\lambda_{i,c} = \sum_{\mu\in\mathcal{M}\,:\,\mu_i=c} \lambda_\mu$ for all $i \in I$ and $c \in C$. In this case, we say that lottery $\lambda$ *induces* random matching $\rho^\lambda$. Observe that $\rho^\lambda_{i,c}$ is the probability that student $i$ will be assigned to school $c$ under $\lambda$. Let $Supp\,(\lambda) \subseteq \mathcal{M}$ be the support of $\lambda$, i.e., $Supp\,(\lambda) = \{\mu \in \mathcal{M} : \lambda_\mu > 0\}$.

We state the following theorem whose proof is an extension of the proof of the standard Birkhoff (1946) - von Neumann (1953) Theorem (also see Kojima and Manea (2010)):

**Theorem 1** ***(School-Choice Birkhoff - Von Neumann Theorem)*** *For any random matching $\rho \in \mathcal{X}$, there exists a lottery $\lambda \in \Delta\mathcal{M}$ that induces $\rho$, i.e., $\rho = \rho^\lambda$.*

Through this theorem's constructive proof and related algorithms in combinational optimization theory, such as the Edmonds (1965) algorithm, one can find a lottery implementing $\rho$ in polynomial time. Thus, without loss of generality, we will focus on random matchings rather than lotteries. A *(school-choice) mechanism* selects a random matching for a given school-choice problem. For problem $[P, \succsim]$, we denote the random matching of a mechanism $\varphi$ by $\varphi\,[P, \succsim]$ and the random matching vector of a student $i$ by $\varphi_i\,[P, \succsim]$.

# 4 Properties

## 4.1 Previous Notions of Fairness

We first define two previously studied notions that are satisfied by many mechanisms in the literature and real life. Throughout this section, we fix a problem $[P, \succsim]$.

We start with the most standard fairness property in school-choice problems as well as other allocation problems. This weakest notion of fairness is related to the treatment of equal students, i.e., students with the same preferences and priorities. We refer to two students $i$, $j \in I$ as *equal* if $P_i = P_j$ and $i \sim_c j$ for all $c \in C$. A random matching $\rho$ *treats equals equally* if for any equal student pair $i, j \in I$, we have $\rho_i = \rho_j$, that is: two students with exactly the same preferences and equal priorities at all schools should be guaranteed the same enrollment chance at every school at a matching that treats equals equally. The real-life school-choice mechanism used earlier in Boston as well as the new NYC/Boston mechanism treat equals equally.

Before introducing the second probabilistic fairness property, we define a deterministic fairness notion. A (deterministic) matching $\mu$ is *stable* if there is no student pair $i, j$ such that $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$.[13] That is: a matching is stable if there is no student who envies the assignment of a student who has lower priority than he does for that school. Whenever such a student pair exists at a matching, we say that there is *justified envy.* Let $\mathcal{S} \subseteq \mathcal{M}$ be the set of stable matchings. A stable matching always exists (Gale and Shapley, 1962).

The second probabilistic fairness property is a direct extension of stability to lottery mechanisms: A random matching $\rho$ is *ex-post stable* if it is induced by a lottery whose support includes only stable matchings, i.e., there exists some $\lambda \in \triangle\mathcal{M}$ such that $Supp(\lambda) \subseteq \mathcal{S}$ and $\rho = \rho^{\lambda}$.

Since recently introduced real-life mechanisms are ex-post stable (and the implemented matchings are stable), ex-post stability has been seen as a key property in previous literature. A characterization of ex-post stability exists for strict priorities (Roth, Rothblum, and Vande Vate, 1993), yet such a characterization is unknown for weak priorities.

## 4.2 A New Notion: Ex-ante Stability

We now formalize the two fairness notions over random matchings that were informally discussed in the Introduction.

[13] The early literature on college admissions and school choice (e.g. Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003)) used the term *fair* instead of *stable*. Subsequent studies have used the term *stable* more often based on the connection of their models with the two-sided model of Gale and Shapley (1962). Since we already have several fairness concepts, we have adopted this terminology to avoid confusion.

We say that a random matching $\rho \in \mathcal{X}$ causes *ex-ante justified envy* of $i \in I$ *toward (lower-priority student)* $j \in I \setminus \{i\}$, with $i \succ_c j$, for $c \in C$ if $\rho_{i,a} > 0$ for some $a \prec_i c$ and $\rho_{j,c} > 0$. A random matching is *ex-ante stable* if it does not cause any ex-ante justified envy.

Observe that an ex-ante stability and stability are equivalent concepts for deterministic matchings. Although ex-ante stability is appealing, it does not impose any restrictions when dealing with fairness issues regarding students with equal priorities.

We say that a random matching $\rho \in \mathcal{X}$ induces *ex-ante discrimination (between equal-priority students)* $i, j \in I$, with $i \sim_c j$, *for* $c \in C$, if $\rho_{i,a} > 0$ for some $a \prec_i c$ and $\rho_{i,c} < \rho_{j,c}$. A random matching is *strongly ex-ante stable* if it eliminates both ex-ante justified envy and ex-ante discrimination.

The elimination of ex-ante discrimination implies equal treatment of equals. Thus, a strongly ex-ante stable random matching satisfies equal treatment of equals. Strong ex-ante stability implies ex-ante stability, but the converse is not true. Theorem 2 (below) shows that a strongly ex-ante stable random matching always exists. For deterministic matchings, elimination of ex-ante discrimination between equal-priority students is equivalent to a *no-envy*[14] requirement among students with equal priority and thus may not always be guaranteed.

We compare ex-ante (and strong ex-ante) stability with the earlier notion, ex-post stability. It turns out that ex-post stability is weaker than ex-ante stability (and strong ex-ante stability), while the converse is not true:

**Proposition 1** *If a random matching is ex-ante stable then it is also ex-post stable. Moreover, any lottery that induces an ex-ante random matching has a support that includes only stable matchings.*

**Proof of Proposition 1.** We prove the contrapositive of the second part of the proposition. The first part of the proposition follows from the second part. Let $\rho \in \mathcal{X}$ and $\lambda \in \Delta\mathcal{M}$ be such that $\rho^\lambda = \rho$. Suppose there exists some unstable matching $\mu \in \mathcal{M} \setminus \mathcal{S}$ such that $\lambda_\mu > 0$. Then there exists a pair $(i, c) \in I \times C$ such that $cP_i\mu_i$ and for some $j \in I$ with $\mu_j = c$ we have $i \succ_c j$. Since $\lambda_\mu > 0$, we have $\rho_{j,c} > 0$ while $\rho_{i,\mu_i} > 0$, $i \succ_c j$, and $cP_i\mu_i$, i.e., $\rho$ is not ex-ante stable. ■

On the other hand, Example 1 (stated in the Introduction) has shown that the converse of the first part of this proposition is not true.

### 4.3 Pareto Efficiency

We define and work with two Pareto efficiency concepts defined over ordinal preferences.

[14]Given a deterministic matching $\mu \in \mathcal{M}$, there exists *no-envy* between a pair of students $i, j \in I$ if $\mu_i\ P_i\ \mu_j$ and $\mu_j\ P_j\ \mu_i$.

For student $i \in I$, random matching vector $\pi_i$ *ordinally (Pareto) dominates* random matching vector $\rho_i$, if $\sum\limits_{aR_ic} \pi_{i,a} \geq \sum\limits_{aR_ic} \rho_{i,a}$ for all $c \in C$ and $\sum\limits_{aR_ib} \pi_{i,a} > \sum\limits_{aR_ib} \rho_{i,a}$ for some $b \in C$, i.e., $\pi_i$ first-order stochastically dominates $\rho_i$ with respect to $P_i$. A random matching $\pi \in \mathcal{X}$ *ordinally (Pareto) dominates* $\rho \in \mathcal{X}$, if for all $i \in I$, either $\pi_i$ ordinally dominates $\rho_i$ or $\pi_i = \rho_i$, and there exists at least one student $j \in I$ such that $\pi_j$ ordinally dominates $\rho_j$. We say that a random matching is *ordinally (Pareto) efficient* if there is no random matching that ordinally dominates it.

We refer to ordinally efficient deterministic matchings as *Pareto efficient.* A random matching is *ex-post (Pareto) efficient* if there exists a lottery that induces this random matching and has its support only over Pareto efficient matchings.

Ordinal efficiency implies ex-post efficiency, while the converse is not true for random matchings (Bogomolnaia and Moulin, 2001). It is well known that even with strict school priorities, ex-post stability and ex-post efficiency are not compatible.

**Proposition 2** ***(Roth, 1982)*** *There does not exist any ex-post stable and ex-post efficient mechanism.*

Since we take fairness notions as given, we will focus on constrained ordinal efficiency and constrained ordinal dominance as the proper efficiency concepts for mechanisms that belong to a particular class.

# 5 Strong Ex-ante Stable School Choice

## 5.1 Fractional Deferred-Acceptance Mechanism

ok Strong ex-ante stability is an appealing stability property since (1) it guarantees all the enrollment chances to a higher-priority student at his preferred school before all lower-priority students (i.e., by *elimination of ex-ante justified envy*) thereby also ensuring ex-post stability, and (2) it treats equal-priority students – not only equal students – fairly by giving them equal enrollment chance at competed schools (i.e., by *elimination of ex-ante discrimination*). We now introduce the central mechanism in the theory of strongly ex-ante stable lotteries. This mechanism employs a *fractional deferred-acceptance (FDA) algorithm.*

The FDA algorithm is in the spirit of the classical student-proposing deferred-acceptance algorithm of Gale and Shapley (1962). In this algorithm, we talk about a *fraction of a student* applying to, being tentatively assigned to, or being rejected by a school. In using such language, we have in mind that upon termination of the algorithm, the fraction of a student permanently assigned to some school will be interpreted as the assignment probability of the student to that school. Hence, fractions in fact represent enrollment chances. In the FDA algorithm, a student fraction, by applying to a school, may seek a certain fraction of one seat at that school. As a result, depending on its quota and the priorities of other applicants, the school may *tentatively* assign a certain fraction (possibly less than the fraction the student is seeking) of a seat to the student and reject any remaining fraction of the student. In the algorithm's description below, when we say *fraction $w$ of student $i$ applies to school $c$*, this means that at most a fraction $w$ of a seat at school $c$ can be assigned to student $i$. As an example, suppose fraction $\frac{1}{3}$ of student 1 applies to school $c$ at some step of the algorithm. School $c$ then may, for example, admit $\frac{1}{4}$ of student $i$ and reject the remaining $\frac{1}{12}$ of him. We next give a more precise description.

**The FDA Algorithm:**

**Step 1:** *Each student applies to his favorite school. Each school $c$ considers its applicants. If the total number of applicants is greater than $q_c$, then applicants are tentatively assigned to school $c$ one by one starting from the highest priority applicants such that equal-priority students, if assigned a fraction of a seat at this school at all, are assigned an equal fraction. Unassigned applicants (possibly some being a fraction of a student) are rejected.*

⋮

*In general,*

**Step s:** *Each student who has a rejected fraction from the previous step, applies to his next-favorite school that has not yet rejected any fraction of him. Each school $c$ considers its tentatively assigned applicants together with the new applicants. Applicant fractions are tentatively assigned to school $c$ starting from the highest priority applicants as follows: For all applicants of the highest priority level, increase the tentatively assigned shares from 0 at an equal rate until there is an applicant who has been assigned all of his fraction. In such a case continue with the rest of the applicants of this priority level by increasing the tentatively assigned shares at an equal rate until there is another applicant who has been assigned all of his fraction. When all applicant fractions of this priority level are served, continue with the next priority level in a similar fashion. If at some point during the process, the whole quota of school $c$ has been assigned, then reject all outstanding fractions of all applicants.*

The algorithm terminates when no unassigned fraction of a student remains. At this point, the procedure is concluded by making all tentative random assignments permanent. We next give a detailed example to illustrate the FDA algorithm.

**Example 3** ***How does the FDA algorithm work?*** Consider the following problem with six students $\{1, 2, 3, 4, 5, 6\}$ and four schools $\{a, b, c, d\}$, two, $b$ and $d$, with a quota of one, and the other two, $a$ and $c$, with a quota of two:

| $\succsim_a$ | $\succsim_b$ | $\succsim_c$ | $\succsim_d$ |
|---|---|---|---|
| $\vdots$ | 6 | 4 | 5 |
| $\vdots$ | 1, 3 | 2, 3, 5 | 3, 6 |
| $\vdots$ | 5 | $\vdots$ | $\vdots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

| $P_1$ | $P_2$ | $P_3$ | $P_4$ | $P_5$ | $P_6$ |
|---|---|---|---|---|---|
| $b$ | $c$ | $d$ | $b$ | $c$ | $d$ |
| $a$ | $b$ | $c$ | $c$ | $d$ | $c$ |
| $\vdots$ | $a$ | $b$ | $\vdots$ | $\vdots$ | $b$ |
| $\vdots$ | $d$ | $a$ | $\vdots$ | $\vdots$ | $a$ |

***Step 1:*** Students 1 and 4 apply to school $b$ (with quota one), which tentatively admits student 1 and rejects student 4. Students 2 and 5 apply to school $c$ (with quota two), which does not reject any of their fractions. Students 3 and 6 apply to school $d$ (with quota one), which tentatively admits $\frac{1}{2}$ of 6 and $\frac{1}{2}$ of 3, and rejects the remaining halves.

***Step 2:*** Having been rejected by school $d$, each outstanding half-fraction of students 3 and 6 applies to the next-favorite school, which is school $c$. Having been rejected by school $b$, student 4 applies to his next choice, which is also school $c$. This means school $c$ considers half-fraction of each of 3 and 6 and one whole of 4 together with one whole of 2 and 5. Among the five students, 4 has the highest priority, and hence, is tentatively placed at school $c$. Next in priority are students 2, 3, and 5 with equal priority, and thus $\frac{1}{3}$ of each is tentatively admitted at school $c$, which exhausts its quota of two. As a consequence, $\frac{1}{2}$ of student 6, $\frac{1}{6}$ of 3, and $\frac{2}{3}$ of each of 2 and 5 are rejected by $c$.

***Step 3:*** The next choice of 2 is $a$, and hence the rejected $\frac{2}{3}$ of him applies to $a$, and is tentatively admitted there. The next choice of 3 and 6 is $b$, and hence $\frac{1}{2}$ of 6 and $\frac{1}{6}$ of 3 apply to $b$, which is currently full and holding the whole of student 1. Since 6 has higher priority than both 1 and 3, the entire applying fraction of 6 is tentatively admitted. Since 1 and 3 share equal priority at $b$, we gradually increase assigned shares of both students from 0 at an equal rate. This implies that $\frac{1}{6}$ of 3 and $\frac{1}{3}$ of 1 are to be tentatively admitted and the remaining $\frac{2}{3}$ of 1 is to be rejected. The next choice of student 5 is $d$, and hence $\frac{2}{3}$ of him applies to $d$, which is currently holding $\frac{1}{2}$ of both of 3 and 6. Since 5 has higher priority than 3 and 6 both of whom have equal priority, the whole $\frac{2}{3}$ of 5 is tentatively admitted, whereas $\frac{1}{6}$ of each of 3 and 6 is tentatively admitted, causing the remaining $\frac{1}{3}$ of each student to be rejected.

***Step 4:*** The next choice of student 1 is $a$, hence the rejected $\frac{2}{3}$ of him applies to $a$, and is tentatively admitted there. For students 3 and 6, the best choice that hasn't rejected either is $b$, and hence $\frac{1}{3}$ of each student applies to $b$. School $b$ is currently full and holding $\frac{1}{2}$ of 6, $\frac{1}{6}$ of 3, and $\frac{1}{3}$ of 1. Since 1 and 3 have equal but lower priority than 6 at $b$, the school holds on to all of the $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ fraction of 6, and only $\frac{1}{12}$ of each of 1 and 3 are tentatively admitted by $b$; while the remaining $\frac{1}{4}$ of 1 and $\frac{5}{12}$ of 3 are rejected.

***Step 5:*** The next choice of 1 and 3 after $b$ is $a$, and hence $\frac{1}{4}$ of 1 and $\frac{5}{12}$ of 3 apply to $a$, which is not filled yet and can accommodate all of these fractions: It is currently holding $\frac{11}{12}$ of 1, $\frac{5}{12}$ of 3, and $\frac{2}{3}$ of 2. Since there are no further rejections, the algorithm terminates and returns the following random matching outcome:

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{11}{12}$ | $\frac{1}{12}$ | 0 | 0 |
| 2 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| **3** | $\frac{5}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |
| **4** | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |
| **6** | 0 | $\frac{5}{6}$ | 0 | $\frac{1}{6}$ |

$\diamond$

While the FDA algorithm is intuitive, the computation of its outcome poses a new challenge that did not exist for its deterministic analogue (i.e., the version proposed by Gale and Shapley). It turns out that in the FDA algorithm a student may end up applying to the same school infinitely many times. Thus, we next observe that the FDA algorithm as explained above may not converge in a finite number of steps. We illustrate this with a simple example.

**Example 4** ***The FDA algorithm may not terminate in a finite number of steps:*** Consider the following simple problem with three students and three schools each with a quota of one:

| $\succsim_a$ | $\succsim_b$ | $\succsim_c$ |
|---|---|---|
| 3 | 1, 2 | ⋮ |
| 1, 2 | 3 | ⋮ |
| | | ⋮ |

| $P_1$ | $P_2$ | $P_3$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $c$ |

***Step 1:*** Students 1 and 2 apply to school $a$, and $\frac{1}{2}$ of each is tentatively admitted (while $\frac{1}{2}$ of each is rejected), since they have the same priority. Student 3 applies to $b$ and is tentatively admitted.

***Step 2:*** Rejected $\frac{1}{2}$ of student 2 next applies to school $c$ and is tentatively admitted. Rejected $\frac{1}{2}$ of student 1 applies to school $b$, where he has higher priority than the currently admitted student 3. Now $\frac{1}{2}$ of student 3 is rejected and $\frac{1}{2}$ of 1 is tentatively admitted.

***Step 3:*** Rejected $\frac{1}{2}$ of student 3 applies to $a$, where he has higher priority than both 1 and 2. As a result $\frac{1}{2}$ of 3 is tentatively admitted whereas $\frac{1}{4}$ of each of 1 and 2 is rejected.

***Step 4:*** Rejected $\frac{1}{4}$ of 2 next applies to school $c$ and is tentatively admitted (in addition to the previously admitted $\frac{1}{2}$ of him). Rejected $\frac{1}{4}$ of student 1 applies to school $b$, where he has higher priority than the currently admitted $\frac{1}{2}$ of student 3. Now $\frac{1}{4}$ of student 3 is rejected and $\frac{1}{4}$ of 1 is tentatively admitted.

⋮

As the procedure goes on, rejected fractions of student 3 by school $b$ continue to apply to school $a$ in turn, leading to fractions of student 3 to accumulate at $a$, and at the same time, causing (a smaller fraction of) student 1 to be rejected by school $a$ at each application. This, in turn, leads student 1 to apply to school $c$ and cause (the same fraction of) 3 to be further rejected. Consequently, all fractions of student 3 accumulate at school $a$ and all those of student 1 at school $b$:

***Step*** $\infty$ : The sum of the admitted fractions of student 3 at school $a$ is 1, which is the sum of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ . The sum of the admitted fractions of student 1 at school $b$ is 1. The sum of the admitted fractions of student 2 at school $c$ is 1. $\diamondsuit$

Even though the FDA algorithm may not terminate in finite time, the above example suggests that its outcome can be computed without getting lost in infinite loops by examining the rejection cycles that might form throughout the steps of the algorithm.

To define the finite version of the FDA algorithm, we need to define a few new concepts. We first define a binary relation between students. Let $i, j \in I$ and $c \in C$. Suppose that at a step of the FDA algorithm, some fraction of student $i$ gets rejected by school $c$, while he still holds some fraction there. On the other hand, by that step, school $c$ temporarily holds some fraction of some other student $j$ who is yet to be rejected by $c$. Then, we say that $i$ is *partially rejected by $c$ in favor of* $j$, and denote it by $j \hookrightarrow_c i$. If either some fraction of $j$ gets rejected by $c$ or all fractions of $i$ get rejected by $c$ at a later step in the algorithm, then the above relationship no longer holds. In this case we say that $j \hookrightarrow_c i$ is no longer *current*.

A *rejection cycle* is a list of distinct students and schools $(i_1, c_1, ..., i_m, c_m)$ such that at a step of the algorithm, we have

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

and all partial rejection relations are current.

Observe that at the moment the cycle occurs, student $i_1$ gets partially rejected by school $c_m$ in favor of student $i_m$. We know that school $c_1$ has not rejected student $i_1$ at any fraction; thus, the next available choice for student $i_1$ is $c_1$. Therefore, student $i_1$ applies "again" to school $c_1$. As a result student $i_2$ gets partially rejected again, and the same sequence of partial rejections reoccur. That is, the algorithm cycles. We refer to this cycle as a *current* rejection cycle as long as all partial rejection relations are current, and we say that $i_1$ *induces* this rejection cycle.

Nonetheless, this cycle either converges to a tentative random matching in the limit or, sometimes, in a finite number of steps when some partial rejections turn into full rejections. Thus, once a cycle is detected, it can be solved as a simple linear programming problem.

We make the following observation, which will be crucial in the definition of the formal FDA algorithm:

**Observation 1** *If a rejection cycle*

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

*is current in the FDA algorithm, then for each student $i_\ell$, the best school that has not rejected a fraction of him is school $c_\ell$; that is, whenever a fraction of $i_\ell$ gets rejected, he next makes an offer to school $c_\ell$.*

For any arbitrary order of students, we state the "formal" FDA algorithm:

**Algorithm 1** ***The FDA Algorithm:***

*Step s.* Fix some student $i_1 \in I$ who has an unassigned fraction from the previous step. He applies to the next best school that has not yet rejected any fraction of him. Let $c_1$ be this school. Two cases are possible:

(a) If the student $i_1$ induces a rejection cycle

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} ... \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1,$$

then we resolve it as follows: For $i_{m+1} \equiv i_1$ and $c_0 \equiv c_m$, $c_1$ tentatively accepts the maximum possible fraction of $i_1$ such that each school $c_\ell$ tentatively accepts

- all fractions of applicants tentatively accepted in the previous step except the ones belonging to the lowest-priority level,
- the total rejected fraction of student $i_\ell$ from school $c_{\ell-1}$, and
- an equal fraction (if possible) among the lowest-priority applicants tentatively accepted in the previous step (including student $i_{\ell+1}$)

so that it does not exceed its quota $q_{c_\ell}$.

(b) If $i_1$ does not induce a rejection cycle, school $c_1$ considers its tentatively assigned applicants from the previous step together with the new fraction of $i_1$. It tentatively accepts these fractions starting from the highest priority. In case its quota is filled in this process, it tentatively accepts an equal fraction (if possible) of all applicants belonging to the lowest accepted priority level. It rejects all outstanding fractions.

We continue until no fraction of a student remains unassigned. At this point, we terminate the algorithm by making all tentative random assignments permanent.$\Diamond$

We illustrate the above algorithm through an example in Appendix A. Since we have defined the algorithm in a sequential fashion, it is not clear whether the procedure is independent of the order of the proposing students. Corollary 1 (below) shows that this statement is true, and thus, its outcome is unique.[15] We refer to the mechanism whose outcome is found through the above FDA algorithm as the *FDA mechanism*.

## 5.2 Results Regarding the FDA Mechanism

We next present some desirable properties of the FDA mechanism and its iterative algorithm:

**Proposition 3** *The FDA algorithm converges to a random matching in a finite number of steps in polynomial time.*

The proof of Proposition 3 is given in Appendix B.

**Theorem 2** *An FDA outcome is strongly ex-ante stable.*

**Proof of Theorem 2.** Let $\pi$ be the FDA algorithm's outcome according to some proposal order of students. We first show that this outcome is a well-defined random matching. Suppose not. Then, there exists a student $i$ who is not matched with probability one at $\pi$. Thus, $\pi$ is substochastic and there exists some school $c$ that is undermatched at $\pi$, i.e., $\sum_j \pi_{j,c} \leq q_c$. At some step, student $i$ makes an offer to $c$ and some fraction of him gets rejected by it, as he ends up with some rejected probability at the end of the algorithm. However, school $c$ only rejects a student if its quota is tentatively filled. Moreover, once it is tentatively filled, it never gets undermatched. However, this contradicts the earlier conclusion that its quota was not filled at $\pi$. Thus, $\pi$ is a bistochastic matrix, i.e., it is a random matching. Next we show that $\pi$ is strongly ex-ante stable. Since in the algorithm, (i) a student fraction always applies to the best school that has not yet rejected him, and (ii) when its quota is filled, a school always prefers higher priority students to the lower-priority ones, a student cannot have ex-ante justified envy toward a lower-priority student. If $\pi$ is not strongly ex-ante stable, then it should be the case that there is ex-ante discrimination between equal-priority students, i.e., there are $i \sim_c j$ for some school $c$ such that $cP_ia$, $\pi_{i,a} > 0$, and yet $\pi_{i,c} < \pi_{j,c}$. Consider the first step after which the (tentative) random matching vector of school $c$ does not change. At this step, some students apply to school $c$ and in return some fractions of some students with equal priority $i'$ and $j'$ are tentatively accepted and some are rejected. The only way $\pi_{i',c} < \pi_{j',c}$ is if no fraction of $i'$ ever gets rejected by school $c$. Thus, $\pi_{i',a'} = 0$ for all $a' \prec_{i'} c$. This contradicts the claim that such a student $i$ exists. ■

[15]This result is analogous to the result regarding the deferred-acceptance algorithm of Gale and Shapley, which can also be executed by students making offers sequentially instead of simultaneously (McVitie and Wilson, 1971).

Our next result states that from a welfare perspective the FDA outcome is the most appealing strongly ex-ante stable matching. This finding can also be interpreted as the random analogue of Gale and Shapley's celebrated result on the constrained Pareto optimality of the student-proposing deferred-acceptance outcome (among stable matchings) for the deterministic two-sided matching context.

**Theorem 3** *An FDA outcome ordinally dominates all other strongly ex-ante stable random matchings.*

The proof of Theorem 3 is also given in Appendix B. Theorem 3 implies that the FDA mechanism is well defined, i.e., its outcome is unique and independent of the order of students making applications in the algorithm.

**Corollary 1** *The FDA algorithm's outcome is independent of the order of students making offers; and thus, it is unique.*

# 6 Ex-ante Stable School Choice

The FDA mechanism satisfies ex-ante stability but sacrifices some efficiency at the expense of finding a random matching that treats equal-priority students fairly. Therefore, we next address how we can achieve more efficient outcomes without sacrificing fairness too much, i.e. by giving up equal treatment of equal-priority students, but maintaining ex-ante stability and equal treatment of equals.

By Proposition 2 we know that there is no mechanism that satisfies ordinal efficiency and ex-ante stability. Thus, we define the following constrained efficiency concept: A mechanism $\varphi$ is *constrained ordinally efficient within its class* if there exist no mechanism $\psi$ in the same class as $\varphi$ and no problem $[P, \succsim]$ such that $\psi[P, \succsim]$ ordinally dominates $\varphi[P, \succsim]$.

We now characterize constrained ordinal efficient mechanisms within the class of ex-ante stable mechanisms. First, we restate a useful result due to Bogomolnaia and Moulin (2001) that characterizes ordinal efficiency. Fix a problem $[P, \succsim]$. For any random matching $\pi \in \mathcal{X}$, we say that $i$ *ex-ante envies $j$ for $b$ due to $c$*, if $\pi_{j,b} > 0$, $\pi_{i,c} > 0$, and $bP_ic$.[16] We denote it as

$$(i, c) \gtrdot^{\pi} (j, b).$$

[16] Under this definition, a student will ex-ante envy himself, if he is assigned fractions from two schools. This is different from the improvement relationship defined by Bogomolnaia and Moulin. Unlike them, we do not rule out this possibility and use it for the constrained efficiency characterization within ex-ante stable random matchings.

A *stochastic improvement cycle* $Cyc = (i_1, c_1, ..., i_m, c_m)$ at $\pi$ is a list of distinct student-school pairs $(i_\ell, c_\ell)$ such that

$$(i_1, c_1) \gtrdot^\pi (i_2, c_2) \gtrdot^\pi ... \gtrdot^\pi (i_m, c_m) \gtrdot^\pi (i_1, c_1).$$

(We use modulo $m$ whenever it is unambiguous for subscripts, i.e., $m+1 \equiv 1$.) Let $w \leq \min_{\ell \in \{1,...,m\}} \pi_{i_\ell, c_\ell}$. Cycle $Cyc$ is *satisfied* with fraction $w$ at $\pi$ if for all $\ell \in \{1, ..., m\}$, a fraction $w$ of the school $c_{\ell+1}$ is assigned to student $i_\ell$ additionally and a fraction $w$ of school $c_\ell$ is removed from his random matching, while we do not change any of the other matching probabilities at $\pi$. Formally, we obtain a new random matching $\rho \in \mathcal{X}$ such that for all $i \in I$ and $c \in C$,

$$\rho_{i,c} = \begin{cases} \pi_{i,c} + w & \text{if } i = i_\ell \text{ and } c = c_{\ell+1} \text{ for some } \ell \in \{1, ..., m\}, \\ \pi_{i,c} - w & \text{if } i = i_\ell \text{ and } c = c_\ell \text{ for some } \ell \in \{1, ..., m\}, \\ \pi_{i,c} & \text{otherwise.} \end{cases}$$

The following is a direct extension of Bogomolnaia and Moulin's result to our domain and our definition of the ex-ante envy relationship. Therefore, we skip its proof.

**Proposition 4** ***(Bogomolnaia and Moulin, 2001)*** *A random matching is ordinally efficient if and only if it has no stochastic improvement cycle.*

## 6.1 Ex-ante Stability and Constrained Ordinal Efficiency

Proposition 4 suggests that if a random matching has a stochastic improvement cycle, then one can obtain a new random matching that ordinally dominates the initial one simply by satisfying this stochastic improvement cycle. Observe, however, that satisfying such a cycle may induce ex-ante justified envy at the new random matching. Consequently, given that our goal is to maintain ex-ante stability, to improve the efficiency of an ex-ante stable random matching, we can only work with those stochastic improvement cycles that respect the ex-ante stability constraints. For this purpose we introduce an envy relationship as follows:

We say that $i$ *ex-ante top-priority envies* $j$ *for* $b$ *due to* $c$, and denote it as

$$(i, c) \blacktriangleright^\pi (j, b),$$

if $(i, c) \gtrdot^\pi (j, b)$ and $i \succsim_b k$ for all $(k, a) \in I \times C$ such that $(k, a) \gtrdot^\pi (j, b)$. That is, $i$ envies $j$ for $b$ due to $c$, and $i$ is the highest-priority student that envies $j$ for $b$.[17]

An *ex-ante stable improvement cycle* $(i_1, c_1, ..., i_m, c_m)$ at $\pi$ is a list of distinct student-school pairs $(i_\ell, c_\ell)$ such that

$$(i_1, c_1) \blacktriangleright^\pi (i_2, c_2) \blacktriangleright^\pi ... \blacktriangleright^\pi (i_m, c_m) \blacktriangleright^\pi (i_1, c_1).$$

[17]Like the ex-ante envy relationship, a student will ex-ante top-priority envy himself if he is assigned fractions from two schools and for the better of the two schools, he is among the highest-priority students ex-ante envying.

We state our main result of this subsection below. Although one direction of this result is easy to prove, the other direction's proof needs extra attention to detail. Our result generalizes Proposition 4 (stated for the equal priority domain by Bogomolnaia and Moulin), and a result by Erdil and Ergin (2008) (stated for the deterministic domain) to the probabilistic school-choice framework:

**Proposition 5** *An ex-ante stable random matching $\rho$ is not ordinally dominated by any other ex-ante stable random matching if and only if there is no ex-ante stable improvement cycle at $\rho$.*

The proof of Proposition 5 is given in Appendix B.

## 6.2 Ex-ante Stable Fraction Trading

Motivated by Proposition 5, we shall use the FDA outcome to obtain a constrained ordinally efficient ex-ante stable matching. Our second proposal, roughly, rests on the following intuition: Since the outcome of the FDA mechanism is ex-ante stable, if we start initially from this random matching and iteratively satisfy ex-ante stable improvement cycles, we should eventually arrive at a constrained ordinally efficient ex-ante stable random matching. Though intuitive, this approach need not guarantee equal treatment of equals. Therefore, in what follows we will also need to pay attention to the ex-ante stable improvement cycles that are to be selected.

Our second proposal, the *fractional deferred-acceptance and trading (FDAT)*, starts from the FDA outcome and satisfies *all* ex-ante stable improvement cycles simultaneously so as to preserve equal treatment of equals to obtain a new random matching. It iterates until there are no new ex-ante stable improvement cycles. Before formalizing this procedure, to fix ideas and point out some potential difficulties, we first illustrate our approach with an example:

**Example 5** ***How does the FDAT algorithm work?*** We use the same problem as in Example 3.

***Step 0.*** We have found the FDA outcome in Example 3 as

$\rho^1 =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{11}{12}$ | $\frac{1}{12}$ | 0 | 0 |
| 2 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| **3** | $\frac{5}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |
| **4** | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |
| **6** | 0 | $\frac{5}{6}$ | 0 | $\frac{1}{6}$ |

***Step 1.*** We form top-priority envy relationships as

$$(1,a) \blacktriangleright^{\rho^1} (3,b), (6,b), (1,b)$$

$$(2,a) \blacktriangleright^{\rho^1} (3,c), (4,c), (5,c), (2,c)$$

$$(3,f) \blacktriangleright^{\rho^1} (5,d), (6,d), (3,d) \qquad \forall f \in \{a,b,c\}$$
$$(3,f) \blacktriangleright^{\rho^1} (2,c), (4,c), (5,c), (3,c) \qquad \forall f \in \{a,b\}$$
$$(3,a) \blacktriangleright^{\rho^1} (1,b), (6,b), (3,b)$$

$$(5,d) \blacktriangleright^{\rho^1} (2,c), (3,c), (4,c), (5,c)$$

$$(6,b) \blacktriangleright^{\rho^1} (3,d), (5,d), (6,d).$$

There is only one ex-ante stable improvement cycle:

$$(3,c) \blacktriangleright^{\rho^1} (5,d) \blacktriangleright^{\rho^1} (3,c)$$

We satisfy this cycle with the maximum possible fraction $\frac{1}{3}$ and obtain:

$\rho^2 =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{11}{12}$ | $\frac{1}{12}$ | 0 | 0 |
| 2 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| **3** | $\frac{5}{12}$ | $\frac{1}{12}$ | 0 | $\frac{1}{2}$ |
| **4** | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ |
| **6** | 0 | $\frac{5}{6}$ | 0 | $\frac{1}{6}$ |

***Step 3.*** There are no new top-priority envy relationships at $\rho^2$, and no new ex-ante stable improvement cycles; thus, $\rho^2$ is the outcome of the FDAT algorithm. $\diamondsuit$

The main difficulty with this approach is determining which ex-ante stable improvement cycle to satisfy, if there are many. This choice may cause fairness violations regarding the equal treatment of equals, or there can be many ways to find a solution respecting equal treatment of equals. Thus, the outcome of the FDAT algorithm as explained above is not uniquely determined. Furthermore, there are also legitimate computational concerns in finding more than one ex-ante stable improvement cycle at a time.[18] We overcome these fairness and computational issues by adapting to our domain a fractional trading algorithm, which was introduced in the operations research literature by Athanassoglou and Sethuraman (2007). It is referred to as the *constrained-consumption algorithm* and was introduced to obtain ordinally efficient allocations in house allocation problems with existing tenants (Abdulkadiroğlu and Sönmez, 1999). Similar algorithms were also previously introduced by Yilmaz (2006, 2009). Our version, the *ex-ante stable consumption (EASC) algorithm*, is embedded in step $s \geq 1$ of the FDAT algorithm as a way to satisfy ex-ante stable improvement cycles simultaneously and equitably. It is explained in detail in Appendix D.

We state the FDAT algorithm formally as follows:

**Algorithm 2** ***The FDAT Algorithm:***

*Step 0.* Run the FDA algorithm. Let $\rho^1$ be its random matching outcome.

⋮

*Step s.* Let $\rho^{\mathrm{s}} \in \mathcal{X}$ be found at the end of step s-1. If there is an ex-ante stable improvement cycle, run the EASC algorithm. Let $\rho^{\mathrm{s+1}}$ be the outcome and continue with Step s+1. Otherwise, terminate the algorithm with $\rho^{\mathrm{s}}$ as its outcome. $\Diamond$

We refer to the mechanism whose outcome is found through this algorithm as the *FDAT mechanism.*

In Appendix E-Example 8, we illustrate the EASC algorithm to show how the formal FDAT algorithm works for the problem in Example 5. Although the execution of the FDAT algorithm is obvious and simple in this example without the implementation of the EASC algorithm in each step, for expositional purposes we reexecute it with the embedded EASC algorithm.[19]

## 6.3 Results Regarding the FDAT Mechanism

**Proposition 6** *The FDAT algorithm converges to a random matching in a finite number of steps in polynomial time.*

**Proof of Proposition 6.** We know that Step 0 of FDA algorithm works in polynomial time (by Proposition 3).

[18] In a worst-case scenario, the number of ex-ante stable improvement cycles at an ex-ante stable matching grows exponentially with the number of students.

[19] In general, without the use of the EASC algorithm or a similar well-defined technique, step $s \geq 1$ of the FDAT algorithm may not be well defined.

Consider each following step of the FDAT algorithm: We can determine all ex-ante top-priority envy relationships in no more than $|I|\,|C|$ comparisons for each student-school pair. Since there are $|I|\,|C|$ student school pairs, the total number of such comparisons is no more than $|I|^2\,|C|^2$. Existence or non-existence of a cycle in such a graph can be verified by depth-first search, which is known to be polynomial time as well. The EASC algorithm is also executable in polynomial time (Athanassoglou and Sethuraman, 2007).

Next, we prove that the number of steps in FDAT is polynomial in the number of students and schools. After each step $\mathrm{t} \geq 1$ of the FDAT algorithm, at least one student $i \in I$ leaves a school $c \in C$ with $\rho^{\text{t-1}}_{i,c} > 0$ with 0 fraction and gets into better schools, i.e., $\rho^{\mathrm{t}}_{i,c} = 0$ and $\sum_{aP_ic} \rho^{\mathrm{t}}_{i,a} > \sum_{aP_ic} \rho^{\text{t-1}}_{i,a}$. (Otherwise, the same ex-ante stable improvement cycle of $\rho^{\text{t-1}}_{i,c}$ would still exist at $\rho^{\mathrm{t}}$, contradicting that the EASC algorithm has converged at step t.) Thus, the FDAT algorithm converges in no more than $|C|\,|I| + 1$ steps (including Step 0). Since each step works in polynomial time, the FDAT algorithm can be implemented in polynomial time and converges in a finite number of steps. ■

**Theorem 4** *The FDAT mechanism is ex-ante stable.*

**Proof of Theorem 4.** Consider each step of the FDAT algorithm.

In Step 0, the outcome of the FDA has no justified envy toward a lower-priority student by Theorem 2.

In Step 1, students in determined ex-ante stable improvement cycles are made better off (in an ordinal dominance sense), while others' welfare is unchanged. Moreover, the students who are made better off are among the highest-priority students who desire a seat at the school where they receive a larger share. That is, for any student $i$ with $\rho^1_{i,c} > \rho^0_{i,c}$, there is some school $b$ with $cP_ib$, and $\rho^1_{i,b} < \rho^0_{i,b}$, and there is no student $j \succ_c i$ such that $\rho^1_{j,a} > 0$ for some school $a$ with $cP_ja$. (Otherwise, $i$ would not ex-ante top-priority envy a student $k$ with $\rho^0_{k,c} > 0$ for $c$ due to $b$, since $j$ would do that due to $a$ or a worse school. Moreover, since $\rho^0$ is ex-ante stable, $\rho^0_{i,c} = 0$. The last two statements would imply $(i, c) \notin \mathcal{A}(\rho^0)$, which in turn implies that $\rho^1_{i,c} = 0$.) Hence, $\rho^1$ is ex-ante stable.

We repeat this argument for each step. Hence, when the algorithm is terminated, the outcome is ex-ante stable. ■

Our next result states that from a welfare perspective the FDAT outcome is among the most appealing ex-ante stable random matchings. Improving upon this matching would necessarily lead to ex-ante justified envy. This finding can also be interpreted as the random analogue of the mechanism proposed by Erdil and Ergin (2008) for a deterministic school-choice model with random tie-breaking. In that context, the outcome of the Erdil-Ergin mechanism has been shown to be constrained ex-post Pareto efficient among ex-post stable matchings.

**Theorem 5** *The FDAT mechanism is constrained ordinally efficient within the ex-ante stable class.*

**Proof of Theorem 5.** Suppose that the FDAT outcome $\rho$ is ordinally dominated by an ex-ante stable random matching for some problem $P$. By Proposition 5, there exists an ex-ante stable improvement cycle at $P$. Thus, this contradicts the fact that $\rho$ is the FDAT outcome. ■

**Theorem 6** *The FDAT mechanism treats equals equally.*

**Proof of Theorem 6.** The FDA mechanism treats equals equally as it is strongly ex-ante stable (by Theorem 2). Thus, two students with the same preferences and priorities have exactly the same random matching vector under the FDA outcome $\rho^0$. Let $i, j$ be two equal students. Then $\rho_i^0 = \rho_j^0$ and $(i, c) \in \mathcal{A}(\rho^0)$ if and only if $(j, c) \in \mathcal{A}(\rho^0)$ for any school $c \in C$. By Athanassoglou and Sethuraman (2007), the EASC algorithm treats equals equally. The last two statements imply that outcome of Step 1, $\rho^1$ treats equals equally. We repeat this argument iteratively for each step, showing that the FDAT outcome treats equals equally. ■

## 6.4 The FDAT Mechanism vs. Probabilistic Serial Mechanism

The way the FDA and FDAT mechanisms treat equal-priority students resembles the probabilistic serial (PS) mechanism of Bogomolnaia and Moulin (2001) proposed for the "random assignment" problem where there are no exogenous student priorities. Loosely speaking, within any given step of the PS algorithm, those students who compete for the available units of the same object are allowed to consume equal fractions until the object is exhausted. Similarly, within any given step of the FDA algorithm those equal-priority students who have applied to the same school are also treated equally in very much the same way. Despite such similarity the two procedures are indeed quite different in general. The difference of the two algorithms comes from the fact that the PS algorithm makes permanent random matchings within each step, whereas the FDA algorithm always makes tentative random matchings till the last step. We can expect to have some efficiency loss due to FDA's strong ex-ante stability property, while the PS mechanism is not strongly ex-ante stable. Even if FDA and PS outcomes are different, one may think that starting from the FDA outcome, fractional trading will somehow establish the equivalence with the PS outcome. However, as the following example shows, the PS outcome does not necessarily ordinally dominate the FDA outcome, and hence, the FDAT outcome, which ordinally dominates the FDA outcome, and the PS outcome are not the same either:

**Example 6** ***Neither FDA nor FDAT is equivalent to the PS mechanism when all students have the same priority:*** Assume there are four students $\{1, 2, 3, 4\}$ and four schools

$\{a, b, c, d\}$ each with a quota of one. All students have equal priorities at all schools. The students' preferences are given as

| $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|---|---|---|---|
| $d$ | $a$ | $d$ | $c$ |
| $c$ | $d$ | $c$ | $b$ |
| $a$ | $c$ | $b$ | $d$ |
| $b$ | $b$ | $a$ | $a$ |

The FDA outcome is:

$\rho^{FDA} =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{1}{3}$ | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| **2** | $\frac{2}{3}$ | 0 | 0 | $\frac{1}{3}$ |
| **3** | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| **4** | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 |

The FDAT outcome is:

$\rho^{FDAT} =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| **2** | 1 | 0 | 0 | 0 |
| **3** | 0 | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |
| **4** | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | 0 |

On the other hand, the PS outcome is:

$\rho^{PS} =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |
| **2** | $\frac{5}{6}$ | $\frac{1}{6}$ | 0 | 0 |
| **3** | 0 | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{2}$ |
| **4** | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 0 |

Observe that $\rho^{FDAT}$ and $\rho^{PS}$ are both ordinally efficient. Moreover, $\rho^{PS}$ does not ordinally dominate $\rho^{FDA}$ (e.g., contrast student 2's random matching vectors under $\rho_2^{FDA}$ vs. $\rho_2^{PS}$). $\Diamond$

# 7 Concluding Comments

In this paper, we have established a framework that generalizes one-to-many two-sided and one-sided matching problems. Such a framework enables the mechanism designer to achieve strong and appealing ex-ante efficiency properties when students are endowed with ordinal preferences as exemplified in the pioneering work of Bogomolnaia and Moulin (2001). On the other hand, fairness considerations play a crucial role in the design of practical school-choice mechanisms, since school districts are vulnerable to possible legal action resulting from a violation of student priorities. We have formulated two natural and intuitive ex-ante fairness notions called strong ex-ante stability and ex-ante stability and have showed that they are violated by prominent school-choice mechanisms such as the current NYC/Boston mechanism. We have proposed two mechanisms that stand out as attractive members of their corresponding classes.

The research on school-choice lotteries is a relatively new area in market design theory and there are many remaining open questions. One important question is about the characterization of ex-post stability when matchings are allowed to be random. Similarly to the results we have established for strong ex-ante stability (Theorem 3) and ex-ante stability (Theorem 5), a characterization of constrained ordinal efficient and ex-post stable random matchings currently remains an important future issue.

Strategic issues regarding lottery matching mechanisms, in general, have not been well understood. In the context of one-sided matching (i.e., the special case of our model where all students have equal priority at all schools) strategy-proofness is essentially incompatible with ordinal efficiency. Therefore, notwithstanding its appeal in terms of various properties including ordinal efficiency, the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) that has triggered a rapidly growing literature on the random assignment problem is not strategy-proof. In the context of school choice, due to the well-known three-way tension among stability, efficiency, and incentives, strategy-proof and stable mechanisms are necessarily inefficient (see e.g. Erdil and Ergin (2008), Abdulkadiroğlu, Pathak, and Roth (2009), and Kesten (2009a)). A mechanism is *strategy-proof* if, for each agent, his random matching vector obtained through the mechanism via his truth-telling ordinally dominates or is equal to the one obtained via his revelation of any untruthful ranking. The current NYC/Boston mechanism, which is strategy-proof, is the most efficient stable mechanism (Gale and Shapley, 1962) when priorities are strict. However, in the school-choice problem with weak priorities, it is not even ex-post efficient within the ex-post stable class of mechanisms. Moreover, it has been shown empirically (Abdulkadiroğlu, Pathak, and Roth, 2009) and theoretically (Kesten, 2009a) to be subject to significant and large welfare losses. As a result of this observation non-strategy-proof mechanisms have been highly advocated and proposed in the recent literature on school choice (see e.g. Erdil and Ergin (2008), Kesten (2009a), and Abdulkadiroğlu, Che, and Yasuda (2008)).

Given the negative results outlined above regarding different fairness and efficiency properties it is probably not surprising that the two mechanisms proposed in this paper are not strategy-proof. This observation follows from the following two impossibility results regarding the existence of strategy-proof mechanisms in our problem domain. We state these observations in the next two remarks. The first remark is a reformulation of a result due to Bogomolnaia and Moulin (2001) for the present context.

**Remark 1** *When $|I| \geq 4$, there is no strategy-proof, ex-ante stable, and constrained ordinally efficient mechanism that also respects equal treatment of equals.*

The next remark shows the incompatibility between strategy-proofness and strong ex-ante stability. Its proof is given Appendix B.

**Remark 2** *When $|I| \geq 3$, there is no strategy-proof and strongly ex-ante stable mechanism.*

# A Appendix: How does the FDA algorithm work when there is a rejection cycle?

**Example 7** ***How does the finite FDA algorithm work?*** Assume there are four students $\{1, 2, 3, 4\}$ and four schools, $\{a, b, c, d\}$ each with a quota of one. The priorities and preferences are given as follows:

| $\succ_a$ | $\succ_b$ | $\succ_c$ | $\succ_d$ |
|---|---|---|---|
| 4 | 1, 2 | 2 | $\vdots$ |
| 2 | 3, 4 | 1, 3 | $\vdots$ |
| $\vdots$ | | 4 | $\vdots$ |
| $\vdots$ | | | $\vdots$ |

| $P_1$ | $P_2$ | $P_3$ | $P_4$ |
|---|---|---|---|
| $c$ | $a$ | $b$ | $b$ |
| $b$ | $c$ | $c$ | $a$ |
| $d$ | $b$ | $d$ | $\vdots$ |
| $a$ | $d$ | $a$ | $\vdots$ |

Students propose according to the order 1, 2, 3, 4 :

***Step 1:*** Student 1 applies to school $c$ and is tentatively admitted.

***Step 2:*** Student 2 applies to school $a$ and is tentatively admitted.

***Step 3:*** Student 3 applies to school $b$ and is tentatively admitted.

***Step 4:*** Student 4 applies to school $b$. The applicants of school $b$ are students 3 and 4 (who have equal priority). Since applications exceed the quota, $\frac{1}{2}$ of each of 3 and 4 are rejected by $b$, while $\frac{1}{2}$ of each of 3 and 4 are tentatively admitted.

***Step 5:*** Student 3 has an outstanding fraction of $\frac{1}{2}$ and applies to his next best school, $c$. The applicants of $c$ are student 1 with whole fraction and student 3 with fraction $\frac{1}{2}$. Each has equal priority at school $c$ whose quota has been exceeded. Thus $\frac{1}{2}$ of each of students 1 and 3 are tentatively admitted at school $c$, while $\frac{1}{2}$ of student 1 is rejected. Since 1 is partially rejected in favor of 3 by $c$, we have $3 \hookrightarrow_c 1$.

***Step 6:*** Student 1 has an outstanding fraction of $\frac{1}{2}$, and applies to his next best school, $b$. School $b$ has three applicants, $\frac{1}{2}$ of 3, $\frac{1}{2}$ of 4, and $\frac{1}{2}$ of 1. Since the quota of the school is exceeded and student 1 has the highest priority among the three applicants, $\frac{1}{2}$ of 1 is tentatively admitted, while $\frac{1}{4}$

of each of 3 and 4 are tentatively admitted, and $\frac{1}{4}$ of each of 3 and 4 are rejected. We have $1 \hookrightarrow_b 4$, and $1 \hookrightarrow_b 3$, hence there is a rejection cycle $(3, c, 1, a)$. The resolution of this cycle is trivial, since once 3 applies to school $c$ again with his outstanding fraction $\frac{1}{4}$, all of this is rejected by $c$ since both 1 and 3 have equal priority at $c$ and they already have $\frac{1}{2}$ fraction each at $c$. Thus, it is no longer true that $3 \hookrightarrow_c 1$, and the cycle is resolved.

***Step 7:*** Student 3 has an outstanding fraction of $\frac{1}{4}$, and applies to his next best school, $d$. This is tentatively accepted by $d$.

***Step 8***: Student 4 has an outstanding fraction of $\frac{3}{4}$, and applies to his next best school, $a$. School $a$ has two applicants, whole fraction of 2 and $\frac{3}{4}$ of 4. Since the quota of $a$ is only one, and student 4 has higher priority than 2 at $a$, then $\frac{3}{4}$ of 4 and $\frac{1}{4}$ of 2 are tentatively admitted to $a$, while $\frac{3}{4}$ of 2 is rejected. We have $4 \hookrightarrow_a 2$.

***Step 9:*** Student 2 has an outstanding fraction of $\frac{3}{4}$ and applies to his next best school, $c$. School $c$ has three applicants, 1 with fraction $\frac{1}{2}$, 3 with fraction $\frac{1}{2}$, and 2 with fraction $\frac{3}{4}$. Since the quota of $c$, which is one, has been exceeded, and 2 has higher priority than each of 1 and 3 who have equal priority, $\frac{3}{4}$ of 2, $\frac{1}{8}$ of each 1 and 3 are tentatively admitted to $c$, while $\frac{3}{8}$ of each of 1 and 3 are rejected. We have $2 \hookrightarrow_c 1$ and $2 \hookrightarrow_c 3$. The former relation induces a new cycle $(1, b, 4, a, 2, c)$. This cycle is not trivial. We use a simple linear program to resolve this cycle:

First, consider the following constraints with unknowns $y_1, x_1, y_2, y_4$ :

$$\begin{aligned} \phi_{1,b} + x_1 + \min\{\phi_{4,b} - y_4, \phi_{3,b}\} + \phi_{4,b} - y_4 &= q_b \\ \phi_{4,a} + y_4 + \phi_{2,a} - y_2 &= q_a \\ \phi_{2,c} + y_2 + \min\{\phi_{1,c} - y_1, \phi_{3,c}\} + \phi_{1,c} - y_1 &= q_c \end{aligned} \tag{1}$$

$$\begin{aligned} y_1 \leq x_1 &\leq \phi_{1,c} + w_1 \\ 0 \leq y_4 &\leq \phi_{4,b} \\ 0 \leq y_2 &\leq \phi_{2,a} \\ 0 \leq y_1 &\leq \phi_{1,c} \end{aligned}$$

where $\phi_{1,b} = \frac{1}{2}$, $\phi_{4,b} = \phi_{3,b} = \frac{1}{4}$ are the fractions of 1, 4, and 3 currently tentatively admitted to $b$, $\phi_{4,a} = \frac{3}{4}$ and $\phi_{2,a} = \frac{1}{4}$ are the fractions of 4 and 2 currently tentatively admitted to $a$, and $\phi_{2,c} = \frac{3}{4}$, $\phi_{3,c} = \phi_{1,c} = \frac{1}{8}$ are the fractions of 2, 3, and 1 currently tentatively admitted to $c$. Fraction $w_1 = \frac{3}{8}$ is the fraction of student 1 that will apply to school $b$ next (if we had continued in the cycle step by step). Recall that $q_b = q_a = q_c = 1$ are the quotas of the three schools. The unknowns, $y_4$, $y_2$, $y_1$, are the limit fractions of students 4, 2, and 1, respectively, to be rejected by schools $b$, $c$, and $a$, when the cycle is resolved. These fractions cannot exceed their endowments at respective schools $b$, $c$, and $a$. Moreover, these fractions of 4 and 2 will be tentatively accepted at schools $a$ and $c$, respectively, in addition to $\phi_{4,a} = \frac{1}{4}$ and $\phi_{2,c} = \frac{1}{4}$. On the other hand, at least $y_1$ units of student 1's fraction will be tentatively admitted to school $b$. Since 1 has an additional outstanding fraction $w_1 = \frac{3}{8}$ units, his tentatively admitted fraction cannot be less than $y_1$. Thus, we represent this fraction by $x_1$, with lower bound $y_1$, and with upper bound $w_1 + \phi_{1,c} = \frac{1}{2}$, which is the total endowment of student 1 that can be distributed in this cycle. Clearly, there are feasible values of these unknowns, such as $x_1 = y_1 = y_2 = y_4 = 0$. These values represent the current situation before the cycle resolution. Cycle will be resolved when $x_1$ is maximized subject to the constraints given in Equation system (1). Hence, when we solve this linear program to resolve the cycle, we obtain:

$$x_1 = 1, \; y_4 = \frac{1}{4}, \; y_2 = \frac{1}{4}, \; y_1 = \frac{1}{8}.$$

At this point, the tentative random matchings of students are: whole fraction of 1 at $b$, whole fraction of 4 at $a$, whole fraction of 2 at $c$. We also have from the previous step: $\frac{1}{4}$ of 3 at $d$.

***Step 10:*** Student 3 has an outstanding fraction of $\frac{3}{4}$, with which he applies to his best school that has not rejected him yet, $d$. Now, the whole fraction of 3 is applying to $d$, which tentatively admits him.

There are no outstanding student fractions left. The algorithm terminates with the outcome

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | 0 | 1 | 0 | 0 |
| **2** | 0 | 0 | 1 | 0 |
| **3** | 0 | 0 | 0 | 1 |
| **4** | 1 | 0 | 0 | 0 |

.

$\Diamond$

# B Appendix: Proofs of the Results Regarding the FDA Mechanism

**Proof of Proposition 3.** First, we prove that a rejection cycle can be resolved in polynomial time. Suppose a rejection cycle occurs at a step when $i_1$ applies to $c_1$ with fraction $w_1$ as

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} \dots \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1,$$

At the step when the cycle occurs, for each $\ell$, let $\phi_{i,c_\ell}$ be the fraction of student $i$ previously tentatively assigned to school $c_\ell$. For each school $c_\ell$ with $\ell > 1$ we have the following constraint to respect the cycle resolution condition:

$$\sum_{i \succ_{c_\ell} i_{\ell+1}} \phi_{i,c_\ell} + y_\ell + \sum_{i \sim_{c_\ell} i_{\ell+1}} \min\left\{\phi_{i_{\ell+1},c_\ell} - y_{\ell+1}, \phi_{i,c_\ell}\right\} = q_{c_\ell} \tag{2}$$

where

$$\phi_{i_\ell,c_{\ell-1}} \geq y_\ell \geq 0 \tag{3}$$

$$\phi_{i_1,c_m} \geq y_1 \geq 0 \tag{4}$$

and $1 \equiv m+1$. Each $y_\ell$ is unknown and is the fraction of student $i_\ell$ that will be rejected by school $c_{\ell-1}$ and tentatively accepted at school $c_\ell$ (in addition to his previous fraction $\phi_{i_\ell,c_\ell}$, if it exists) when the cycle is resolved. On the other hand, school $c_\ell$ still holds on to the higher-priority students' fractions $\sum_{i \succ_{c_\ell} i_{\ell+1}} \phi_{i,c_\ell}$ when the cycle is resolved. School $c_\ell$ also makes sure that each student at the same priority level as $i_{\ell+1}$ is held at the same fraction $\phi_{i_{\ell+1},c_\ell} - y_{\ell+1}$, unless he did not have that much fraction to start with. The sum of all tentatively accepted fractions will be $q_{c_\ell}$.

For school $c_1$, the constraint is different, because student $i_1$ is applying now with a fraction $w_1$ and will apply in the future with his rejected fractions from $c_m$, which is $y_1$. Thus,

$$\sum_{i \succ_{c_1} i_2} \phi_{i,c_1} + x_1 + \sum_{i \sim_{c_1} i_2} \min \{\phi_{i_2,c_1} - y_2, \phi_{i,c_1}\} = q_{c_1} \tag{5}$$

where

$$w_1 + y_1 \geq x_1 \geq y_1. \tag{6}$$

Here, $x_1$ is the fraction in $w_1 + y_1$ (the total fraction of $i_1$ rejected by $c_m$) that will be tentatively accepted at $c_1$, when the cycle is resolved.

To resolve the cycle, we maximize $x_1$ subject to the constraints in Equations 2 - 6. At least a feasible solution exists for these constraints, that is, $x_1 = 0$ with $y_\ell = 0$ for all $\ell$. Thus, this is a linear programming problem in a compact set, and it has a well-defined solution with at least one of the inequality constraints holding at the upper-bound. Hence, either for at least one $\ell$, $y_\ell = \phi_{i_\ell,c_{\ell-1}}$, i.e., the student $i_\ell$ gets fully rejected by school $c_{\ell-1}$, or $w_1+y_1 = x_1$, i.e., no outstanding fraction of student $i_1$ remains. If the former case is true, then the cycle is resolved for good by the rejection of the whole of one student from a school in the cycle. If the latter is true, then the cycle is temporarily resolved with no outstanding fractions belonging to the students in the cycle, i.e., with no full rejections. Moreover, it is well known that this linear programming problem can be solved in polynomial time (Khachian, 1979).

Next, if a cycle does not occur, then similarly the step of the algorithm can be resolved in polynomial time.

Thus, all we need to show is that the number of steps in the FDA algorithm is polynomial in $|I|$ and $|C|$. Observe that in each step of the FDA algorithm, students get weakly worse off, since they only make proposals to a school that has not rejected a fraction of themselves. After all $|I|$ students make offers, at least one student gets rejected by one school and has an outstanding fraction, or the algorithm converges, whether or not a cycle occurs. Since there are $|C|$ schools, the algorithm converges in at most $|I|\,|C|$ steps, and since each step is polynomial, the FDA algorithm mechanism works in polynomial finite time. ■

**Proof of Theorem 3.** We will use the method of contradiction in our proof. Suppose this is not true for some school-choice problem. Fix a problem $[P, \succsim]$. Let $\pi \in \mathcal{X}$ be the FDA algorithm's outcome random matching for some order of students making offers, and $\rho \in \mathcal{X}$ be a strongly ex-ante stable random assignment that is not stochastically dominated by $\pi$. This means that

$$\begin{array}{c}\text{there exist } i_0 \in I \text{ and } a_0 \in C \text{ such that } 0 \neq \rho_{i_0,a_0} > \pi_{i_0,a_0} \\ \text{where } a_0 P_{i_0} e_0 \text{ for some } e_0 \in C \text{ with } 0 \neq \pi_{i_0,e_0} > \rho_{i_0,e_0}.\end{array} \tag{7}$$

We will construct a finite sequence of student-school pairs as follows:

*Construction of a trading cycle from $\pi$ to $\rho$:* Statement 7 implies that there exists $i_1 \in I \backslash \{i_0\}$ such that $\rho_{i_1,a_0} < \pi_{i_1,a_0} \neq 0$. Then strong ex-ante stability of the FDA outcomes implies that $i_1 \succsim_{a_0} i_0$, for otherwise $\pi$ would have induced ex-ante justifiable envy of $i_1$ toward $i_0$ for $a_0$ (in case $i_1 \succ_{a_0} i_0$) or $\pi$ would have ex-ante discriminated $i_0$ and $i_1$ at $a_0$ (in case $i_1 \sim_{a_0} i_0$). Then, since $\rho_{i_1,a_0} < \pi_{i_1,a_0}$, $\rho_{i_0,a_0} > \pi_{i_0,a_0}$, and $\rho$ is strongly ex-ante stable, in order for $\rho$ not to have ex-ante justified envy of $i_1$ toward $i_0$ for $a_0$ (in case $i_1 \succ_{a_0} i_0$) and $\rho$ not to have ex-ante discrimination between $i_0$ and $i_1$ for $a_0$ (in case $i_1 \sim_{a_0} i_0$), there must exist some $a_1 \in C \backslash \{a_0\}$ such that $0 \neq \rho_{i_1,a_1} > \pi_{i_1,a_1}$ where $a_1 P_{i_1} a_0$. (More precisely, there are two cases:

Case (1) $i_1 \succ_{a_0} i_0$: Suppose by contradiction that for all $b \in C$ with $b P_{i_1} a_0$, we have $\rho_{i_1,b} \leq \pi_{i_1,b}$. Then by feasibility there is $c \in C$ with $a_0 P_{i_1} c$ and $\rho_{i_1,c} > \pi_{i_1,c}$. But then $i_1$ would ex-ante justifiably envy $i_0$ for $a_1$ at $\rho$, contradicting $\rho$ is strongly ex-ante stable.

Case (2) $i_1 \sim_{a_0} i_0$ : Since $\pi_{i_0,e_0} \neq 0$ and $a_0 P_{i_0} e_0$ (by Statement 7 above), we have $0 \neq \pi_{i_1,a_0} \leq \pi_{i_0,a_0} \neq 0$. Thus, $\rho_{i_1,a_0} < \pi_{i_1,a_0}$ and $\rho_{i_0,a_0} > \pi_{i_0,a_0}$ imply that $\rho_{i_1,a_0} < \rho_{i_0,a_0}$. Then, no ex-ante discrimination at $\rho$ between $i_0$ and $i_1$ for $a_0$ implies that there is no $d \in C$ where $a_0 P_{i_1} d$ with $\rho_{i_1,d} \neq 0$. Then such an $a_1$ should exist for $i_1$.)

Observe that $i_1$ satisfies the same Statement 7 above as $i_0$ does using $a_1$ instead of $a_0$, and $a_0$ instead of $e_0$, and $i_1 \succsim_{a_0} i_0$, i.e.,

$$\rho_{i_1,a_1} > \pi_{i_1,a_1}\ ,\ a_1 P_{i_1} a_0 \text{ with } \rho_{i_1,a_0} < \pi_{i_1,a_0} \neq 0 \text{ and } i_1 \succsim_{a_0} i_0.$$

Thus, as we continue iteratively we obtain a finite sequence of students and schools such that each pair $(a_{\ell-1}, i_\ell)$ (subscripts are modulo $n+1$, so that $n+1 \equiv 0$) appears only once in the sequence,

$$e_0, i_0, a_0, i_1, a_1, ..., i_n, a_n$$

and each $i_\ell$ satisfies Condition 7 replacing $i_\ell$ with $i_0$, $a_\ell$ with $a_1$ and $a_{\ell-1}$ with $e_0$, and additionally satisfying $i_\ell \succsim_{a_{\ell-1}} i_{\ell-1}$, i.e.,

$$\rho_{i_\ell,a_\ell} > \pi_{i_\ell,a_\ell}\ ,\ a_\ell P_{i_\ell} a_{\ell-1} \text{ with } \rho_{i_\ell,a_{\ell-1}} < \pi_{i_\ell,a_{\ell-1}} \neq 0 \text{ and } i_\ell \succsim_{a_{\ell-1}} i_{\ell-1} \tag{8}$$

and finally, by finiteness of schools and students, we have

$$a_n \equiv e_0 \text{ and yet } i_n \neq i_0,$$

where $e_0$ can be chosen as defined in Condition 7. This sequence describes a special probability trading cycle from $\pi$ to $\rho$ for some better schools, so that $\rho$ cannot be ordinally dominated by $\pi$. $\Diamond$

Observe that there can be many such cycles, some of them overlapping. And each such cycle has at least two agents and two schools. Suppose there are $m^*$ such cycles $Cyc^1, ...Cyc^m, ..., Cyc^{m^*}$and let $I^1, I^2, ..., I^m, ..., I^{m^*}$ be the sets of students and $C^1, C^2, ..., C^m, ..., C^{m^*}$ be the corresponding sets of schools in these cycles, respectively. Let $I^*$ be the union of all above student sets and $C^*$ be the union of all above school sets. We will prove some claims that will facilitate the proof of the theorem:

*Claim 1: Take a cycle $Cyc^m = (i_0, a_0, ..., i_n, a_n)$. There is no $a_\ell \in C^m$ and no $b \in C$ such that for student $i_{\ell+1}$, we have $a_\ell P_{i_{\ell+1}} b$ and $\rho_{i_{\ell+1},b} \neq 0$.*

*Proof of Claim 1:* Suppose, on the contrary, there are $a_\ell \in C^m$ and $b \in C$ such that $a_\ell P_{i_{\ell+1}} b$ and $\rho_{i_{\ell+1},b} \neq 0$. We also have, $\rho_{i_{\ell+1},a_\ell} < \pi_{i_{\ell+1},a_\ell} \neq 0$ by construction of cycle $Cyc^m$ (see Statement 8 above). We also have by construction, $0 \neq \rho_{i_\ell,a_\ell} > \pi_{i_\ell,a_\ell}$, $a_\ell P_{i_\ell} a_{\ell-1}$, $\rho_{i_\ell,a_{\ell-1}} < \pi_{i_\ell,a_{\ell-1}} \neq 0$, and, finally $i_{\ell+1} \succsim_{a_\ell} i_\ell$ (see Statement 8 above). Consider two cases:

Case (1') $i_{\ell+1} \succ_{a_\ell} i_\ell$ : Since $\rho_{i_\ell,a_\ell} \neq 0$, $\rho_{i_{\ell+1},b} \neq 0$ and $a_\ell P_{i_{\ell+1}} b$, student $i_{\ell+1}$ ex-ante justifiably envies $i_\ell$ for $a_\ell$ at $\rho$, contradicting $\rho$ is strongly ex-ante stable.

Case (2'): $i_{\ell+1} \sim_{a_\ell} i_\ell$ : By the strong ex-ante stability of $\pi$, there is no ex-ante discrimination between $i_{\ell+1}$ and $i_\ell$ for $a_\ell$ at $\pi$. Since $a_\ell P_{i_\ell} a_{\ell-1}$ and $\pi_{i_\ell,a_{\ell-1}} \neq 0$, we must have $\pi_{i_\ell,a_\ell} \geq \pi_{i_{\ell+1},a_\ell}$. Then, we have $\rho_{i_\ell,a_\ell} > \pi_{i_\ell,a_\ell} \geq \pi_{i_{\ell+1},a_\ell} > \rho_{i_{\ell+1},a_\ell}$. Recall that $\rho_{i_{\ell+1},b} \neq 0$ for $a_\ell P_{i_{\ell+1}} b$. The last two statements imply that $\rho$ ex-ante discriminates between $i_\ell$ and $i_{\ell+1}$ at $a_\ell$, contradicting that $\rho$ is strongly ex-ante stable. $\Diamond$

Consider the sequence of offers and rejections in the FDA algorithm that leads to $\pi$. Let $i_\ell \in I^m$ for a cycle $Cyc^m$ (without loss of generality, let $(i_0, a_0, i_1, a_1, ..., i_n, a_n)$ be this cycle) be the *last* student in $I^*$ to apply and get a positive fraction under $\pi$ from the next school in his cycle (i.e., for $i_\ell$, this school is $a_\ell \in C^m$). Let t be this step of the algorithm. We prove the following claim:

*Claim 2: The total sum of student fractions that school $a_{\ell-1}$ has tentatively accepted until the beginning of step t of the FDA algorithm is equal to its quota, i.e., school $a_{\ell-1}$ is filled at the beginning of step t.*

*Proof of Claim 2:* Consider agent $i_{\ell-1}$. We have $\pi_{i_{\ell-1},a_{\ell-2}} > 0$ by construction of $Cyc^m$. By the choice of student $i_\ell$, student $i_{\ell-1}$ should have applied to school $a_{\ell-2}$ at some step s ¡ t. We also have $a_{\ell-1} P_{i_{\ell-1}} a_{\ell-2}$ by construction of $Cyc^m$. Then, in the FDA algorithm, $i_{\ell-1}$ should have applied to $a_{\ell-1}$ first at some step r ¡ s. This is true as he can apply to $a_{\ell-2}$ in the algorithm only after having been rejected by school $a_{\ell-1}$. A school can reject a student only if it has tentatively accepted student fractions summing up to its quota. Since $a_{\ell-1}$ remains to be filled after it becomes filled in the algorithm, the claim follows. $\Diamond$

Thus, by Claim 2, $a_{\ell-1}$ is full at the beginning of step t just before student $i_\ell$ applies. Then, there exists some student $j \in I$ with $i_\ell \succsim_{a_\ell} j$ such that some fraction of $j$ was tentatively accepted by school $a_\ell$ before step t and some fraction of $j$ gets kicked out of school $a_\ell$ at the end of step t (so that by the choice of $i_\ell$, some fraction of his gets in $a_{\ell-1}$). Since the FDA algorithm converges to a well-defined random matching, there is some $b \in C$ such that $a_{\ell-1} P_j b$ and $\pi_{j,b} \neq 0$. We prove the following claim:

*Claim 3:* We have $j \notin I^*$.

*Proof of Claim 3:* Suppose not, i.e., $j$ is in some cycle. By the choice of student $i_\ell$, the ordered four-tuple $b(= a_{\ell-2}), j\,(= i_{\ell-1})\,, a_{\ell-1}, i_\ell$ cannot be part of $Cyc^m$, i.e., $j$ cannot be accepted by $b$ after being rejected by $a_{\ell-1}$ in the FDA algorithm and yet $\rho_{j,b} < \pi_{j,b}$ (i.e., see Statement 8 for the construction of a cycle). But then by the choice of school $b$, $\rho_{j,b} \geq \pi_{j,b} \neq 0$. However, Claim 1 applied for school $a_{\ell-1}$ and student $j(= i_{\ell-1})$ and the fact that $a_{\ell-1} P_j b$ together imply that $\rho_{j,b} = 0$, contradicting the previous statement. Thus, $j \notin I^*$. $\Diamond$

We are ready to finish the proof of the theorem. Since school $a_{\ell-1}$ is full at the beginning of step t (by Claim 2), there is student $i_{\ell-1} \in I^m \setminus \{i_\ell\}$, i.e., preceding $a_{\ell-1}$ in $Cyc^m$, with $0 \neq \rho_{i_{\ell-1},a_{\ell-1}} > \pi_{i_{\ell-1},a_{\ell-1}}$ who applied to school $a_{\ell-2} \prec_{i_{\ell-1}} a_{\ell-1}$ after being rejected by school $a_{\ell-1}$. Moreover, by the choice of $i_\ell$, student $i_{\ell-1}$ applies to $a_{\ell-2}$ before step t (for the last time), and hence, he got rejected by $a_{\ell-1}$ before step t. Moreover, $i_{\ell-1} \neq j$ (by Claim 3). Thus, $j \succsim_{a_{\ell-1}} i_{\ell-1}$. We will establish a contradiction, and complete the proof of the theorem. Two cases are possible:

Case (1"'): $j \succ_{a_{\ell-1}} i_{\ell-1}$ : Since $\rho_{j,b} \neq 0$, strong ex-ante stability of $\rho$ implies that $\rho_{i_{\ell-1},a_{\ell-1}} = 0$ leading to a contradiction to the fact that $0 \neq \rho_{i_{\ell-1},a_{\ell-1}}$.

Case (2"'): $j \sim_{a_{\ell-1}} i_{\ell-1}$ : Recall again that $\pi_{i_{\ell-1},a_{\ell-2}} > 0$ and $a_{\ell-1} P_{i_{\ell-1}} a_{\ell-2}$, $\pi_{j,b} > 0$, and $a_{\ell-1} P_j b$. But then, $i_{\ell-1}$ gets rejected by $a_{\ell-1}$ at the FDA algorithm at the same step as $j$ gets rejected with some fraction, which is step t (since $\rho$ does not ex-ante discriminate $j$ and $i_{\ell-1}$ at $a_{\ell-1}$, they should have equal fractions at $a_{\ell-1}$ prior to step t), and thus $i_{\ell-1}$ applies to school $a_{\ell-2}$ after step t, contradicting the choice of student $i_\ell$. ■

**Proof of Remark 2.** Let $\varphi$ be a strongly ex-ante stable mechanism. Consider the following problem with three students $1, 2, 3$, and three schools $a, b, c$, each with quota one:

| $P_1$ | $P_2$ | $P_3$ |
|---|---|---|
| $a$ | $a$ | $b$ |
| $b$ | $c$ | $a$ |
| $c$ | $b$ | $c$ |

| $\succsim_a$ | $\succsim_b$ | $\succsim_c$ |
|---|---|---|
| 3 | 1 | $\vdots$ |
| 1, 2 | $\vdots$ | $\vdots$ |
| | $\vdots$ | $\vdots$ |

There is a unique strongly ex-ante stable random matching that is given as follows:

$\rho =$

| | **a** | **b** | **c** |
|---|---|---|---|
| **1** | 0 | 1 | 0 |
| **2** | 0 | 0 | 1 |
| **3** | 1 | 0 | 0 |

Thus, $\varphi[P, \succsim] = \rho$.

However, if student 1 submits the preferences

| $P'_1$ |
|---|
| $a$ |
| $c$ |
| $b$ |

instead of $P_i$, then the unique strongly ex-ante stable random matching will be

$\rho' =$

| | **a** | **b** | **c** |
|---|---|---|---|
| **1** | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| **2** | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ |
| **3** | 0 | 1 | 0 |

Hence, $\varphi[(P'_1, P_{-1}), \succsim] = \rho'$. Observe that there can be von Neumann - Morgenstern utility functions of Student 1 that may make $\rho'_1$ more desirable than $\rho_1$.

If $|I| > 3$, we can make the example hold by embedding it in a problem with $I = \{1, 2, ..., |I|\}$ and $C = \{a, b, c, d_4, ..., d_{|I|}\}$ where each student $i \in \{4, ...., |I|\}$ is ranking school $d_i$ as his first choice, and each student $i \in \{1, 2, 3\}$ is ranking each school $d_i$ lower than schools $a, b, c$. Under any ex-ante strongly stable matching, each $i \in \{4, 5, ..., |I|\}$ will be matched with $d_i$ and $\{1, 2, 3\}$ will be mapped with $\{a, b, c\}$. ■

# C Appendix: Proof of Proposition 5

**Proof of Proposition 5.** **"Only if" part:** Let $\rho$ be an ex-ante stable random matching with an ex-ante stable improvement cycle $Cyc = (i_1, a_1, ..., i_m, a_m)$. Let $i_{m+1} \equiv i_1$ and $a_{m+1} \equiv a_1$. Let $\pi$ be the random matching obtained by satisfying this cycle with some feasible fraction. Then $\pi$ ordinally dominates $\rho$. Since each student $i_\ell$ envies student $i_{\ell+1}$ for $a_{\ell+1}$ due to $a_\ell$ at $\rho$, and $i_\ell$ is a highest $a_l$-priority student ex-ante envying a student with a positive probability at school $a_{\ell+1}$. Thus, either (1) $i_\ell$ is at the same priority level with $i_{\ell+1}$ for $a_{\ell+1}$, or (2) $i_\ell$ is at a lower-priority level than $i_{\ell+1}$ for $a_{\ell+1}$ but any $i \sim_{a_{\ell+1}} i_{\ell+1}$ does not ex-ante envy himself or $i_{\ell+1}$ for $a_{\ell+1}$ at $\rho$, that is: $i$ is not assigned with a positive probability to a worse school than $a_{\ell+1}$ at $\rho$. Thus, when we satisfy the cycle $Cyc$, there will be no ex-ante justified envy toward a lower-priority student and $\pi$ is ex-ante stable.

**"If" part:** Let $\rho$ be an ex-ante stable random matching. Let $\pi \neq \rho$ be an ex-ante stable random matching that ordinally dominates $\rho$. We will construct a particular ex-ante stable improvement cycle at $\rho$.

Let $I' = \{i \in I : \rho_i \neq \pi_i\}$. Clearly, $I' \neq \emptyset$. Note that for all $i' \in I'$, $\pi_{i'}$ stochastically dominates $\rho_{i'}$. Thus, whenever $\pi_{i',a} > \rho_{i',a}$ for some $i' \in I'$ and $a \in C$, then there is $j' \in I'$ with $\pi_{j',a} < \rho_{j',a}$; moreover, since $\pi_{j'}$ stochastically dominates $\rho_{j'}$, there is $b \in C$ with $bP_{j'}a$ and $\pi_{j',b} > \rho_{j',b}$. Let $C' = \{c \in C : \pi_{i,c} > \rho_{i,c}$ for some $i \in I'\}$. Clearly, $C' \neq \emptyset$.

Consider the following directed graph: Each pair student-school pair $(i,c) \in I' \times C'$ with $\rho_{i,c} \neq 0$ is represented by a node. Fix a school $c \in C'$. Let each student-school pair $(i,c)$ in this graph containing school $c$ be pointed to by every student-school pair containing a student envying student $i$ for school $c$ and has the highest priority among such envying students in $I'$. We repeat this for each $c \in C'$.

Note that no student-school pair in the resulting graph points to itself, and each student-school pair in this graph is pointed to by at least one other student-school pair. Moreover, each student-school pair $(i,c)$ in this graph can only be pointed to by a student-school pair that contains a different school than $c$. Then there is at least one cycle of student-school pairs $Cyc = (i_1, a_1, i_2, a_2, ..., i_m, a_m)$ with $(i_m, a_m) \equiv (i_0, a_0)$ and $m \geq 2$. By construction, we have $(i_\ell, a_\ell) \gtrdot^\rho (i_{\ell+1}, a_{\ell+1})$ for $\ell = 0, \ldots, m-1$. Note also that cycle $Cyc$ contains at least two distinct students. Then cycle $Cyc$ is a stochastic improvement cycle.

Now consider school $a_{\ell+1}$ of the pair $(i_{\ell+1}, a_{\ell+1})$ in cycle $Cyc$. Suppose, for a contradiction, that student $i_\ell$ does not ex-ante top-priority envy $i_{\ell+1}$ for $a_{\ell+1}$ due to $a_\ell$. Then there is a student-school pair $(j,d)$ with $j \notin I'$, which is not represented in our graph, such that $(j,d) \blacktriangleright^\rho (i_{\ell+1}, a_{\ell+1})$. In particular, $j \succ_{a_{\ell+1}} i$ for any $i \in I'$ such that $(i,d) \gtrdot^\rho (i_{\ell+1}, a_{\ell+1})$ for any $d \in C'$. Let $k \in I'$ such that $\pi_{k,a_{\ell+1}} > \rho_{k,a_{\ell+1}}$. Since $j \succ_{a_{\ell+1}} k$ and $\rho_{j,d} = \pi_{j,d}$, student $j$ justifiably ex-ante envies $k$ at $\pi$. This contradicts the ex-ante stability of $\pi$. ■

# D Appendix: The EASC Algorithm

The description of the algorithm mostly follows the constrained consumption algorithm of Athanassoglou and Sethuraman (2007) with a few modifications:

Given an ex-ante stable matching $\rho$, we first define the following set:

$$\mathcal{A}(\rho) = \{(i,c) \in I \times C : \rho_{i,c} > 0 \text{ or } (i,a) \blacktriangleright^{\rho} (j,c) \text{ for some } j \in I \text{ and } a \in C\}.$$

Given an initial ex-ante stable random matching $\rho$, our adaptation of the constrained consumption algorithm finds a random matching $\pi$ such that (1) $\pi$ ordinally dominates or is equal to $\rho$ and (2) $(i,c) \notin \mathcal{A}(\rho) \Longrightarrow \pi_{i,c} = 0$.

It is executed through a series of *flow networks*, each of which is a directed graph from an artificial *source node* to an artificial *sink node*, denoted as $\sigma$ and $\tau$, respectively. We will carry the assignment probabilities from source to sink over this flow network, so that the eventual flow will always determine a feasible random matching. The initial network is constructed as follows:

The *nodes* of the network are (1) source $\sigma$ and sink $\tau$; (2) each school $c \in C$; and (3) for each $i \in I$ and $\ell \in \{1, ..., |C|\}$, $i_{(\ell)} \in I \times \{1, ..., |C|\}$; i.e., the $\ell^{\text{th}}$ node of student $i$ is a node, where this node corresponds to the $\ell^{\text{th}}$ choice of student $i$ among the schools.

Let $N = I \times \{1, ..., |C|\} \cup C \cup \{\sigma, \tau\}$ be the set of nodes of the network.

An *arc* from node $x$ to node $y$ is represented as $x \to y$. Let $\omega_{x \to y}$ be the *capacity* of arc $x \to y$.[20] The arcs have the following load capacities:

(1) Each arc $\sigma \to i_{(\ell)}$ has the capacity $\rho_{i,c}$, where school $c$ is the $\ell^{\text{th}}$ choice of student $i$.

(2) Each arc $i_{(\ell)} \to c$ has the capacity $\infty$, if $(i,c) \in \mathcal{A}(\rho)$, and $c$ is ranked $\ell^{\text{th}}$ or better at the student $i$'s preferences; and 0, otherwise.

(3) Each arc $c \to \tau$ has the capacity $q_c$, the quota of school $c$.

(4) Any arc between any other two nodes has capacity zero.

Thus, the arcs with positive load capacities are directed from the source $\sigma$ to the student nodes, from the student nodes to feasible school nodes with respect to $\mathcal{A}(\rho)$, and from school nodes to the sink $\tau$.

Let $\Gamma = \langle N, \omega \rangle$ denote this network. We define additional concepts for such a network.

A *cut* of the network is a subset of nodes $K \subseteq N$ such that $\sigma \in K$ and $\tau \in N \backslash K$. The *capacity* of a cut $K$ is the sum of the capacities of the arcs that are directed from nodes in $K$ to nodes in $N \backslash K$, and it is denoted as $\Omega(K)$, that is: $\Omega(K) = \sum_{x \in K, y \in N \backslash K} \omega_{x \to y}$. A *minimum cut* $K^*$ is a minimum capacity cut, i.e., $K^* \in \arg\min_{\{\sigma\} \subseteq K \subseteq N \backslash \{\tau\}} \Omega(K)$. A *flow* of the network is a list $\phi = (\phi_{x \to y})_{x,y \in N}$ such that (1) for each $x, y \in N$, $\phi_{x \to y} \leq \omega_{x \to y}$, i.e., the flow cannot exceed the capacity, and (2)

[20]Without loss of generality, we focus on rational numbers as load capacities.

for all $x \in N\backslash\{\sigma,\tau\}$, $\sum_{y\in N}\phi_{y\to x} = \sum_{y\in N}\phi_{x\to y}$, i.e., total incoming flow to a node should be equal to the total outgoing flow. Let $\Phi$ be the set of flows. The *value* of a flow $\phi$ is the total outgoing flow from the source, i.e., $\Omega(\phi) = \sum_{y\in N}\phi_{\sigma\to y}$. A *maximum flow* $\phi^*$ is a flow with the highest value, i.e., $\phi^* \in \arg\max_{\phi\in\Phi}\Omega(\phi)$. Observe that in our network $\Gamma$, the maximum flow value is equal to $|I|$.

The algorithm solves iterative *maximum flow-minimum cut problems*, a powerful tool in graph theory and linear programming. The corresponding duality theorem is stated as follows:

**Theorem 7** ***(Ford and Fulkerson, 1956), (Maximum Flow-Minimum Cut Theorem)*** *The value of the maximum flow is equal to the capacity of a minimum cut.*

There are various polynomial-time algorithms, such as the *Edmonds and Karp (1972) algorithm*, which can determine a minimum cut and maximum flow.

The ex-ante stable consumption algorithm updates the network starting from $\Gamma$ by updating the capacity of some of the source arcs $\omega_{\sigma\to i_{(\ell)}}$ over *time*, which is a continuous parameter $t \in [0,1]$. It starts from $t = 0$ and increases up to $t = 1$. Thus, let's relabel the source arc weights as a function of time $t$ as $\omega^t_{\sigma\to i_{(\ell)}}$ by setting $\omega^0_{\sigma\to i_{(\ell)}} \equiv \omega_{\sigma\to i_{(\ell)}}$ for each arc $\sigma \to i_{(\ell)}$. No other arc capacity is updated. Let $\Gamma^t$ be the corresponding flow network at time $t$.

There will also be iterative steps in the algorithm with start times $t^1 = 0 \le t^2 \le ... \le t^{\mathrm{n}} \le 1 = t^{\mathrm{n}+1}$, for steps 1,...,n, respectively. All assignment activity in step m occurs in the time interval $(t^{\mathrm{m}}, t^{\mathrm{m}+1}]$.

This algorithm is in the class of eating algorithms introduced by Bogomolnaia and Moulin (2001), and $t$ also represents the assigned fraction of each student, since each student is assumed to be assigned at a uniform speed of 1. This activity is referred to as *eating a school*. Each school is assumed to be a perfectly divisible object with $q_c$ copies.

We update the feasible assignment set $\mathcal{A}(\rho)$ in each step. Let $\mathcal{A}^{\mathrm{m}}(\rho)$ be the feasible student-school pairs at step m=1,...,n. We have $\mathcal{A}^1(\rho) = \mathcal{A}(\rho) \supseteq \mathcal{A}^2(\rho) \supseteq ... \supseteq \mathcal{A}^{\mathrm{n}}(\rho)$.

At each step m, let $b_i \in C$ be the *best feasible* school for student $i$, that is, $(i, b_i) \in \mathcal{A}^{\mathrm{m}}(\rho)$, and $b_i R_i c$ for all $c$ with $(i,c) \in \mathcal{A}^{\mathrm{m}}(\rho)$. Also, let $e_i \in C$ be the *endowment* school of student $i$, that is, if $R_i(c)$ is the rank of school $c$ for $i$, then $\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(e_i))}} > 0$ and $b_i P_i e_i R_i c$ for all $c$ with $\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(c))}} > 0$. Observe that $e_i$ may not exist for a student $i$, which case is denoted as $e_i = \emptyset$. In the algorithm we describe here, each student consumes the best school feasible for him at $t$ while his endowment of a worse school decreases.

We are ready to state the algorithm, a slightly modified version of the Athanassoglou and Sethuraman (2007) algorithm:

**Algorithm 3** ***The EASC Algorithm:***

Suppose that until step m $\geq 1$, we determined $t^{\mathrm{m}}$, $\left\{\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(\ell)}}\right\}_{i\in I,\ell\in\{1,\ldots,|C|\}}$, and $\mathcal{A}^{\mathrm{m}}(\rho)$.

***Step m:*** We determine $t^{\mathrm{m}+1}$, $\left\{\omega^{t}_{\sigma\to i_{(\ell)}}\right\}_{i\in I,\ell\in\{1,\ldots,|C|\}}$ for all $t\in(t^{\mathrm{m}},t^{\mathrm{m}+1}]$, and $\mathcal{A}^{\mathrm{m}+1}(\rho)$ as follows: Initially time satisfies $t=t^{\mathrm{m}}$. Let $\{b_i,e_i\}_{i\in I}$ be determined given $\mathcal{A}^{\mathrm{m}}(\rho)$ and $\left\{\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(\ell)}}\right\}$.

Then $t$ continuously increases. At $t$ the arc capacities $\omega^{(t)}\left(\sigma\to i_{(\ell)}\right)$ are updated as follows for each $i\in I$ and $c\in C$ :

$$\omega^{t}_{\sigma\to i_{(R_i(c))}}:=\begin{cases}\max\left\{t-\sum\limits_{\ell=1}^{R_i(b_i)-1}\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(\ell)}},\ \omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(b_i))}}\right\} & \text{if } c=b_i \text{ and } e_i\neq\emptyset,\\ \min\left\{\sum\limits_{\ell=1}^{R_i(b_i)}\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(\ell)}}+\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(e_i))}}-t,\ \omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(e_i))}}\right\} & \text{if } c=e_i,\\ \omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(c))}} & \text{otherwise.}\end{cases}$$

That is, each student $i$ consumes his best feasible school $b_i$ with uniform speed by trading away fractions from his endowment school $e_i$, if it exists and the consumption fraction of the best school exceeds his initial consumption of $\omega^{t^{\mathrm{m}}}_{\sigma\to i_{(R_i(b_i))}}$.

Time $t$ increases until one of the following two events occurs:

- $t<1$, and yet
  - the endowment school fraction endowed to some student reaches to zero, i.e., $\omega^{(t)}_{\sigma\to e_i}=0$ for some $i\in I$: We update

$$t^{\mathrm{m}+1}:=t,$$
$$\mathcal{A}^{\mathrm{m}+1}(\rho):=\mathcal{A}^{\mathrm{m}}(\rho);$$

  or

  - any further increase in $t$ will cause the maximum flow capacity in the network to fall, i.e., for $t^{\varepsilon}>t$ and arbitrarily close to $t$, the network $\Gamma^{(t^{\varepsilon})}$ has a maximum flow capacity less than $|I|$ *(*which can be determined by an algorithm such as Edmonds-Karp*)*: This means that if some student were to consume his best feasible school anymore, some ex-ante stability constraint will be violated. Let $K$ be a minimum cut of $\Gamma^{t^{\varepsilon}}$. Any student $i$ with $i_{(R_i(b_i))}\in K$ and $i_{(R_i(e_i))}\notin K$, $i$ is one of such students. Thus, we update

$$t^{\mathrm{m}+1}:=t,$$
$$\mathcal{A}^{\mathrm{m}+1}(\rho):=\mathcal{A}^{\mathrm{m}}(\rho)\setminus\left\{(i,b_i):e_i\neq\emptyset,\ i_{(R_i(b_i))}\in K,\text{ and } i_{(R_i(e_i))}\notin K\right\}.$$

  We continue with Step m+1.

- $t=1$: The algorithm terminates. The outcome of the algorithm $\pi \in \mathcal{X}$ is found as follows: Let $\phi$ be a maximum flow of the network $\Gamma^1$ (i.e., the final network at time t=1). Then, we set for all $i \in I$ and $c \in C$,

$$\pi_{i,c} := \sum_{\ell=1}^{|C|} \phi_{i_{(\ell)} \to c},$$

i.e., the total flow from student $i$ to school $c$. $\Diamond$

Athanassoglou and Sethuraman (2007) proved that this algorithm with $\mathcal{A}(\rho) = I \times C$ (i.e., the case in which all schools are feasible to be assigned to each student) converges to a unique ordinally efficient random matching such that it treats equals equally whenever $\rho$ treats equals equally; and it Pareto dominates, or is equal to $\rho$. Their statements can be generalized to the case in which $\mathcal{A}(\rho) \subseteq I \times C$ such that the outcome of the above algorithm $\pi$ is constrained ordinally efficient in the class of random matchings $\chi \in \mathcal{X}$ satisfying $\chi_{i,c} > 0 \Rightarrow (i,c) \in I \times C$. Moreover, $\pi$ is also ex-ante stable whenever $\rho$ is ex-ante stable; it ordinally dominates, or is equal to $\rho$; and it treats equals equally whenever $\rho$ treats equals equally. We skip these proofs for brevity.

# E Appendix: How is the EASC algorithm embedded in the FDAT algorithm?

**Example 8** We illustrate the functioning of the FDAT algorithm with the EASC algorithm using the same problem in Example 3 (and Example 5):

***Step 0.*** We found the FDA outcome in Example 3 as

$\rho^1 =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{11}{12}$ | $\frac{1}{12}$ | 0 | 0 |
| 2 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| **3** | $\frac{5}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{6}$ |
| **4** | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ |
| **6** | 0 | $\frac{5}{6}$ | 0 | $\frac{1}{6}$ |

***Step 1.*** We form the feasible student-school pairs for matching as

$$\mathcal{A}^{1.1}\left(\rho^1\right) = \{(1,a),(1,b),(2,a),(2,c),(3,a),(3,b),(3,c),(3,d),(4,c),(5,c),(5,d),(6,a),(6,c)\}.$$

We execute the EASC algorithm as follows:

***Step 1.1.*** Time is set as $t^{1.1} = 0$ : Given that $i_{(\ell)}$ represents the $\ell^{th}$ choice school of student $i$, we form the flow network with the positive weights obtained from the endowment random matching $\rho^1$ as for all $i \in I$ and for all schools $f \in C$, we set the arc capacities

$$\omega^0_{\sigma \to i_{(R_i(f))}} = \rho^1_{i,f},$$

where $R_i(f)$ is the ranking of school $f$ in $i$'s preferences. Next, for all $i \in I$ and $f \in C$, if $(i, f) \in A^1(\rho^1)$, we set the arc capacities of the flow network as

$$\omega^0_{i_{(\ell)} \to f} = \infty,$$

for all ranks $\ell \leq R_i(f)$. Finally, for all $f \in C$, we set the arc capacities

$$\omega^0_{f \to \tau} = q_f.$$

Figure 1 shows this network for $t \in \left[0, \frac{1}{12}\right]$.

Moreover, given these constraints, the best available schools and endowment schools are

| students ($i$) | best school ($b_i$) | endow. school ($e_i$) |
|---|---|---|
| 1 | $b$ | $a$ |
| 2 | $c$ | $a$ |
| 3 | $d$ | $c$ |
| 4 | $c$ | $\emptyset$ |
| 5 | $c$ | $d$ |
| 6 | $d$ | $b$ |

We start increasing time $t$ starting from $t^1 = 0$, thus, each student starts consuming his best available school by trading away from his endowment school (whenever $e_i \neq \emptyset$): that is, the capacity of each arc $\sigma \to i_{(R_i(b_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(b_i))}} = \max\left\{ t - \sum_{\ell=1}^{R_i(b_i)-1} \omega^0_{\sigma \to i_{(\ell)}},\ \omega^0_{\sigma \to i_{(R_i(b_i))}} \right\},$$

and the capacity of each arc $\sigma \to i_{(R_i(e_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(e_i))}} = \min\left\{ \sum_{\ell=1}^{R_i(b_i)} \omega^0_{\sigma \to i_{(\ell)}} + \omega^0_{\sigma \to i_{(R_i(e_i))}} - t,\ \omega^0_{\sigma \to i_{(R_i(e_i))}} \right\},$$

as long as a feasible random assignment can be obtained in the network, i.e., the value of the maximum flow of the network is $|I| = 6$ or the capacity of the endowment school arc does not go to zero.

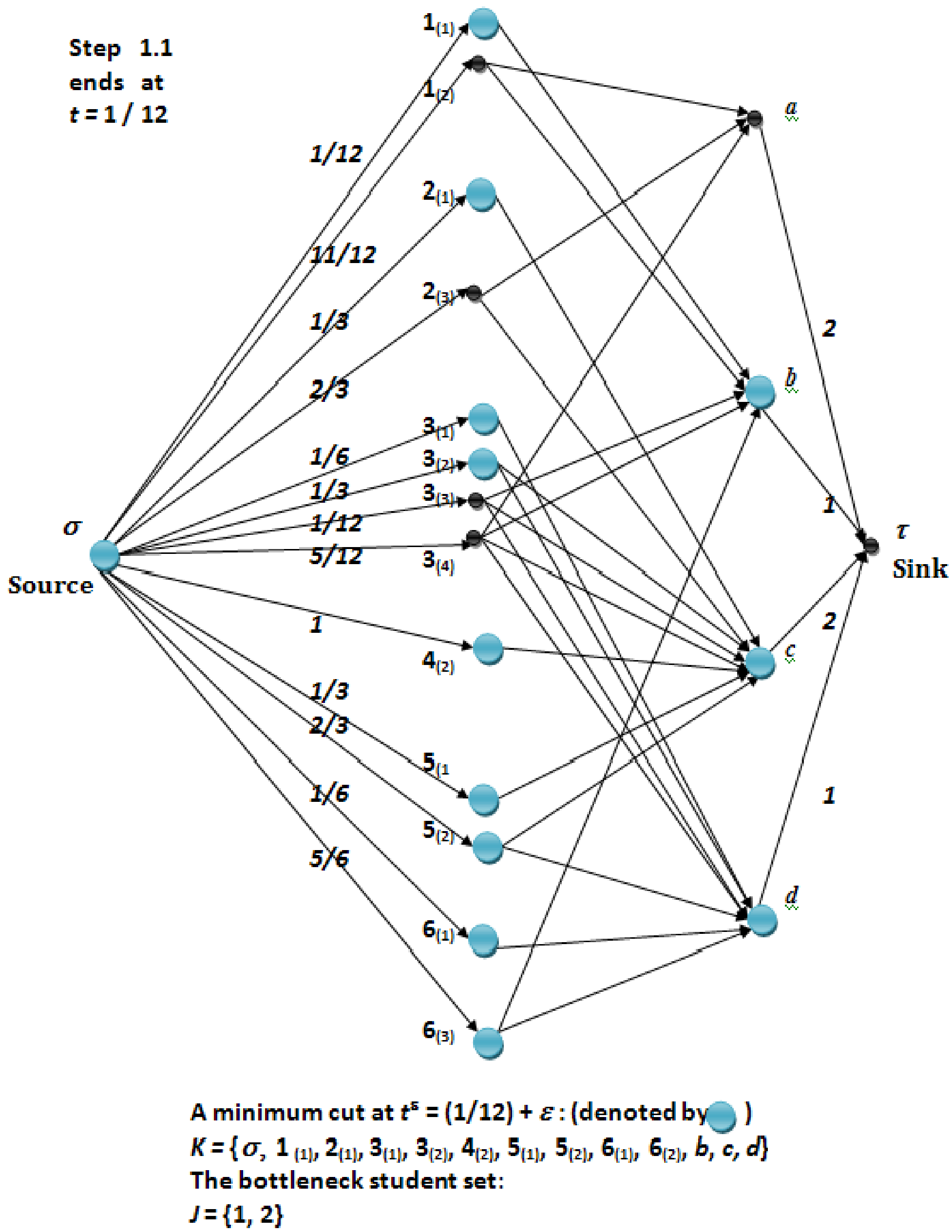


Figure 1: The consumption network for Example 8 at Step 1.1 for times $t \in \left[0, \frac{1}{12}\right]$.

The first condition is satisfied at $t = \frac{1}{12}$: If $t$increases above $\frac{1}{12}$, the value of the maximum flow falls below 6, because of the bottleneck set of agents $J = \{1, 2\}$. At this $t$, there is an excess demand for 1 and 2's best schools, but other agents do not demand 1 and 2's endowment school. Thus, 1 and 2 can no longer trade their endowment school in exchange for a fraction of their best schools. To see that {1,2} is a bottleneck set, we find a minimum cut $K$ as seen in Figure 1 for network at $t = \frac{1}{12}$. Each student's representative nodes for his best school and his endowment school are in $K$, except for students 1 and 2. Their nodes for best schools are in $K$, but not their nodes for endowment schools. Also their endowment school $a$ is not in $K$. Thus, Step 1.1 ends, and students 1 and 2 can no longer consume their best schools $b$ and $c$, respectively. (Observe that the network at $t = \frac{1}{12}$ is identical to the network at $t = 0$.)We set:

$$t^{1.2} = \frac{1}{12},$$
$$\mathcal{A}^{1.2}\left(\rho^1\right) = \mathcal{A}^{1.1}\left(\rho^1\right) \setminus \{(1, b), (2, c)\}.$$

***Step 1.2.*** Time is set as $t^{1.2} = \frac{1}{12}$. The best and endowment schools are updated as

| students ($i$) | best school ($b_i$) | endow. school ($e_i$) |
|---|---|---|
| 1 | $a$ | $\emptyset$ |
| 2 | $a$ | $\emptyset$ |
| 3 | $d$ | $c$ |
| 4 | $c$ | $\emptyset$ |
| 5 | $c$ | $d$ |
| 6 | $d$ | $b$ |

Time increases until $t = \frac{1}{6}$, when there is a new bottleneck set of students with minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 3_{(2)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $6_{(R_6(b_6))} = 6_{(R_6(d))} = 6_{(1)} \in K$ and $6_{(R_6(e_6))} = 6_{(R_6(b))} = 6_{(3)} \notin K$, and there is no other student such that his node for his best (available) school is in $K$ while his node for his endowment school is not, we determine the new bottleneck set as

$$J = \{6\}.$$

Thus, we update

$$t^{1.3} = \frac{1}{6},$$
$$\mathcal{A}^{1.3}\left(\rho^1\right) = \mathcal{A}^{1.2}\left(\rho^1\right) \setminus \{(6, d)\}.$$

At this point the capacities of the source-agent nodes are set still as their initial values at $\omega^0$ (seen in Figure 1).

***Step 1.3.*** Time is set as $t^{1.3} = \frac{1}{6}$. The best and endowment schools are updated as

| students ($i$) | best school ($b_i$) | endow. school ($e_i$) |
|---|---|---|
| 1 | $a$ | $\emptyset$ |
| 2 | $a$ | $\emptyset$ |
| 3 | $d$ | $c$ |
| 4 | $c$ | $\emptyset$ |
| 5 | $c$ | $d$ |
| 6 | $b$ | $\emptyset$ |

At this step, we observe actual trading of fractions of schools $c$ and $d$ between students 3 and 5, since all other students have no endowment schools to trade: time $t$ increases until $\frac{1}{2}$ at which point only the following arc capacities are changing, while the others are still at $\omega^0$ level:

$$\begin{aligned}\omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(b_3))}} = \omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(d))}} = \omega^{\frac{1}{2}}_{\left(\sigma\to 3_{(1)}\right)} &= \\ &= \max\left\{t - \sum_{\ell=1}^{R_3(b_3)-1} \omega^{\frac{1}{6}}_{\sigma\to 3_{(\ell)}},\ \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(b_3))}}\right\} \\ &= \max\left\{\frac{1}{2} - 0, \frac{1}{6}\right\} = \frac{1}{2};\end{aligned}$$

$$\begin{aligned}\omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(e_3))}} = \omega^{\frac{1}{2}}_{\sigma\to 3_{(R_3(c))}} = \omega^{\frac{1}{2}}_{\sigma\to 3_{(2)}} &= \\ &= \min\left\{\sum_{\ell=1}^{R_3(b_3)} \omega^{\frac{1}{6}}_{\sigma\to 3_{(\ell)}} + \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(e_3))}} - t,\ \omega^{\frac{1}{6}}_{\sigma\to 3_{(R_3(e_3))}}\right\} \\ &= \min\left\{\frac{1}{6} + \frac{1}{3} - \frac{1}{2}, \frac{1}{3}\right\} = 0;\end{aligned}$$

$$\begin{aligned}\omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(b_5))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(c))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(1)}} &= \\ &= \max\left\{t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{\frac{1}{6}}_{\sigma\to 5_{(\ell)}},\ \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_3(b_3))}}\right\} \\ &= \max\left\{\frac{1}{2} - 0, \frac{1}{3}\right\} = \frac{1}{2};\end{aligned}$$

$$\omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(d))}} = \omega^{\frac{1}{2}}_{\sigma\to 5_{(2)}} =$$
$$= \min\left\{\sum_{\ell=1}^{R_5(b_5)} \omega^{\frac{1}{6}}_{\sigma\to 5_{(\ell)}} + \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_5(e_5))}} - t,\ \omega^{\frac{1}{6}}_{\sigma\to 5_{(R_5(e_5))}}\right\}$$
$$= \min\left\{\frac{1}{3} + \frac{2}{3} - \frac{1}{2}, \frac{2}{3}\right\} = \frac{1}{2}.$$

Since the endowment school's matching probability reaches zero for student 3, the step ends and we update:

$$t^{1.4} = \frac{1}{2},$$
$$\mathcal{A}^{1.4}\left(\rho^1\right) = \mathcal{A}^{1.3}\left(\rho^1\right).$$

***Step 1.4.*** Time is set to $t^{1.4} = \frac{1}{2}$, only student 3's endowment school changed as $e_3 = b$. But at this time there is a minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(d))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$t^{1.5} = \frac{1}{2},$$
$$\mathcal{A}^{1.5}\left(\rho^1\right) = \mathcal{A}^{1.4}\left(\rho^1\right) \setminus \{(3,d)\}.$$

***Step 1.5.*** Time is set to $t^{1.5} = \frac{1}{2}$, only student 3's best school changed as $e_3 = c$. But at this time there is a minimum cut

$$K = \left\{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(c))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$t^{1.5} = \frac{1}{2},$$
$$\mathcal{A}^{1.5}\left(\rho^1\right) = \mathcal{A}^{1.4}\left(\rho^1\right) \setminus \{(3,c)\}.$$

***Step 1.6.*** Time is set to $t^{1.6} = \frac{1}{2}$, student 3's best school changed as $b_3 = b$ and his endowment school changed as $e_3 = a$. Time $t$ increases until $\frac{7}{12}$, when further increasing $t$ would create a bottleneck set of students with minimum cut

$$K = \left\{\sigma, 1_{(1)}, 2_{(1)}, 3_{(1)}, 3_{(3)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, 6_{(3)}, b, c, d\right\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(b))} = 3_{(3)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(a))} = 3_{(4)} \notin K$, and there is no other student with this property, we have the bottleneck set at

$$J = \{3\}.$$

Observe that in the interval $t \in (\frac{1}{2}, \frac{7}{12}]$, student 3 does not consume his best school more than his capacity. This interval serves as the continuation of the trading between student 3 and 5 regarding schools $c$ and $d$ that has started at Step 1.3. Although 3 has already traded all his endowment of $\frac{1}{3}c$ away in return to get $\frac{1}{3}d$, student 5 has not fully gotten $\frac{1}{3}d$ and traded away $\frac{1}{3}c$. Thus, the market has not cleared yet. Increase in $t$ helps the market to clear, since now we have

$$\begin{aligned}
\omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(b_5))}} &= \omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(c))}} = \omega^{\frac{7}{12}}_{\sigma\to 5_{(1)}} = \\
&= \max\left\{t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{\frac{1}{2}}_{\sigma\to 5_{(\ell)}},\ \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_3(b_3))}}\right\} \\
&= \max\left\{\frac{7}{12} - 0, \frac{1}{2}\right\} = \frac{7}{12};
\end{aligned}$$

$$\begin{aligned}
\omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(e_5))}} &= \omega^{\frac{7}{12}}_{\sigma\to 5_{(R_5(d))}} = \omega^{\frac{7}{12}}_{\sigma\to 5_{(2)}} = \\
&= \min\left\{\sum_{\ell=1}^{R_5(b_5)} \omega^{\frac{1}{2}}_{\sigma\to 5_{(\ell)}} + \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}} - t,\ \omega^{\frac{1}{2}}_{\sigma\to 5_{(R_5(e_5))}}\right\} \\
&= \min\left\{\frac{1}{2} + \frac{1}{2} - \frac{7}{12}, \frac{1}{2}\right\} = \frac{5}{12}.
\end{aligned}$$

while all other arc capacities remain the same. We update as

$$\begin{aligned}
t^{1.7} &= \frac{7}{12}, \\
\mathcal{A}^{1.7}\left(\rho^1\right) &= \mathcal{A}^{1.6}\left(\rho^1\right) \setminus \{(3, b)\}.
\end{aligned}$$

***Step 1.7.*** Time is set to $t^{1.7} = \frac{7}{12}$, student 3's best school changed as $b_3 = a$ and he no longer has an endowment school, i.e., $e_3 = \emptyset$. Time $t$ increases until $\frac{2}{3}$, when further increasing $t$ would create a bottleneck set of students with minimum cut

$$K = \left\{\sigma, 4_{(2)}, 5_{(1)}, c\right\}.$$

Since $5_{(R_5(b_5))} = 5_{(R_3(c))} = 5_{(1)} \in K$ and $5_{(R_5(e_5))} = 5_{(R_5(d))} = 5_{(2)} \notin K$, and there is no other student with this property, we have the bottleneck set as

$$J = \{5\}.$$

Similar to Step 1.6, trade of $c$ from 3 to 5 has continued at this step in return of $d$, and it can be verified that the only updated arc capacities are as follows:

$$\omega^{\frac{2}{3}}_{\sigma \to 5_{(1)}} = \frac{2}{3},$$
$$\omega^{\frac{2}{3}}_{\sigma \to 5_{(2)}} = \frac{1}{3}.$$

We update as

$$t^{1.8} = \frac{2}{3},$$
$$\mathcal{A}^{1.8}\left(\rho^1\right) = \mathcal{A}^{1.7}\left(\rho^1\right) \setminus \{(5, c)\}.$$

***Step 1.8.*** Time is set to $t^{1.8} = \frac{2}{3}$, and student 5's best school is updated as $b_5 = d$ and he no longer has an endowment school, i.e., $e_5 = \emptyset$. Since no student has any endowment school, no more trade takes place in this step, time $t$ increases to 1, and the ex-ante stable consumption algorithm terminates with

$\rho^2 =$

| | **a** | **b** | **c** | **d** |
|---|---|---|---|---|
| **1** | $\frac{11}{12}$ | $\frac{1}{12}$ | 0 | 0 |
| 2 | $\frac{2}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| **3** | $\frac{5}{12}$ | $\frac{1}{12}$ | 0 | $\frac{1}{2}$ |
| **4** | 0 | 0 | 1 | 0 |
| **5** | 0 | 0 | $\frac{1}{6}$ | $\frac{1}{3}$ |
| **6** | 0 | $\frac{5}{6}$ | 0 | $\frac{1}{6}$ |

***Step 2.*** We have the feasible student-school set

$$\mathcal{A}^{2.1}\left(\rho^2\right) = \{(1,a), (1,b), (2,a), (2,c), (3,a), (3,b), (3,c), (3,d), (4,c), (5,c), (5,d), (6,b), (6,d)\}$$
$$= \mathcal{A}^{1.1}\left(\rho^1\right).$$

It is easy to check that there are no feasible ex-ante stable improvement cycles and the FDAT algorithm terminates with outcome $\rho^2$. $\Diamond$

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