Clearing Supply and Demand Under Bilateral Constraints*

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Abstract

In a moneyless market, a non storable, non transferable homogeneous commodity is reallocated between agents with single-peaked preferences. Agents are either suppliers or demanders. Transfers between a supplier and a demander are feasible only if they are *linked*, and the links form an arbitrary bipartite graph. Typically, supply is short in one segment of the market, while demand is short in another,

Information about individual preferences is private, and so is information about feasible links: an agent may unilaterally close one of her links if it is in her interest to do so.

Our *egalitarian transfer* solution rations only the long side in each market segment, equalizing the net transfers of rationed agents as much as permitted by the bilateral constraints. It elicits a truthful report of *both* preferences and links: removing a feasible link is never profitable to either one of its two agents. Together with efficiency, and a version of equal treatment of equals, these properties are characteristic.

Keywords: Bipartite graph, bilateral trade, Strategy-proofness, Equal treatment of equals, Single-peaked preferences.

JEL codes: C72, D63, D61, C78, D71.

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1 Introduction

There are markets where transactions cannot be accompanied by monetary transfers. The examples studied in the recent literature include the barter of indivisible goods (Shapley and Scarf 1974, Ma 1998, Papai 2007), medical labor markets (Roth and Peranson 1999), school choice (Pathak and Sönmez 2008), house allocation (Ergin 2002) and several other markets which can be modeled as matching problems (Gale and Shapley 1962, Roth and Sotomayor 1990). In the absence of a price signal, direct decentralized agreements between participants may be impractical[1], or fail to achieve an efficient allocation of resources[2]. If a centralized mechanism is able to collect unbiased information about private characteristics of the concerned agents and to implement an efficient outcome, it stands as a convincing alternative to market clearing driven by bilateral agreements.

We study a simple moneyless market balancing the supply and demand for a non-storable, not freely-disposable, and non transferable commodity. Think of a group of service providers with limited control over their load of customers on a given day, so that some providers will receive more customer requests than they care to handle, while the load of other providers falls short of their ideal level. Emergency departments (ED) of hospitals routinely divert incoming patients away when they reach their capacity (NJHA 2009). The premature babies are transferred to other neonatal intensive care facilities when there are no vacant incubators in the hospital where the baby was born (BBC 2007, Priest 2008). Airlines transfer travelers around when they cannot honor their bookings. Similar opportunities for mutually advantageous spreading of the total work load arise routinely between hotels of comparable quality, or taxi companies in a given city, salesmen sharing customers, teachers sharing students, etc.. In such situations cash transfers are typically ruled out because they are not part of the organization's culture (as between co-workers), or because they would generate high transaction costs (hotel managers or taxi operators). In other situations there are ethical or legal reasons for ruling them out: public hospitals cannot entertain a kickback for referring a patient away.

Despite the absence of money, the process of clearing supply against demand retains the intrinsic characteristic of markets: individual preferences is the private information of each participant, and a centralized mechanism should elicit this information correctly. Hospitals declaring they have

[1] As in the case of the market for medical residents, see Roth and Peranson 1999.

[2] As when successive mutually profitable new matches result in a cycle: see Roth and Sotomayor (1990).

reached "red status" signal they cannot provide adequate service to an incremental patient, a largely subjective statement unverifiable by an outsider[3]; the ideal number of students in a class is not the same for all instructors, and the same applies to all manners of workload.

In our model the commodity (customers, patients) is homogenous and comes in divisible amounts. The former assumption is a realistic simplification in the taxi, hotel or student examples; ditto in the hospital example if we restrict attention to a given type of emergency patients (say, obstetrics, or post-natal care). The latter assumption is mostly technical (see concluding comments). Preferences of each agent are single-peaked around his ideal/target level (in particular, preferences are convex), and the market participants are either *suppliers* (agents whose initial endowment exceeds their ideal consumption of the commodity) or *demanders* (whose endowment is below their target consumption).

The richness of our model is to allow for arbitrary feasibility constraints on transfers between suppliers and demanders, and to view these constraints as private information as well. A centralized mechanism must elicit from the participants the set of feasible *links* (pairs of one supplier and one demander between which transfers are feasible); agents cannot report an unfeasible link, but are free to “close” unilaterally a feasible link. Such constraints are pervasive in our motivating examples: a given hospital can only divert patients to “nearby" hospitals with adequate facilities; transfers between salespersons are constrained by their proficiency in various languages, and so on. In our model the bipartite graph of links is endogenous, because agents will close some links if it is in their interest to do so[4].

We show that a centralized organization of the market is compatible with truthful revelation of both individual preferences and feasible links (in the strong sense of dominant strategy). This is relevant to some debates in the patients allocation example, where there is evidence that decentralized diversion is wasteful, and some attempts at centralization are being developed[5].

Our *egalitarian transfer* mechanism is simple and well known in the

[3]NJHA 2009 notes that "Diversions may be overridden by the emergency physician in charge when medical judgement indicates that the diverting hospital can handle a certain patient better than the alternative hospital."

[4]In recent literature on buyer-seller networks (e.g., Kranton and Minehart 2000, Corominas-Bosch 2004), agents can similarly pay to establish a link with one another. Our model is however very different in that monetary transfers are ruled out.

[5]REDDINET (http://www.reddinet.com) is a medical communications network linking hospitals in several California counties, for the purpose of improving the efficiency of patients' allocation.

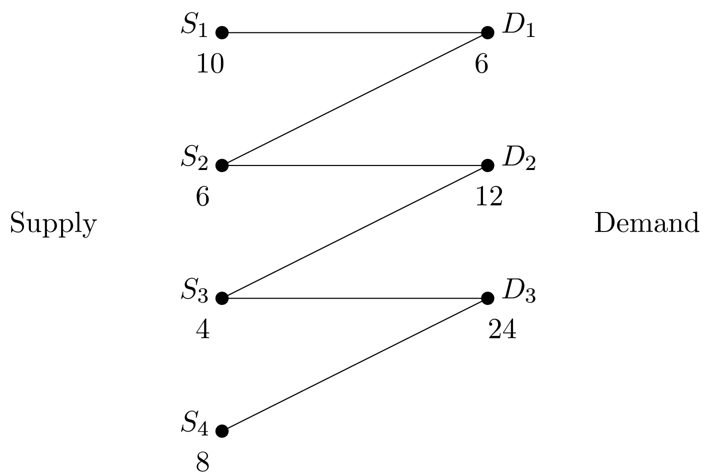

Figure 1: Short supply and short demand co-exist

special case of our model where transfers are not restricted: there are no bilateral constraints, any supplier can transfer commodity to any demander. Then the short side of the market gets the ideal transfer, and the long side is uniformly rationed (Barbera and Jackson 1995; Klaus *et al.* 1998)[6]. Under bilateral constraints two complications arise.

First, short supply and short demand typically coexist in the same problem, but in two segments of the market that do not interact in any efficient outcome. Here is a numerical example

Clearly S_1 is a *captive market* for D_1, a short demand against S_1's long supply. Similarly $\{D_2, D_3\}$ is captive of $\{S_2, S_3, S_4\}$, who are the short supply against $\{D_2, D_3\}$'s long demand. Note that D_1 and S_2 achieve their ideal consumption by a transfer of 6 units. However this transfer would shut out S_1 who can only send her surplus to D_1. It is more efficient to transfer 6 units from S_1 to D_1, then let S_2, S_3, S_4 give their 18 units to demanders 2 and 3.

A familiar graph-theoretical result, the Gallai-Edmonds decomposition (Ore 1962), determines the partition of the market in up to three segments, and the corresponding structure of Pareto optimal allocations: in one seg-

[6]If the sum of demanders' peaks is larger than the sum of suppliers' peaks, we speak of *short supply*. Each supplier unloads an amount of commodity equal to her peak, and the total supply is rationed among demanders according to the *uniform rationing* method. Symmetrically, if the demand is short, each demander gets her ideal amount of commodity and the suppliers are uniformly rationed.

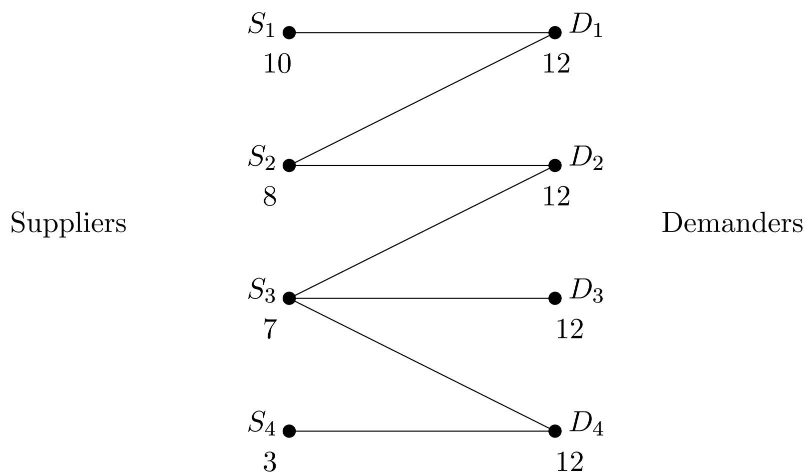

Figure 2: Agents on the short side are not treated identically

ment supply is overdemanded, and the corresponding receivers are rationed; in the second segment supply is underdemanded, and the corresponding receivers eat more than their ideal share; and in the third segment (not present in the example above), supply exactly balances demand.

The second complication is that agents in the long side do not simply get either their ideal transfer or a common transfer, as in Sprumont (1991). Consider the following example:

Here the entire supply is short against a long demand. Absent the bilateral constraints, each demander would receive 7 units. Under the above constraints, the most egalitarian distribution is 10 units for D_1, 8 units for D_2, and 5 units for each of D_3 and D_4.

In a segment of the market with short supply (or short demand), the feasibility constraints take the form of core stability in a certain cooperative game, and the egalitarian transfers are the egalitarian solution (Dutta and Ray 1989) of this game. The compact definition of our solution is as the Lorenz dominant profile of transfers within the Pareto set (Proposition 2)[7].

Thus our solution is much more difficult to compute than in the absence of bilateral constraints (Klaus *et al.* 1998). First we must find the Gallai Edmonds decomposition, next we must solve a finite number of submodular linear systems (see section 4). Remarkably, the essential features of unconstrained uniform rationing are preserved, namely *Efficiency* (Pareto

[7] This is also true in the unconstrained model (de Frutos and Masso 1995).

Optimality), and *Strategyproofness* (truthful revelation of one's preferences is a dominant strategy). The fairness properties of *equal treatment of equals* and *no envy* are also satisfied, provided we adapt their definition to take the constraints into account (see section 5). Last but not least, in our mechanism every participant has no incentive to close a feasible link, as expressed by the *link monotonicity* property: whenever a link ij becomes feasible, ceteris paribus, neither supplier i nor demander j can be worse off.

Our main result (theorem 2, section 6), characterizes the egalitarian transfer mechanism by the combination of efficiency, strategyproofness, voluntary trade (no one prefers to walk out of the market, a consequence of link monotonicity), and (constrained) equal treatment of equals.

In the absence of bilateral constraints, and with agents' endowments known, a parallel result, with a stronger version of the standard equal treatment of equals, was established in Klaus *et al.* (1998).

In the companion paper Bochet *et al.* (2009), we analyze with the same techniques a one-sided version of the present model, that is a constrained generalization of Sprumont's fair division model (Sprumont 1991). The suppliers are now passive, they *must* unload a given amount of the commodity among the set of *receivers*, who each have a private ideal consumption level. See the concluding comments in section 6.

2 Preferences and feasible allocations

We have a set S of suppliers with generic element i, and a set D of demanders with generic element j. A set of transfers of the single commodity from suppliers to demanders results in a vector $(x, y) \in \mathbb{R}_+^S \times \mathbb{R}_+^D$ where x_i (resp. y_j) is supplier i's (resp. demander j's) *net transfer, with* $\sum_S x_i = \sum_D y_j$.

Supplier i (demander j) has *single-peaked preferences* over her *net transfer* x_i with peak s_i (respectively over her net transfer y_j with peak d_j)[8]. We write R_i, R_j for such preferences[9], and $\mathcal{R}$ for the set of single peaked preferences over $\mathbb{R}_+$.

The commodity can only be transferred between certain pairs of supplier i, demander j. The bipartite graph G, a subset of $S \times D$, represents these constraints: $ij \in G$ means that a transfer is possible between $i \in S$ and

[8] For every s_i', s_i'' we have $s_i' < s_i'' \leq s_i \Rightarrow s_i'' P_i s_i'$, and $s_i \leq s_i'' < s_i' \Rightarrow s_i'' P_i s_i'$.

[9] Note that because a supplier is normally endowed with a finite amount of the commodity, her net trade cannot be arbitrarily large, and similarly the net trade of a demander should be capped. This however will not matter since all relevant net trades will take place in the intervals $[0, s_i]$ and $[0, d_j]$.

$j \in D$. We assume throughout that the graph G is connected, else we can treat each connected component of G as a separate problem.

We use the following notation. For any subsets $T \subseteq S$, $C \subseteq D$ the restriction of G is $G(T, C) = G \cap \{T \times C\}$ (not necessarily connected). The set of demanders compatible with the suppliers in T is $f(T) = \{j \in D | G(T, \{j\}) \neq \varnothing\}$. The set of suppliers compatible with the demanders in C is $g(C) = \{i \in S | G(\{i\}, C) \neq \varnothing\}$.

A transfer of goods from S to D is realized by a G-flow φ, i.e., a vector $\varphi \in \mathbb{R}_+^G$ such that $\varphi_{ij} > 0 \Rightarrow ij \in G$. We write $(x(\varphi), y(\varphi))$ for the transfers implemented by φ, namely:

$$\text{for all } i \in S : x_i(\varphi) = \sum_{j \in f(i)} \varphi_{ij}; \text{ for all } j \in D : y_j = \sum_{i \in g(j)} \varphi_{ij} \tag{1}$$

We say that the net transfers (x, y) are *feasible* if they are implemented by some G-flow. We write $\mathcal{A}(G)$ for the set of feasible net transfers, and define similarly $\mathcal{A}(G(T, C))$ for any $T \subseteq S$, $C \subseteq D$. These sets are described in our first result.

Lemma 1: *For any $S' \subseteq S$, $D' \subseteq D$ the three following statements are equivalent:*

i) $(x, y) \in \mathcal{A}(G(S', D'))$

ii) for all $T \subseteq S'$, $x_T \leq y_{f(T)}$ *and* $x_{S'} = y_{D'}$

iii) for all $C \subseteq D'$, $y_C \leq x_{g(C)}$ *and* $y_{D'} = x_{S'}$

Proof: This is a standard application of the Marriage Lemma[10].

For a given profile of preferences $R \in \mathcal{R}^{S \cup D}$, we speak of the *economy* (S, D, G, R) or simply (G, R) when this causes no confusion. All our results in the next two sections, as well as the definition of our solution, only depend upon the profile of peaks s, d, and not upon the full preference profile R. To signal such simplification, we will speak of a *problem* (S, D, G, s, d) or simply (G, s, d).

3 Pareto optimality

We write $\mathcal{PO}(G, R)$ for the set of Pareto optimal net transfers: it contains the feasible net transfer (x, y) if and only if for any other $(x', y') \in \mathcal{A}(G)$ we have

$$\{\text{for all } i, j\text{: } x'_i R_i x_i \text{ and } y'_j R_j y_j\} \Rightarrow \{\text{for all } i, j\text{: } x'_i I_i x_i \text{ and } y'_j I_j y_j\}$$

[10] See Ahuja *et al.* (1993)

To describe Pareto optimal allocations, we use a variant of the Gallai-Edmonds decomposition for bipartite graphs. This result depends upon the problem (G, s, d) and not on the aspects of preferences other than peaks.

When we speak of the (sub)problem $(G(S', D'), s, d)$, we mean that the suppliers in S' will be transferring goods to the agents in D' along $G(S', D')$, so that only the S' and the D' coordinates of s, d matter.

Definition *We say that the problem* $(G(S', D'), s, d)$
i) is balanced if $(s, d) \in A(G(S', D'))$.
ii) has short-supply if for all $T \subseteq S'$, $s_T < d_{f(T)}$.
iii) has short-demand if for all $C \subseteq D'$, $d_C < s_{g(C)}$.

In a balanced problem the net transfer (s, d) is feasible; in a problem with short-demand some net transfers (d, y) with $y \lneqq s$, are feasible (Lemma 1); in a problem with short-supply some net transfers (x, s) with $x \lneqq d$ are feasible (Lemma 1).

The next result says that any allocation problem (G, s, d) can be decomposed in three subproblems, one of each type.

Lemma 2: *For any problem* (G, s, d) *where* G *is connected, and* $s, d \geq 0$, *there exists unique partitions* S_+, S_0, S_- *of* S, *and* D_+, D_0, D_- *of* D *such that*
i) $G(S_-, D_0) = G(S_-, D_-) = G(S_0, D_-) = \varnothing$
ii) $(G(S_0, D_0), s, d)$ *is balanced;*
iii) $(G(S_+, D_-), s, d)$ *has short-supply*
iv) $(G(S_-, D_+), s, d)$ *has short-demand.*

There are algorithms polynomial in the number of nodes $|S| + |D|$ to compute the GE decomposition (see Ore, 1962).

Note that up to two of the pairs $(S_0, D_0), (S_+, D_-)$, or (S_-, D_+) may be empty. One example is given in figure 2 section 1. Another one is when there are no feasibility constraints: $G = S \times D$. As in footnote 7 section 1, we have: if $s_S < d_D$ then $S = S_+, D = D_-$; if $d_D < s_S$ then $S = S_-, D = D_+$; if $s_S = d_D$ then $S = S_0, D = D_0$.

Proof: The Gallai-Edmonds decomposition of a bipartite graph (Ore 1962), gives precisely the statements when for all $i \in S$, $s_i = 1$ and for all $j \in D$, $d_j = 1$. When for each $i \in S$, s_i is a positive integer, we make s_i copies of agent i. Similarly, when for each $j \in D$, d_j is a positive integer, we make d_j copies of agent j.

Then we connect all copies of agents i to all copies of j if and only if $ij \in G$. Again the statements follow by the GE decomposition of this new bipartite graph. By a common rescaling of s, d, we cover the case where these

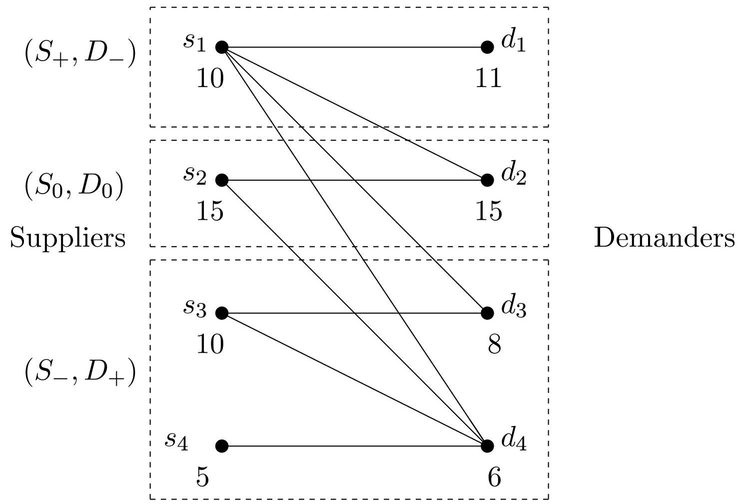

Figure 3: Decomposition with a balanced subgraph

numbers are rational and positive, and by a straightforward limit argument that of real numbers as well, including possibly zero for some peaks.

For future reference (proof of Proposition 4, step 2) we note that the elements of the partition can be defined as the solutions of simple maximization problems.

Define $\mathcal{L} = \arg\max_{T \subseteq S}\{s_T - d_{f(T)}\}$ if there is at least one T such that $s_T > d_{f(T)}$, $\mathcal{L} = \varnothing$ else. As $T \to s_T - d_{f(T)}$ is supermodular, $\mathcal{L}$ is stable by intersection and union, and S_- is its smallest element, while $S_- \cup S_0$ is its largest element. Define similarly $\mathcal{M} = \arg\max_{C \subseteq D}\{d_C - s_{g(C)}\}$ if there is at least one C such that $d_C > s_{g(C)}$, $\mathcal{M} = \varnothing$ else. Then $\mathcal{M}$ is stable by intersection and union, D_- is its smallest element, and $D_- \cup D_0$ its largest element. We omit the straightforward proof.■

We give three examples illustrating the decomposition.

Example 1. *In Figure 1 in Section 1, the partitions are* $S_+ = \{2, 3, 4\}$, $S_- = \{1\}$; $D_+ = \{1\}$, $D_- = \{2, 3\}$ *(there is no* S_0, D_0*).*

Example 2. *In Figure 2 in Section 1, the partitions are* $S_+ = \{1, 2, 3, 4\}$, $D_- = \{1, 2, 3, 4\}$ *(there is no* S_-, D_+, S_0, D_0*).*

Example 3. *In the example of Figure 3, the GE decomposition is* $D_- = \{d_1\}$, $D_+ = \{d_3, d_4\}$, $D_0 = \{d_2\}$, $S_+ = \{s_1\}$, $S_- = \{s_3, s_4\}$, $S_0 = \{s_2\}$. *For another example consider a variant of Figure 4 in which the* $d_2' = 17$ *instead*

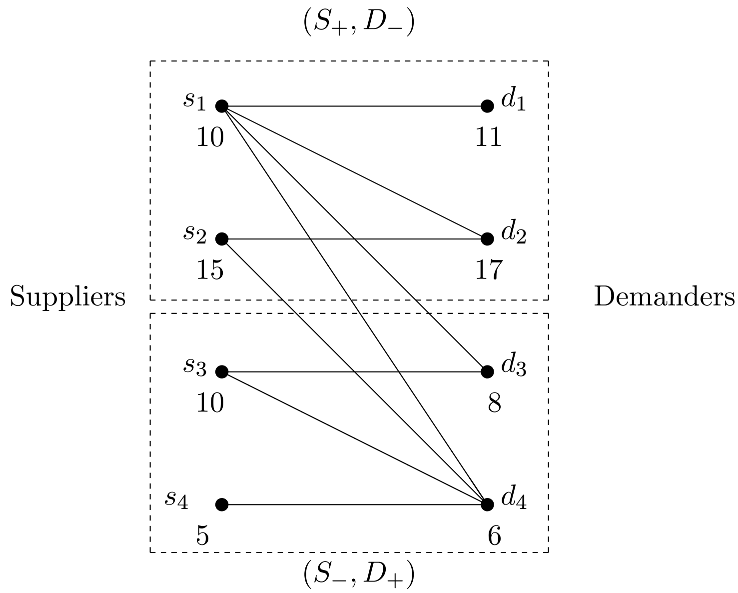

Figure 4: Decomposition without a balanced subgraph

of 15*. This is shown in Figure 4. Now* (S_+, D_-) *is the upper part of the graph while* (S_-, D_+) *is the lower part.*

Figures 3 and 4 illustrate a general property, an immediate consequence of Lemma 2: for any graph and any pair of suppliers i_1, i_2 such that $f(i_1) \subset f(i_2)$, we have $\{i_2 \in S_- \Rightarrow i_1 \in S_-\}$ and $\{i_1 \in S_+ \Rightarrow i_2 \in S_+\}$; for any graph and any pair of demanders j_1, j_2 such that $g(j_1) \subset g(j_2)$, we have $\{j_2 \in D_- \Rightarrow j_1 \in D_-\}$ and $\{j_1 \in D_+ \Rightarrow j_2 \in D_+\}$.

We are now ready to describe the key facts about the set $\mathcal{PO}(G, R)$ of Pareto optimal transfers. For a vector of transfers $(x, y) \in \mathbb{R}^S_+ \times \mathbb{R}^D_+$, we write its projection on $\mathbb{R}^{S'}_+ \times \mathbb{R}^{D'}_+$ as $(x_{[S']}, y_{[D']})$.

Proposition 1: *In the economy* (G, R),
i) if the net transfer (x, y) *implemented by the* G*-flow* φ *is Pareto optimal, then transfers occur only between* S_+ *and* D_-*,* S_0 *and* D_0*,* S_- *and* D_+:

$$\varphi_{ij} > 0 \Rightarrow (i, j) \in (S_0 \times D_0) \cup (S_+ \times D_-) \cup (S_- \times D_+)$$

ii) $(x, y) \in \mathcal{PO}(G, R)$ *if and only if*

$$x_{[S_0]} = s_{[S_0]},\ y_{[D_0]} = d_{[D_0]} \text{ (hence } x_{S_0} = y_{D_0}) \tag{2}$$

$$x_{[S_+]} \geq s_{[S_+]},\ y_{[D_-]} \leq d_{[D_-]} \text{ and } x_{S_+} = y_{D_-}$$

$$x_{[S_-]} \leq s_{[S_-]},\ y_{[D_+]} \geq d_{[D_+]} \text{ and } x_{S_-} = y_{D_+}$$

Proof in the Appendix.

We will pay special attention to the following subset of $\mathcal{PO}(G,R)$, defined by the property that the short side gets its optimal transfer:

$$x_{[S_0]} = s_{[S_0]},\, y_{[D_0]} = d_{[D_0]}$$

$$x_{[S_+]} = s_{[S_+]},\, y_{[D_-]} \le d_{[D_-]} \text{ and } s_{S_+} = y_{D_-}$$
$$x_{[S_-]} \le s_{[S_-]},\, ,y_{[D_+]} = d_{[D_+]} \text{ and } x_{S_-} = d_{D_+} \tag{3}$$

(Note that by Lemma 2, the inequalities $x_{[S_-]} \le s_{[S_-]}$ and $y_{[D_-]} \le d_{[D_-]}$ cannot be all equalities). We will denote by $\mathcal{PO}^*(G,s,d)$, the set of allocations defined by (3). In the sequel we focus on allocations in $\mathcal{PO}^*(G,s,d)$,because under the Voluntary trade property, they are the only allocations Pareto optimal for any choice of preferences in $\mathcal{R}$ with peaks (s,d).

4 The egalitarian transfer solution

First we introduce some additional notation. For any finite set N and any $z \in \mathbb{R}^N$, z^* denotes the *order statistics* of z, obtained by rearranging the coordinates of z in increasing order: $z^{*1} \le z^{*2} \le \cdots \le z^{*n}$. Given two $z, w \in \mathbb{R}^N$, recall that z *Lorenz dominates* w, written z LD w, if for all $k, 1 \le k \le n$

$$\sum_{a=1}^{k} z^{*a} \ge \sum_{a=1}^{k} w^{*a}$$

We say that z is *Lorenz dominant* in the set A if z LD z' for all $z' \in A$. Lorenz dominance is a partial ordering, so not every set, even convex and compact, admits a Lorenz dominant element. On the other hand, in a convex set A there can be at most one Lorenz dominant element.

Now we define a family of descending algorithms, one of which define our solution below. These algorithms apply to the two subproblems $(G(S_-,D_+),s,d)$ and $(G(S_+,D_-),s,d)$, and we start by the former. For any $C \subseteq D_+$ we simply write $g(C)$ instead of $g(C) \cap S_-$. Fix a continuous weakly increasing path of net supplies $\mu \in \mathbb{R}_+ \cup \{\infty\} \to \gamma(\mu) \in \mathbb{R}_+^{S_-}$ such that $\gamma(0) = 0, \gamma(\infty) = s_{[S_-]}$. The system of inequalities with variable μ

$$\gamma_{g(C)}(\mu) \ge d_C \text{ for all } C \subseteq D_- \tag{4}$$

hods for $\mu = \infty$, even with strict inequalities because of short demand. In view of $\gamma_{S_-}(0) \le d_{D_+}$ there is a $\mu^1, 0 \le \mu^1 < \infty$, that is the smallest μ

such that (4) holds true, equivalently μ^1 is the largest μ such that one of the inequalities in (4) is tight.

As $C \to \gamma_{g(C)}(\mu^1) - d_C$ is submodular, the equality $\gamma_{g(C)}(\mu^1) = d_C$ is stable by union and intersection[11] of the sets C. We call C^1 the largest such subset and set $T^1 = g(C^1)$. By Lemma 1, the allocation $(\gamma_{[T^1]}(\mu^1), d_{[C^1]})$ is in $\mathcal{A}(G(T^1, C^1))$, i.e. we can give $\gamma(\mu^1)$ to the agents in C^1 by using all the resources in T^1 and no more.

In the restricted problem $(G(S_- \diagdown T^1, D_+ \diagdown C^1), s, d)$ we set $g^1(C) = g(C) \diagdown T^1$ for all $C \subseteq D_+ \diagdown C^1$. Then

$$\gamma_{g^1(C)}(\mu^1) \geq d_C \text{ for all } C \subseteq D_+ \diagdown C^1$$

because $\gamma_{g^1(C)}(\mu^1) + \gamma_{T^1}(\mu^1) = \gamma_{g(C \cup C^1)}(\mu^1) \geq d_C + d_{C^1}$. In fact $\gamma_{g^1(C)}(\mu^1) > d_C$ because $\gamma_{g^1(C)}(\mu^1) = d_C$ would imply that $C \cup C^1$ is tight at μ_1, contradicting the definition of C^1. We also have $\gamma_{S_- \diagdown T^1}(0) \leq d_{D_+ \diagdown C^1}$, so we can repeat the argument above in the restricted problem $(G(S_- \diagdown T^1, D_+ \diagdown C^1), s, d)$ to find the largest number $\mu^2, \mu^2 < \mu^1$, at which one of the inequalities $\gamma_{g^1(C)}(\mu) \geq d_C$ becomes an equality. We call C^2 the largest such subset of $D_+ \diagdown C^1$ and set $T^2 = g^1(C^2) = g(C^2) \diagdown g(C^1)$. We can achieve the allocation $\gamma_{[T^2]}(\mu^2)$ for the agents in T^2 by using all the resources in C^2 and no more.

Continuing in this fashion, we define a partition $C^1, C^2, \cdots,$ of D_+, a partition $T^1, T^2, \cdots,$ of S_-, and a strictly decreasing sequence $\mu^1 > \mu^2 > \cdots$, such that the allocation $x_{[S_-]} = (\gamma_{[T^1]}(\mu^1), \gamma_{[T^2]}(\mu^2), \cdots), y_{[D_+]} = d_{[D_+]}$ is obtained by assigning for all k the resources in C^k to the agents in $T^k = g(C^k) \diagdown g(C^1 \cup \cdots \cup C^{k-1})$.

The descending algorithm for $G(S_+, D_-)$ are defined similarly by means of a weakly increasing path $\mu \in \mathbb{R}_+ \cup \{\infty\} \to \delta(\mu) \in \mathbb{R}_+^{D_-}$ such that $\delta(0) = 0, \delta(\infty) = d_{[D_-]}$.

Proposition 2: *For any problem (G, s, d), the set $\mathcal{PO}^*(G, s, d)$, contains a Lorenz dominant element $(\overline{x}, \overline{y}) = \mathcal{E}(s, d)$, that we call the egalitarian transfer solution. The allocation $\overline{x}_{[S_-]}$ is obtained by the descending algorithm in $(G(S_-, D_+), s, d)$ along the path*

$$\gamma_i(\mu) = min\{\mu, s_i\} \text{ for all } i \in S_- \tag{5}$$

[11] Take two such subsets C, C' and compute $\gamma_{g(C \cup C')}(\mu^1) + \gamma_{g(C \cap C')}(\mu^1) \leq \gamma_{g(C)}(\mu^1) + \gamma_{g(C')}(\mu^1) = d_C + d_{C'} = d_{C \cup C'} + d_{C \cap C'}$ where the former inequality comes from $g(C \cup C') = g(C) \cup g(C'), g(C \cap C') \subseteq g(C) \cap g(C')$.

The allocation $\overline{y}_{[D_-]}$ is obtained by the descending algorithm in $(G(S_+, D_-), s, d)$ along the path

$$\delta_j(\mu) = min\{\mu, d_j\} \text{ for all } j \in D_- \tag{6}$$

(recall that $\overline{x}_{[S_+]} = s_{[S_+]}$ and $\overline{y}_{[D_+]} = d_{[D_+]}$). Proof in the Appendix.

Example 4. *In the absence of bilateral constraints, i.e., if $G = S \times D$, we already noticed, immediately after Lemma 2, that the decomposition reduces to one of the three (S_-, D_+), (S_+, D_-), or (S_0, D_0). Our solution simply applies the uniform rationing method of Sprumont (1991) to the long side of the market. A similar solution is discussed and characterized in Klaus et al. (1998). In their model, agents' endowments are known and preferences over "consumption" are reported. In our model, agents report their preferences regarding net trades. We do not assume that the planner has any knowledge regarding endowments or preferences.*

Example 5. *In the example of Figure 3, we have $S = S_+, D = D_-$, and the descending algorithm stops at $\mu^1 = 10, \mu^2 = 8, \mu^3 = 5$.*

Example 6. *School assignment*
In Figure 5, there are 8 schools in 4 different neighborhoods. In each neighborhood one school is overcrowded and the other is attended below its capacity. Each student can be transferred to a school in the same or an adjacent neighborhood. School S_1 has 27 excess students, which can be sent to schools D_1 and D_2. In the remaining of the graph, the number of excess students is below the vacancies,and schools D_3 and D_4 absorb all the excess from S_2, S_3, and S_4. The decomposition is $S_- = \{S_1\}, D_+ = \{D_1, D_2\}; S_+ = \{S_2, S_3, S_4\}, D_- = \{D_3, D_4\}$. The egalitarian transfer rule gives

$$(x_1, x_2, x_3, x_4) = (25, 5, 10, 15) \text{ and } (y_1, y_2, y_3, y_4) = (10, 15, 20, 10)$$

5 Properties of the egalitarian transfer rule

We discuss the incentives and equity properties of the egalitarian transfer rule, the basis of our characterization result in the next section. Those properties bear on the profile of individual preferences R, therefore instead of a problem (S, D, G, s, d), we consider now the economy (S, D, G, R) (or simply (G, R)). We use the notation $s[R_i], d[R_j]$ for the peak transfer of supplier i and demander j.

Definition: *Given the agents (S, D), a rule selects for every economy $(G, R) \in 2^{S \times D} \times \mathcal{R}^{S \cup D}$ a feasible allocation $\psi(G, R) \in \mathcal{A}(G)$.*

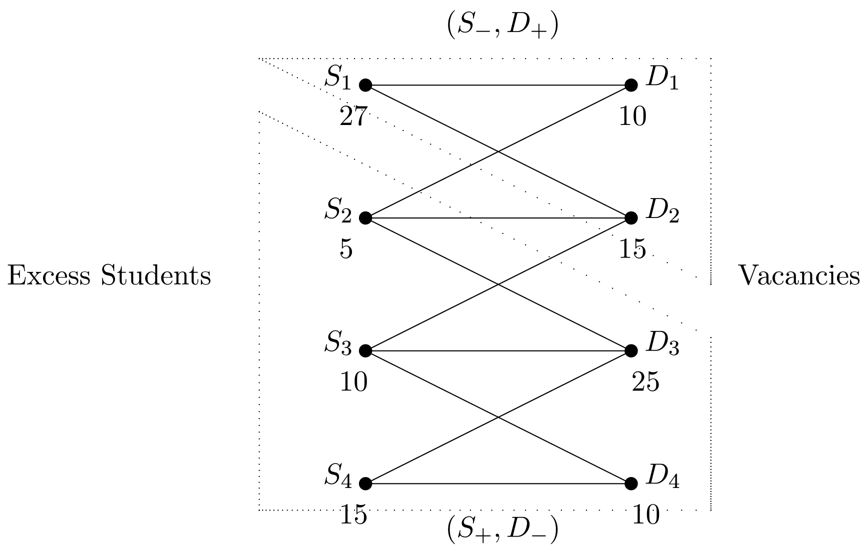

Figure 5: The school assignment example

We define first the *link monotonicity* property, requiring that an agent on either side of the market cannot be hurt by the access to new links. As discussed in the introduction, this ensures that no agent has an incentive to close a feasible link, revealing all feasible links to the manager is a dominant strategy:

Link monotonicity: *A rule* ψ *is link monotonic if for any economy* $(G, R) \in 2^{S\times D} \times \mathcal{R}^{S\cup D}$*, and any* $i \in S, j \in D$*, we have* $\psi_i(G \cup \{ij\}, R) R_i \psi_i(R, G)$*. And a similar statement where we exchange the role of demanders and suppliers.*

We show below that the egalitarian transfer rule is link monotonic. On the other hand the addition of a link ij may well hurt agents other than i, j. In the example in Figure 6 with short demand our rule picks the allocation $x_1 = 3, x_2 = 1$, and after the addition of the link S_2D_1 it gives $x_1 = x_2 = 2$.

Proposition 3: *The egalitarian transfer rule is link-monotonic.*

Proof in the Appendix.

In the rest of the section we discuss properties for which the graph G is fixed, so we write a rule simply as $\psi(R)$ for $R \in \mathcal{R}^{S\cup D}$. The next incentive property is the familiar *strategyproofness*. It is useful to decompose it into a monotonicity and an invariance condition.

Monotonicity: *A rule* ψ *is monotonic if one's net transfer is weakly increasing in her reported peak: for all* $R \in \mathcal{R}^{S\cup D}$*,* $i \in S, j \in D$ *and* $R'_i, R'_j \in \mathcal{R}$

$$s[R'_i] \leq s[R_i] \Rightarrow \psi_i(R'_i, R_{-i}) \leq \psi_i(R); \text{ and } d[R'_j] \leq d[R_j] \Rightarrow \psi_j(R'_j, R_{-j}) \leq \psi_j(R)$$

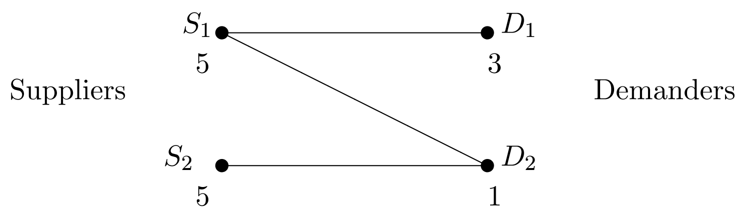

Figure 6: A new link may hurt non-involved agents

Invariance: *A rule ψ is invariant if for all $R \in \mathcal{R}^{S \cup D}$, $i \in S, j \in D$ and $R'_i, R'_j \in \mathcal{R}$*

$$\{s[R_i] < \psi_i(R) \text{ and } s[R'_i] \leq \psi_i(R)\} \text{ or } \{s[R_i] > \psi_i(R) \text{ and } s[R'_i] \geq \psi_i(R)\} \tag{7}$$
$$\Rightarrow \psi_i(R'_i, R_{-i}) = \psi_i(R)$$

and a similar statement when $j \in D$ reports $R'_j \in \mathcal{R}$ with a peak on the same side of $\psi_j(R)$ as the peak of R_j.

Strategyproofness: *A rule ψ is strategyproof if for all $R \in \mathcal{R}^{S \cup D}$, $i \in S, j \in D$ and $R'_i, R'_j \in \mathcal{R}$*

$$\psi_i(R) R_i \psi_i(R'_i, R_{-i}) \text{ and } \psi_j(R) R_j \psi_j(R'_j, R_{-j})$$

Each one of monotonicity or invariance implies *own-peak-only:* my net transfer only depends upon the peak of my preferences, and not on the way I compare transfers across my peak.

The next Lemma connects these three properties and Pareto optimality.

Lemma 3: *Monotonicity and invariance*
i) If a rule is monotonic and invariant, it is strategy-proof;
ii) An efficient and strategyproof rule is monotonic and invariant.

Proof: We omit the easy argument proving statement *i*), just as in the Sprumont (1991) model.

Statement ii) Fix an efficient (Pareto optimal) and strategyproof rule ψ, a preference profile $R \in \mathcal{R}^{S \cup D}$, a supplier $i \in S$, and an alternative preference $R'_i \in \mathcal{R}$. We use the notation $s_i = s[R_i], s'_i = s[R'_i]$, $R' = (R'_i, R_{-i})$, and $(s, d), (s', d)$ are the profiles of peaks at R and R' respectively; finally $x_i = \psi_i(R), x'_i = \psi_i(R')$.

We prove monotonicity for a supplier i (and omit the entirely similar argument for a demander). Suppose $s'_i \leq s_i$. We want to show $x'_i \leq x_i$. Distinguish two cases.

Case 1: $i \in S_-(s,d)$. Assume first $s'_i > x_i$. Then the decomposition at (s',d) is unchanged, $S_-(s,d) = S_-(s',d)$ so by efficiency (Proposition 1) $x'_i \leq s'_i$. Assume $x_i < x'_i$; then we have $x_i < x'_i \leq s'_i \leq s_i$, and we get a contradiction of strategyproofness (SP) for agent i at profile R. Assume next $s'_i \leq x_i$. Then $x_i < x'_i$ would give $s'_i < x'_i$, contradicting SP for agent i at R'.

Case 2: $i \in (S_0 \cup S_+)(s,d)$. Then efficiency gives $s_i \leq x_i$, so $x_i < x'_i$ would give $s'_i \leq s_i \leq x_i < x'_i$, hence a violation of SP for agent i at R'.

We show invariance next, again in the case of a supplier i. Under the premises of property (7) inside the left bracket, if $\psi_i(R') > \psi_i(R)$ we have $s'_i \leq \psi_i(R) < \psi_i(R')$, hence a violation of SP for agent i at R'. If $\psi_i(R') < \psi_i(R)$ we can find a preference R^*_i with peak $s^*_i = s_i$ such that $\psi_i(R')P^*_i\psi_i(R)$. Then, $\psi_i(R^*_i, R_{-i}) = \psi_i(R)$, so agent i with preferences R^*_i benefits by reporting s'_i. The proof under the premises of (7) inside the right bracket is identical.■

Proposition 4

The egalitarian transfer rule is monotonic and invariant, hence strategyproof as well.

Proof in the Appendix.

We now turn to equity properties. The familiar equity test of no envy must be adapted to our model because of the feasibility constraints. If supplier 1 envies the net transfer x_2 of supplier 2, it might not be possible anyway to give him x_2 because the demanders connected to agent 1 do not allow it. Alternatively, if we can exchange the net transfers of 1 and 2, this typically requires to construct a new flow and alter some of the other agents' allocations. In either case we submit that agent 1 has no legitimate claim against the allocation x.

An envy argument by agent 1 against agent 2 is legitimate only if it is feasible to improve upon agent 1's allocation without altering the allocation of anyone other than agent 2.

No envy: *Fix $G \in 2^{S\times D}$. A rule ψ satisfies no envy if for any preference profile $R \in \mathcal{R}^{S\cup D}$ and any $i_1, i_2 \in S$ such that $\psi_{i_2}(R)P_{i_1}\psi_{i_1}(R)$, there exists no $(x,y) \in A(G)$ such that*

$$\psi_i(R) = x_i \text{ for all } i \in S \setminus \{i_1, i_2\};\ \psi_j(R) = y_j \text{ for all } j \in D$$

$$\text{and } x_{i_1}P_{i_1}\psi_{i_1}(R) \tag{8}$$

and a similar statement where we exchange the role of demanders and suppliers.

Note that if i_1, i_2 have identical connections, $i_1 j \in G \Leftrightarrow i_2 j \in G$, then no envy implies $\psi_{i_1}(R) P_{i_1} \psi_{i_2}(R)$.

The familiar horizontal equity property must be similarly adapted to account for the bilateral constraints on transfers.

Equal treatment of equals: *Fix $G \in 2^{S \times D}$. A rule ψ satisfies ETE if for any preference profile $R \in \mathcal{R}^{S \cup D}$ and any $i_1, i_2 \in S$* such that $s_{i_1}[R_{i_1}] = s_{i_2}[R_{i_2}]$, *there exists no $(x, y) \in A(G)$ such that*

$$\psi_i(R) = x_i \text{ for all } i \in S \setminus \{i_1, i_2\} \ \psi_j(R) = y_j \text{ for all } j \in D$$
$$\text{and } |x_{i_1} - x_{i_2}| < |\psi_{i_1}(R) - \psi_{i_2}(R)| \tag{9}$$

and a similar statement where we exchange the role of demanders and suppliers.

Again, if i_1, i_2 have identical connections and preferences, ETE implies $\psi_{i_1}(R) = \psi_{i_2}(R)$.

Our definition of ETE is with regard to net trades. As is well-known this is stronger than a version of ETE stated in terms of preferences such as the one in Ching (1994). While the weaker version is implied by no envy, this is not the case for ETE. However under Pareto optimality, No envy implies ETE as we show below.

Proposition 5

i) No envy plus Pareto optimality imply equal treatment of equals;

ii) The egalitarian rule $\mathcal{E}$ satisfies no envy.

Proof: Statement *i*). Suppose the rule ψ violates ETE and check it violates no envy and/or Pareto optimality. Fix a profile R and two suppliers $1, 2$ such that $s_1[R_1] = s_2[R_2] = s^*$, and there exists (x, y) satisfying (9). Assume without loss $\psi_1(R) < \psi_2(R)$. Note that $x_1 + x_2 = \psi_1(R) + \psi_2(R)$ so only two cases are possible: $\psi_1(R) < x_1 \leq x_2 < \psi_2(R)$, or $\psi_1(R) < x_2 \leq x_1 < \psi_2(R)$. Assume the first case. If $s^* \geq \psi_2(R)$, supplier 1 envies 2 via (x, y); similarly $s^* \leq \psi_1(R)$ implies a violation of no envy. If $x_1 \leq s^* \leq x_2$, the profile of transfers (x, y) is Pareto superior to $\psi(R)$ (for both agents). If $x_2 < s^* < \psi_2(R)$, the profile $(x', y), x'_2 = s^*, x'_1 = x_1 + x_2 - s^*$, $x'_k = x_k$ else, is a convex combination of (x, y) and $\psi(R)$, so it is feasible ($\mathcal{A}(G)$ is convex), and Pareto superior to $\psi(R)$ (for both agents). The case $\psi_1(R) < s^* < x_1$ leads to a similar violation of PO.

In the second case, observe that the profile $(x', y), x'_1 = x'_2 = \frac{1}{2}(x_1 + x_2), x'_k = x_k$ else, is a convex combination of (x, y) and $\psi(R)$, so it is feasible and we are back to the first case.

Statement *ii*). Let ψ be the egalitarian rule, and R be a profile at which supplier 1 envies supplier 2 via (x, y). We have $x_1 + x_2 = \mathcal{E}_1(R) + \mathcal{E}_2(R)$

and (x, y) coincides with $\mathcal{E}$ elsewhere. As $\mathcal{E}(R)$ Lorenz dominates x we must have $|x_2 - x_1| > |\mathcal{E}_2(R) - \mathcal{E}_1(R)| > 0$. If $x_2 - x_1$ and $\mathcal{E}_2(R) - \mathcal{E}_1(R)$ have the same sign, then we have $x_1 < \mathcal{E}_1(R) < \mathcal{E}_2(R) < x_2$ (or a symmetric condition by exchanging 1 and 2). Now $\mathcal{E}_2(R) P_1 \mathcal{E}_1(R)$ implies $s[R_1] > \mathcal{E}_1(R)$, hence $\mathcal{E}_1(R) P_1 x_1$, contradiction. If $x_2 - x_1$ and $\mathcal{E}_2(R) - \mathcal{E}_1(R)$ have opposite signs, convexity of $\mathcal{A}(G)$ implies that $(x', y), x'_1 = x'_2 = \frac{1}{2}(x_1 + x_2), x'_k = \mathcal{E}_k(R)$ else, is feasible. So (x', y) Lorenz dominates $\mathcal{E}(R)$, a contradiction.■

6 Characterization result

Our last axiom is a basic incentive property stating that each agent is entitled to keep her endowment of the commodity and refuse to trade.

Voluntary trade: *Fix $G \in 2^{S \times D}$. A rule ψ guarantees voluntary trade if for all $R \in \mathcal{R}^{S \cup D}$, $i \in S \cup D$, we have $\psi_i(R) R_i 0$.*

Note that link monotonicity implies voluntary trade.

Theorem: *The egalitarian transfer rule $\mathcal{E}$ is characterized by Pareto optimality, strategyproofness, voluntary trade, and equal treatment of equals.*
Proof in the Appendix.

7 Concluding comments

Summary: Our model generalizes to a considerable extent the standard one-sided division model introduced by Sprumont (1991) and its two-sided version considered by Klaus *et al.* (1998). This extension generates several hurdles because of additional feasibility constraints imposed by the bipartite graph. The division of the graph in three submarkets in which there is excess supply, balancedness, and excess demand respectively, gives the structure of the Pareto optimal allocations. Then the feasibility constraints are captured by a system of submodular upper bounds on coalitional shares in the excess supply segment of the market, and a system of supermodular lower bounds in the excess demand segment. Finally equal treatment of equals must be restricted to those equalizing transfers that do not affect the shares of agents not involved in the transfer. After those new features are properly incorporated, our *egalitarian transfer* solution is characterized by the combination of efficiency, strategyproofness and equal treatment of equals. We conjecture that the egalitarian transfer solution is also robust against coordinated misreport of preferences by any subgroups of agents, i.e. the solution is group strategyproof.

Companion paper: In Bochet *et al.* (2009) we study the generalization of Sprumont's model to arbitrary bipartite graphs. The suppliers are now passive, they *must* unload a given amount of the commodity among the set of *receivers*, who each have a private ideal consumption level. Thus the receivers may have to consume more or less than their peak. This one-sided model is simpler, on the other hand we endow each receiver with some capacity constraints (from above and below), which complicates the analysis.

To describe the set of Pareto optimal allocations, we use the same decomposition as in Lemma 2, but its interpretation is different. Receivers in D_- absorb only the resources of S_+, and end up consuming less than their peak, while those in D_+ absorb all resources in S_-, and end up consuming more than their peak.

We characterize a rule similar to the egalitarian transfer rule by means of efficiency, strategyproofness and equal treatment of equals.

Extensions: First, following Sasaki (1997), Ehlers and Klaus (2003) for the division model under single peaked preferences (Sprumont, 1991), we can think of a "discrete" variant where indivisible units have to be traded between sellers and demanders. Both papers above offer a characterization of the randomized uniform rule, and it is likely that their result can be adapted to our model with bilateral constraints. Second, we have considered here only rules which treat agents as symetrically as possible given the bilateral constraints. But exogenous priority rights may apply to agents on the long side of the market (those who are rationed), in which case we want to understand what incentive compatible rules respect these constraints.

In the one-sided fair division model of Sprumont (1991), the rich family of allotment rules (Barbera, Jackson and Neme, 1997) preserves the incentive properties of the egalitarian rule while allowing a very different treatment of the agents. Similarly the family of fixed paths rules (Moulin, 1999) is characterized by the combination of efficiency, strategyproofness, resource monotonicity and consistency. Further research questions include extending both families to our model.

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8 Appendix: proofs

8.1 Proposition 1

Statement (i) We fix $(x, y) \in \mathcal{PO}(G, R)$ and a G-flow φ implementing (x, y) ((1)). To show that condition $i)$ is satisfied we proceed in 2 steps.

Step (i).1 We show

$$ij \in G(S_+, D_0 \cup D_+) \Rightarrow \varphi_{ij} = 0 \tag{10}$$

The proof is by contradiction. Pick $i^*j^* \in G(S_+, D_0 \cup D_+)$ such that $\varphi_{i^*j^*} > 0$. We construct first a *transfer path* $i^*j^*, j^*i_1, i_1j_1, j_1i_2, \cdots, j_{K-1}i_K$, entirely in $G(S_0 \cup S_-, D_0 \cup D_+)$ except for the first edge i^*j^*, and such that *i)* $\varphi_{i_kj_k} > 0$ for every odd edge i_kj_k; and *ii)* $x_{i_K} < s_{i_K}$ or $y_{j_{K-1}} > d_{j_{K-1}}$. Note that if $y_{j_{K-1}} > d_{j_{K-1}}$ then the last edge $j_{K-1}i_K$ is even and i_K can be chosen arbitrarily. Note also that some agents may appear multiple times in the path.

If $y_{j^*} > d_{j^*}$, the transfer path is simply i^*j^*, j^*i_1 for an arbitrary i_1 in $S_0 \cup S_-$. Suppose next $y_{j^*} \leq d_{j^*}$. Then by Lemma 2 the set $T^1 = g(j^*) \cap (S_0 \cup S_-)$ is non empty: if $j^* \in D_+$ because $(G(S_-, D_+), s, d)$ has short-demand, and if $j^* \in D_0$ because $(G(S_0, D_0), s, d)$ is balanced. If there exists $i_1 \in T^1$ such that $x_{i_1} < s_{i_1}$, then the transfer path ends with i_1. Assume for all $i_1 \in T^1$, $x_{i_1} \geq s_{i_1}$, and observe from Lemma 2 that $d_{j^*} \leq s_{T^1}$. Combining this with $y_{j^*} \leq d_{j^*}$ and $x_{T^1} \geq s_{T^1}$, and with the fact that some of the net transfer of agent j^* comes from i^*, outside $S_0 \cup S_-$, we see that some of the net transfer of coalition T^1 goes to other agents than j^*. By statement *i)* in Lemma 2, these agents are in $D_0 \cup D_+$, hence there exists $i_1 \in T^1$ and $j_1 \in (D_0 \cup D_+) \diagdown \{j^*\}$ such that $\varphi_{i_1j_1} > 0$. If j_1 can be chosen such that $y_{j_1} > d_{i_1}$, then the transfer path ends with i_1j_1, j_1i_2, where i_2 is arbitrary in $S_0 \cup S_-$. Otherwise we have $y_{j_1} \leq d_{j_1}$, and we consider $T^2 = g(\{j^*, j_1\}) \cap (S_0 \cup S_-)$. If there exists $i_2 \in T^2$ such that $x_{i_2} < s_{i_2}$, then our transfer path will end at i_2. Else for all $i_2 \in T^2$, $x_{i_2} \geq s_{i_2}$. Then we have

$$y_{j^*} + y_{j_1} \leq d_{j^*} + d_{j_1} \leq s_{T^2} \leq x_{T_2}$$

(where the second inequality comes from Lemma 2). Some of the net transfer of $\{j^*, j_1\}$ comes from outside $S_0 \cup S_-$, hence there exists $i_2 \in T^2$ and $j_2 \in (D_0 \cup D_+) \diagdown \{j^*, j_1\}$ such that $\varphi_{i_2j_2} > 0$. Repeating this construction, after finitely many steps we must reach an i_K such that $x_{i_K} < s_{i_K}$, or j_{K-1} such that $y_{j_{K-1}} > d_{j_{K-1}}$

With our transfer path in hand, we now look for a Pareto improvement of (x, y). This is easy if $x_{i^*} > s_{i^*}$, because we can reduce the net transfer of i^* by a small amount, and at the same time increase supplier i_K's transfer, or decrease demander j_{K-1}'s transfer, without changing that of any other agent (along or outside the path). We simply take away an ε-flow on i^*j^*, and add it to the (possibly nil) flow on j^*i_1, then take it away from i_1j_1, add it to j_1i_2, $\cdots$, until we finally either take it away from $i_{K-1}j_{K-1}$ (if

$y_{j_{K-1}} > d_{j_{K-1}}$) or add it to $j_{K-1}i_K$ (if $x_{i_K} < s_{i_K}$). Of course ε must be smaller than the flow on any odd edge. Thus we have a contradiction.

Assume next $x_{i^*} \leq s_{i^*}$. Then we construct a second transfer path $i^*j'_1, j'_1i'_1, i'_1j'_2, j'_2i'_2, \cdots, j'_Li'_L$, entirely in $G(S_+, D_-)$ such that $i)$ $\varphi_{i'_lj'_l} > 0$ for every even edge $j'_li'_l$ up to $j'_Li'_L$ and $x_{i'_L} > s_{i'_L}$, or $ii)$ $y_{j'_L} < d_{j'_L}$ and $\varphi_{i'_lj'_l} > 0$ for every even edge $j'_li'_l$ up to $j'_{L-1}i'_{L-1}$. The argument is similar to the one above: because $(G(S_+, D_-), s, d)$ has short-supply, the set $C^1 = f(i^*) \cap D_-$ is non empty; if it contains j'_1 such that $y_{j'_1} < d_{j'_1}$, the path stops at $j'_1i'_1$ where i'_1 is arbitrary; else we have $y_{C^1} \geq d_{C^1}$, that we combine with $s_{i^*} < d_{C^1}$ ($(G(S_+, D_-), s, d)$ has short supply) and $x_{i^*} \leq s_{i^*}$ to deduce $x_{i^*} < y_{C^1}$, hence some of the net transfer of C^1 comes from other suppliers than i^* and we can find $j'_1 \in C^1, i'_1 \in S_+ \diagdown \{i^*\}$ such that $\varphi_{i'_1j'_1} > 0$; if i'_1 can be chosen such that $x_{i'_L} > s_{i'_L}$, the path ends right there, otherwise we consider $C^2 = f(\{i^*, i'_1\}) \cap D_-$, for which we have $x_{i^*} + x_{i'_1} < s_{i^*} + s_{i'_1} < d_{C^2}$, and so on.

Check that the concatenation of the two transfer paths leads to a Pareto improvement, thus concluding the proof of Step (i).1. If in the first transfer path $x_{i_K} < s_{i_K}$ and in the second $x_{i'_L} > s_{i'_L}$, we move an ε-flow from i'_L to i_K: we take ε away from $\varphi_{i'_Lj'_L}$, add it to $\varphi_{i'_{L-1}j'_L}$, and so on, until we take ε away from $\varphi_{i^*j^*}$, add it to $\varphi_{i_1j^*}$, and continue as two paragraphs above until we reach i_K. Similarly if $y_{j_{K-1}} > d_{j_{K-1}}$ and $x_{i'_L} > s_{i'_L}$, we take ε away from the flow on all odd edges in the path $i'_Lj'_L, j'_Li'_{L-1}, i'_{L-1}j'_{L-1}, \cdots, j'_1i^*, i^*j^*, \cdots, i_{K-1}j_{K-1}$, so the net transfer of both i'_L and j_{K-1} decrease. The argument is similar when $y_{j'_L} < d_{j'_L}$.

Step (i).2

$$ij \in G(S_+ \cup S_0, D_+) \Rightarrow \varphi_{ij} = 0$$

The proof mimics that of Step (i).1, hence is omitted.

Statement (ii) We fix $(x, y) \in \mathcal{PO}(G, R)$ and a G-flow φ implementing (x, y). From statement $i)$ the agents in D_+ receive all their commodities from S_- and nothing else, hence $y_{D_+} = x_{S_-}$. Those in D_- get all their commodities from S_+ and nothing else, so $y_{D_-} = x_{S_+}$. Thus the commodities in S_0 go to agents in D_0; as $(G(S_0, D_0), s, d)$ is balanced, Pareto optimality requires each of these agents to obtain exactly their peak share. We show next $x_{[S_+]} \geq s_{[S_+]}$ and $y_{[D_-]} \leq d_{[D_-]}$, using once again a "transfer path" argument.

Suppose $i_1 \in S_+$ is such that $x_{i_1} < s_{i_1}$. We construct, exactly like in step (i).1 (second construction), a path $i^*j'_1, j'_1i'_1, i'_1j'_2, j'_2i'_2, \cdots, j'_Li'_L$, entirely in $G(S_+, D_-)$ such that $i)$ $\varphi_{i'_lj'_l} > 0$ for every even edge $j'_li'_l$ up to $j'_Li'_L$ and $x_{i'_L} > s_{i'_L}$, or $ii)$ $y_{j'_L} < d_{j'_L}$ and $\varphi_{i'_lj'_l} > 0$ for every even edge $j'_li'_l$ up to

$j'_{L-1}i'_{L-1}$. The same transfer argument contradicts the Pareto optimality of (x, y). This proves $x_{[S_+]} \geq s_{[S_+]}$.

Next we suppose $j_1 \in D_-$ is such that $y_{j_1} > d_{j_1}$. We construct a transfer path $j_1 i_1, i_1 j_2, \cdots, j_K i_K$, entirely in $G(S_+, D_-)$ and such that *i)* $\varphi_{i_k j_k} > 0$ for every odd edge $j_k i_k$; and *ii)* $x_{i_K} > s_{i_K}$ or $y_{j_{K-1}} < d_{j_{K-1}}$. This will allow a Pareto improving transfer in the usual way. To build the first edge of the path, consider the non empty set $T^1 = \{i \in S_+ | \varphi_{ij_1} > 0\}$. If it contains i_1 such that $x_{i_1} > s_{i_1}$ we can stop. Otherwise we have $x_{[T^1]} \leq s_{[T^1]}$, but we just proved $x_{[S_+]} \geq s_{[S_+]}$ so $x_{[T^1]} = s_{[T^1]}$; moreover $x_{T^1} = y_{j_1}$ by definition of T^1, so using the fact that $(G(S_+, D_-), s, d)$ has short supply we have $d_{j_1} < y_{j_1} = s_{T^1} < d_{f(T^1)}$, and we conclude that $f(T^1) \diagdown j_1$ is non empty. Take an arbitrary j_2 in that set and a corresponding $i_1 \in T^1$ such that $i_1 j_2 \in G$. If $y_{j_2} < d_{j_2}$ our path is $j_1 i_1, i_1 j_2, j_2 i_2$ with an arbitrary i_2, else $y_{j_2} \geq d_{j_2}$ and we consider $T^2 = \{i \in S_+ | \varphi_{ij_1} + \varphi_{ij_2} > 0\}$, the set of suppliers with a positive flow to j_1 or to j_2 or both. If T^2 contains i_2 such that $x_{i_2} > s_{i_2}$, our path stops at i_2. Otherwise we have $x_{[T^2]} = s_{[T^2]}$ and $x_{T^2} = y_{j_1} + y_{j_2}$, so

$$d_{j_1} + d_{j_2} < y_{j_1} + y_{j_2} = s_{T^2} < d_{f(T^2)}$$

hence $f(T^1) \diagdown \{j_1, j_2\}$ is non empty. And so on.The proof of the "only if" statement is complete.

Suppose now that an allocation (x, y) defined in (2) is Pareto dominated by some $(x', y') \in \mathcal{A}(G)$. As each supplier in S_+ gets at least her peak transfer at x, the same is true at x': $x_{[S_+]} = x'_{[S_+]}$. Similarly, as each demander in D_- gets at most her peak transfer at y, then at y', by single-peakedness of preferences, $y_{[D_-]} \leq y'_{[D_-]}$.

Suppose $y_{[D_-]} \underset{\neq}{\leq} y'_{[D_-]}$. Because D_- can only receive commodity from S_+ we have $y'_{D_-} \leq x'_{S_+} = x_{S_+} = y_{D_-}$, contradiction. A symmetric argument shows $x_{[S_-]} = x'_{[S_-]}$.

8.2 Proposition 2

The descending algorithms along the paths (5),(6), define the Lorenz dominant element in $\mathcal{PO}^*(G, s, d)$.

Proposition 1 says that the set $\mathcal{PO}^*(G, s, d)$ (system (3)) is the cartesian product of three sets: $\{(s_{[S_0]}, d_{[D_0]})\}$, $\mathcal{PO}^*(G(S_-, D_+), s, d)$, and $\mathcal{PO}^*(G(S_+, D_-), s, d)$. To prove that $(\overline{x}, \overline{y})$, defined by the two algorithms (5),(6) is Lorenz dominant in $\mathcal{PO}^*(G, s, d)$, it is therefore enough to show separately that $(\overline{x}_{[S_-]}, d_{[D_+]})$, is Lorenz dominant in $\mathcal{PO}^*(G(S_-, D_+), s, d)$ and $(s_{[S_0]}, \overline{y}_{[D_-]})$ is Lorenz dominant in $\mathcal{PO}^*(G(S_+, D_-), s, d)$. We prove the former and omit the similar argument for the latter.

We simplify notation by writing the suppliers' net transfers in S_- as x, instead of $x_{[S_-]}$. Recall that the definition of $\overline{x}$ involves two parallel partitions of D_+ and S_- : $D_+ = C^1 \cup \cdots \cup C^k \cup \cdots$, and $S_- = T^1 \cup \cdots \cup T^k \cup \cdots$, where $T^k = g(C^k) \diagdown g(C^1 \cup \cdots \cup C^{k-1})$; moreover in $\overline{x}$, the demanders in C^k receive transfers only from the suppliers in T^k, and those transfers are the optimal ones. Finally the net transfer of supplier $i \in T^k$ is $\overline{x}_i = \min\{\mu^k, s_i\}$.

We further partition T^k as follows

$$A^k = \{i \in T^k | \overline{x}_i < s_i \Leftrightarrow \mu^k < s_i\};\ B^k = \{i \in T^k | \overline{x}_i = s_i \Leftrightarrow \mu^k \geq s_i\}$$

The set A^1 is non empty because $\overline{x}_{T^1} = \sum_{T^1} \min\{\mu^1, s_i\} = d_{C^1} < s_{T^1}$. The set A^2 is non empty because

$$s_{T^2} \geq \sum_{T^2} \min\{\mu^1, s_i\} > d_{C^2} = \sum_{T^2} \min\{\mu^2, s_i\}$$

where the strict inequality is explained in the definition of the ascending algorithm.

Repeating this argument shows $A^k \neq \varnothing$ for all k. Next we label the agents in S_- as $\{1, \cdots, |S_-|\}$ in such a way that the sequence $\overline{x}_i$ is weakly decreasing and moreover

- the first $|A_1|$ terms cover A^1
- the next terms cover a possibly empty subset $\widetilde{B}^1$ of B^1
- the next $|A^2|$ terms cover A^2
- the next terms cover a possibly empty subset $\widetilde{B}^2$ of $B^1 \cup B^2$

and so on. This is possible because in A^k everyone gets μ^k and the sequence μ^k decreases strictly. Before A^k we need not pick any coordinate in $B^{k'}, k' \geq k$, because such an agent receives no more than μ^k.

We fix now an arbitrary $(x, d_{[D_+]}) \in \mathcal{PO}^*(G(S_-, D_+), s, d)$ and check that x is Lorenz dominated by $\overline{x}$. We use the notation $x^*(i) = \sum_{j=|S_-|}^{|S_-|-i+1} x^{*j}$ (recall x^* is the order statistics of x), so that $x_T \leq x^*(|T|)$ for all T. If $T \subset S_-$ is such that $x_T = x^*(|T|)$ we say that T is an *x-tail*. From our labeling of S_-, any subset $\{\overline{x}_1, \cdots, \overline{x}_i\}$ is an $\overline{x}$-tail. We want to prove $x^*(t) \geq \overline{x}^*(t)$ for all $t = 1, \cdots, |S_-|$.

By feasibility $x_{T^1} \geq d_{C^1} = \overline{x}_{T^1}$ and by Pareto optimality and Proposition 1, $x \leq \overline{x}$ in B^1. Therefore $x_T \geq \overline{x}_T$ for all $T \subseteq T^1$ such that $T^1 \diagdown T \subseteq B^1$. In particular

$$x_T \geq \overline{x}_T \text{ for all } T \text{ s. t. } A^1 \subseteq T \subseteq A^1 \cup \widetilde{B}^1 \tag{11}$$

If the above T is an $\overline{x}$-tail (i.e., if $T \diagdown A^1$ contains the largest elements of $\widetilde{B}^1$), (11) gives $\overline{x}^*(|T|) \leq x_T \leq x^*(|T|)$. Next we note that $\frac{x^*(t)}{t}$ decreases weakly in t, so that for $t \leq |A_1|$ we have

$$\frac{x^*(t)}{t} \geq \frac{x^*(|A_1|)}{|A_1|} \geq \frac{x(A_1)}{|A_1|} \geq \frac{\overline{x}(A_1)}{|A_1|} = \frac{\overline{x}^*(t)}{t}$$

where the equality is because $\overline{x}$ is egalitarian in A^1. We have proved the desired inequality $x^*(t) \geq \overline{x}^*(t)$ up to $t = |A^1 \cup \widetilde{B}^1|$.

Next consider T^2. Feasibility implies $x_{T^1 \cup T^2} \geq d_{C^1 \cup C^2} = \overline{x}_{T^1 \cup T^2}$ and Pareto optimality gives $x \leq \overline{x}$ in $B^1 \cup B^2$. Therefore

$$x_T \geq \overline{x}_T \text{ for all } T \text{ s.t. } A^1 \cup \widetilde{B}^1 \cup A^2 \subseteq T \subseteq A^1 \cup \widetilde{B}^1 \cup A^2 \cup \widetilde{B}^2 \tag{12}$$

Again if we choose for T an $\overline{x}$-tail, the inequality $x^*(t) \geq \overline{x}^*(t)$ follows at once for t s.t. $|A^1 \cup \widetilde{B}^1 \cup A^2| \leq t \leq |A^1 \cup \widetilde{B}^1 \cup A^2 \cup \widetilde{B}^2|$. If t is such that $t = |A^1 \cup \widetilde{B}^1| + a \leq |A^1 \cup \widetilde{B}^1 \cup A^2|$, we pick an $\overline{x}$-tail $T, A^1 \cup \widetilde{B}^1 \subset T \subset A^1 \cup \widetilde{B}^1 \cup A^2$, with $|T| = t$. Because $\overline{x}$ is egalitarian in A^2, we have

$$\overline{x}^*(t) = \overline{x}_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|}\overline{x}_{A^2} = (1 - \frac{a}{|A^2|})\overline{x}_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|}\overline{x}_{A^1 \cup \widetilde{B}^1 \cup A^2}$$

We claim

$$x^*(t) \geq x_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|}x_{A^2} = (1 - \frac{a}{|A^2|})x_{A^1 \cup \widetilde{B}^1} + \frac{a}{|A^2|}x_{A^1 \cup \widetilde{B}^1 \cup A^2} \tag{13}$$

which will imply $x^*(t) \geq \overline{x}^*(t)$ because $x_T \geq \overline{x}_T$ is true both for $A^1 \cup \widetilde{B}^1$ and $A^1 \cup \widetilde{B}^1 \cup A^2$.

The claim follows from the following fact: if X, Y, Z are three disjoint subsets, we have

$$x^*(|X| + |Y|) \geq x_X + \frac{|Y|}{|Y| + |Z|}x_{Y \cup Z}$$

Indeed $\frac{|Y|}{|Y|+|Z|}x_{Y \cup Z}$ is no more than the sum of the $|Y|$ largest terms in $x_{[Y \cup Z]}$, and x_X is no more than the sum of the $|X|$ largest terms in $x_{[X]}$. Applying this inequality to $X = A^1 \cup \widetilde{B}^1, Y = T \diagdown (A^1 \cup \widetilde{B}^1)$ and $Z = (A^1 \cup \widetilde{B}^1 \cup A^2) \diagdown T$ gives (13). This shows $x^*(t) \geq \overline{x}^*(t)$ fro all $t \leq |A^1 \cup \widetilde{B}^1 \cup A^2 \cup \widetilde{B}^2|$. A straightforward induction argument concludes the proof.

8.3 Proposition 3

The egalitarian transfer rule is link-monotonic.

Fix the economy (G, R), and $i^* \in S$, $j^* \in D$ such that $i^*j^* \notin G$. Set $G' = G \cup \{i^*j^*\}$, we must show $\mathcal{E}_{i^*}(R, G')R_{i^*}\mathcal{E}_{i^*}(R, G)$ and $\mathcal{E}_{j^*}(R, G')R_{j^*}\mathcal{E}_{j^*}(R, G)$.

Write $S_+, D_+, .., $ and $S'_+, D'_+, ..,$ for the Gallai-Edmonds decompositions of G and G'. If $i^* \in S_+ \cup S_0$ and $j^* \in D_+ \cup D_0$, then these two decompositions coincide and $\mathcal{E}(R, G) = \mathcal{E}(R, G')$. From now on we assume $i^* \in S_-$.

Case 1: $j^* \in D_+$

The GE decompositions in G and G' are identical, so we can restrict attention to $(S_-, D_+, G(S_-, D_+))$, because the links of D_+ with $S_+ \cup S_0$ play no role. The egalitarian allocation denoted (x, y) at $G(S_-, D_+)$ gives (see section 4) the partitions $C^1, C^2, \cdots,$ of D_+, and $T^1, T^2, \cdots,$ of S_-, as well as the strictly decreasing sequence $\mu^1 > \mu^2 > \cdots$:

$$x_{[S_-]} = (\gamma_{[T^1]}(\mu^1), \gamma_{[T^2]}(\mu^2), \cdots), y_{[D_+]} = d_{[D_+]}$$

where $\gamma_i(\mu) = \min\{\mu, s_i\}$. The key fact is that we assign for all k the resources in C^k to the agents in $T^k = g(C^k) \diagdown g(C^1 \cup \cdots \cup C^{k-1})$ and only to those.

Assume $i^* \in T^k, j^* \in C^{k'}$. If $k \leq k'$, we see that the partitions C^l, T^l and the sequence μ^l are unchanged by the addition of link i^*j^*: for instance $g'(C^1) = g(C^1)$ and $g'(C) \geq g(C)$ for all C, so C^1, T^1, and μ^1 are unchanged; then $g'(C^2) \diagdown T^1 = g(C^1) \diagdown T^1$ and $g'(C) \diagdown T^1 \geq g(C) \diagdown T^1$ for all C, so C^2, T^2, and μ^2 are unchanged; and so on. Thus the egalitarian allocation does not change.

Assume next $k > k'$, so that the partitions and parameters may change at $G'(S_-, D_+)$. Let x' be the new egalitarian allocation for S_-, we assume $x'_{i^*} < x_{i^*} = \min\{\mu^k, s_{i^*}\}$, and derive a contradiction of the Lorenz optimality of x'. Set $A = C^1 \cup \cdots \cup C^k$, $B = T^1 \cup \cdots \cup T^k$ and note two facts: $g'(A) = B$, and the egalitarian allocation x at G is such that $\min\{\mu^k, s_i\} \leq x_i$ for all $i \in B$.

Let φ' be the G'-flow implementing the allocation (x', d). As in the proof of Proposition 1, we construct a *transfer path* $i^*j_1, j_1i_1, i_1j_2, j_2i_2, \cdots, j_Ti_T$, entirely in $G'(B, A)$ such that *i)* $\varphi'_{i_tj_t} > 0$ for every even edge i_tj_t; and *ii)* $x'_{i_T} > \min\{\mu^k, s_{i_T}\}$.

Set $A^1 = f'(i^*) \cap A$. We claim that $B^1 = \{i_1 \in B \setminus \{i^*\} : \varphi'_{i_1j_1} > 0 \text{ for some } j_1 \in A^1\}$ is non-empty. Otherwise we have

$$d_{A^1} = x'_{i^*} < x_{i^*} \leq d_{f(i^*) \cap A} \leq d_{A^1}$$

If there exists $i_1 \in B^1$ such that $x'_{i_1} > \min\{\mu^k, s_{i_1}\}$, then the path stops at $j_1 i_1$. Otherwise, let $A^2 = f'(B^1 \cup \{i^*\}) \cap A$ and consider the set $B^2 = \{i_2 \in B \setminus B^1 \cup \{i^*\} : \varphi'_{i_2 j_2} > 0$ for some $j_2 \in A^2\}$. It is non empty, otherwise we have

$$d_{A^2} = x'_{B^1 \cup \{i^*\}} < \sum_{B^1 \cup \{i^*\}} \min\{\mu^k, s_i\} \leq x_{B^1 \cup \{i^*\}} \leq d_{f(B^1 \cup \{i^*\}) \cap A} \leq d_{A^2}$$

If there exists $i_2 \in B^2$ such that $x'_{i_2} > \min\{\mu^k, s_{i_2}\}$, then then the path stops at $j_2 i_2$. Otherwise, we continue iteratively until we find an i_T such that $x'_{i_T} > \min\{\mu^k, s_{i_T}\}$.

By Pareto optimality, we have $x'_{i_T} \leq s_{i_T}$, therefore $x'_{i_T} > \mu^k$, while by assumption $x'_{i^*} < \min\{\mu^k, s_{i^*}\}$. Now our transfer path provides a feasible Pigou-Dalton transfer at x' between i^* and i_T: we take away from φ' an ε-flow from $j_T i_T$, add it to $i_{T-1} j_T$, take it from $j_{T-1} i_{T-1}$, etc.., until we add it to $i^* j_1$. This contradicts the Lorenz optimality of x'.

Case 2: $j^* \in D_- \cup D_0$

Now adding the link $i^* j^*$ can change the GE decomposition. If in the new decomposition, $i^* \in S'_+ \cup S'_0$ then i^* cannot be worse off because he gets his peak transfer. Now we assume $i^* \in S'_-$, implying $j^* \in D'_+$. We restrict attention to $G'(S'_-, D'_+)$.

Write (x', d) the egalitarian transfer solution at G'. The corresponding partitions are T'^k, C'^k and the parameters μ'^k. Assume, without loss of generality, that in the descending algorithm at G', $i^* \in T'^2$ and $x'_{i^*} = \min\{\mu'^2, s_{i^*}\}$. Then, $j^* \in C'^k$ for some $k \geq 2$ (because $G'(T'^k, C'^l) = \varnothing$ if $k > l$).

We let (x, y) be the egalitarian allocation at G and let φ be a G-flow which implements it. We assume $x_{i^*} > \min\{\mu'^2, s_{i^*}\}$, and derive a contradiction of the Lorenz optimality of (x, y). We construct again, a *transfer path* $i^* j_1, j_1 i_1, i_1 j_2, j_2 i_2, \cdots, j_T i_T$, entirely in $G(S'_- \setminus T'^1, D'_+)$ such that *i)* $\varphi_{i^* j_1}, \varphi_{i_{t-1} j_t} > 0$ for every odd edge $i_{t-1} j_t$; and *ii)* $x_{i_T} < \min\{\mu'^2, s_{i_T}\}$. Note that some agents may appear multiple times in the path.

Let $A^1 = \{j_1 \in D : \varphi_{i^* j_1} > 0\}$. Observe that $A^1 \subseteq D'_+ \setminus C'^1$, and $d_{A^1} = x_{i^*}$. Set $B^1 = g'(A^1) \cap \{S'_- \setminus T'^1\}$. We have $i^* \in B^1$ and claim $B^1 \setminus i^* \neq \emptyset$: otherwise in G', $A^1 \subseteq D'_+ \setminus C'^1$ is only connected to i^* and to some suppliers outside $S'_- \setminus T'^1$; the transfers to A^1 in any flow achieving x' are entirely borne by $G'(S'_- \setminus T'^1, D'_+)$, therefore $x'_{i^*} \geq d_{A^1}$; now $d_{A^1} = x_{i^*}$ gives $x'_{i^*} \geq x_{i^*}$, contradicting our assumption $x'_{i^*} < x_{i^*}$.

If there exists $i_1 \in B^1 \setminus i^*$ such that $x_{i_1} < \min\{\mu'^2, s_{i_1}\}$, then the path stops at $j_1 i_1$. Otherwise, let $A^2 = \{j_2 \in D : \varphi_{i_1 j_2} > 0$ for some $i_1 \in B^1\}$.

We have as above $A^2 \subseteq D'_+ \setminus C^1$ and $d_{A^2} = x_{B^1}$. Consider the set $B^2 = g'(A^2) \cap \{S'_- \setminus T'^1\}$. We have $B^1 \subseteq B^2$ and we claim $B^2 \setminus B^1 \neq \emptyset$. Otherwise in G', $A^2 \subseteq D'_+ \diagdown C'^1$ is only connected to B^1 and to some suppliers outside $S'_- \setminus T'^1$; the transfers to A^2 in any flow achieving x' are entirely borne by $G'(S'_- \setminus T'^1, D'_+)$, therefore $x'_{B^1} \geq d_{A^2} = x_{B^1}$. But we have assumed $x_i \geq \min\{\mu'^2, s_i\}$ for all $i \in B^1 \setminus i^*$, and $B^1 \subseteq S'_- \setminus T'^1$ implies $\min\{\mu'^2, s_i\} \geq x'_i$. Therefore $x'_{B^1} \geq x_{B^1}$ implies $x'_i \geq x_i$, contradiction.

If there exists $i_2 \in B^2$ such that $x_{i_2} < \min\{\mu'^2, s_{i_2}\}$, then the path stops at $j_2 i_2$. Otherwise, we continue in this fashion until we complete the construction of the announced path. We proceed then to take away an ε-flow from $i^* j_1$, add it to $j_1 i_1$, etc.., until finally adding it to $j_T i_T$. This achieves a Pigou-Dalton transfer from i^* to i_T, in contradiction of the Lorenz optimality of x.

8.4 Proposition 4

The egalitarian transfer rule is monotonic and invariant, hence strategyproof as well.

Proof Because the egalitarian transfer rule is peak-only, it is enough to speak of the profiles of peaks, instead of the full fledged preferences. We fix a supplier $i \in S$ and a benchmark profile (s, d), with corresponding egalitarian transfers (x, y).

We consider a change of peak by agent i to s'_i, and we write $s'_{i'} = s_{i'}$ for all $i' \neq i$, so that $s' = (s'_i, s_{-i})$ with corresponding allocation (x', y'). As usual we omit the entirely similar argument for a change of peak by a demander j.

Step 1 Assume $i \in S_+(s, d)$ so $x_i = s_i$. If $s'_i < s_i$ the GE decomposition (Lemma 2) does not change, so $i \in S_+(s', d)$ and $x'_i = s'_i < x_i$. If $s'_i > s_i$ and still the GE decomposition does not change, then $i \in S_+(s', d)$ and $x'_i = s'_i > x_i$. Consider the critical report s^*_i, $s^*_i > s_i$, if any, at which the GE decomposition and the status of agent i change. By Lemma 2 *ii*), there is no change in the decomposition at s'_i as long as $s'_T < d_{f(T) \cap D_-(s,d)}$ for all $T \subseteq S_+(s, d)$. Thus s^*_i is the smallest number such that

$$s_{T \diagdown i} + s^*_i = d_{f(T) \cap D_-(s,d)} \tag{14}$$

for some subset T of $S_+(s, d)$ containing i. Let T^* be the largest T contained in $S_+(s, d)$ and satisfying (14) (well defined by the usual submodularity argument). Recall from the proof of Lemma 2 that $\widetilde{T} = (S_- \cup S_0)(s, d)$ is the largest solution of $\arg\max_{T \subseteq S}\{s_T - d_{f(T)}\}$. Writing $s^* = (s^*_i, s_{-i})$, we have

$$s_{\widetilde{T}} - d_{f(\widetilde{T})} = \max_{T \subseteq S} \{s_T - d_{f(T)}\} \text{ and } s^*_{T^*} - d_{f(T^*) \cap D_-(s,d)} = 0$$

and $T^*, \widetilde{T}$ are disjoint. Therefore $\max_{T \subset S}\{s^*_T - d_{f(T)}\} = \max_{T \subset S}\{s_T - d_{f(T)}\}$ and the largest solution of $\arg\max_{T \subseteq S}\{s^*_T - d_{f(T)}\}$ is now $(S_- \cup S_0)(s,d) \cup T^* = (S_- \cup S_0)(s^*,d)$. Moreover $S_-(s,d)$ is still a solution of $\arg\max_{T \subseteq S}\{s^*_T - d_{f(T)}\}$, therefore it is the smallest. So $i \in S_0(s^*,d)$ and we have

$$x^*_i = s^*_i \geq x_i \geq s_i \tag{15}$$

We shall prove in step 3 below that for any $s'_i > s^*_i$ we have $x'_i \geq s^*_i$, thus completing the proof of monotonicity when $i \in S_+(s,d)$.

Step 2. Assume $i \in S_-(s,d)$, so $x_i \leq s_i$.

If $s'_i > s_i$ the GE decomposition (Lemma 2) does not change. By Proposition 2, $(x'_{[S_-]}, d_{[D_+]})$ is Lorenz dominant in $\mathcal{PO}^*(G(S_-, D_+), s', d)$, written simply as $\mathcal{PO}^*(s',d)$, of which $\mathcal{PO}^*(s,d)$ is a subset. If $(x'_{[S_-]}, d_{[D_+]}) \in \mathcal{PO}^*(s,d)$, it is Lorenz dominant there too, so it coincides with $(x_{[S_-]}, d_{[D_+]})$ and $x'_i = x_i$. If $(x'_{[S_-]}, d_{[D_+]}) \in \mathcal{PO}^*(s,d)$, it is because of the inequality $x'_i > s_i$, and then $x_i \leq s_i$ implies $x'_i > x_i$ as desired.

If $s'_i < s_i$ distinguish two cases. *Case 1:* $s'_i > x_i$. Then the decomposition does not change and we have $\mathcal{PO}^*(s',d) \subseteq \mathcal{PO}^*(s,d)$; moreover $(x_{[S_-]}, d_{[D_+]})$ is Lorenz dominant in $\mathcal{PO}^*(s,d)$ and belongs to $\mathcal{PO}^*(s',d)$, so it is Lorenz dominant there as well and we conclude $x' = x$.

Case 2: $s'_i \leq x_i$. If the decomposition does not change, $i \in S_-(s',d)$ gives $x'_i \leq s'_i$ and the desired inequality $x'_i \leq x_i$. It remains to consider those values of s'_i at which the GE decomposition and the status of i change. As in Step 1, the critical (largest) such value is s^*_i such that

$$d_C = s_{g(C) \cap S_-(s,d) \setminus \{i\}} + s^*_i$$

for some subset C of $D_+(s,d)$ containing i. Let C^* be the largest such C contained in $D_+(s,d)$. As in Step 1 we write $s^* = (s^*_i, s_{-i})$ and we have: $(D_- \cup D_0)(s^*,d)$ is the largest solution of $\arg\max\{d_C - s_{g(C)}\}$, while C^* satisfies $d_{C^*} = s_{g(C^*)}$, and these two sets are disjoint. Therefore $(D_- \cup D_0)(s^*,d) = (D_- \cup D_0)(s,d) \cup C^*$, and $i \in D_0(s^*,d)$. We omit the details. The symmetrical conclusion to (15) obtains, namely

$$x^*_i = s^*_i \leq x_i \leq s_i \tag{16}$$

Step 3 Consider now a shift from s_i to s'_i when $i \in S_0(s,d)$. If $s'_i > s_i$, we have $i \in S_-(s',d)$, because any solution of $\arg\max_{T\subseteq S}\{s'_T - d_{f(T)}\}$ contains i and $S_-(s',d)$ is the smallest. Then in the downward shift of supplier i's peak starting at s'_i, the critical value at which the status of i changes is s_i (Step 2), so by (16) $s_i = x_i \leq x'_i$ as desired. Symmetrically $s'_i < s_i$ implies $i \in S_+(s',d)$, because no solution of $\arg\max_{T\subseteq S}\{s'_T - d_{f(T)}\}$ contains i and $(S_- \cup S_0)(s',d)$ is the largest. Moreover s_i is the critical value starting from s'_i (Step 1), so (15) gives $x'_i \leq s_i = x_i$.

It remains to look at a shift from s_i to s'_i such that $i \in S_+(s,d)$ and $i \in S_-(s',d)$. This requires $s'_i > s_i$ and then the critical value s^*_i starting from s_i is the same as the critical value starting from s'_i. Therefore (15) and (16) imply

$$s_i \leq x_i \leq s^*_i \leq x'_i \leq s'_i$$

Step 4 We check finally the invariance property. In the premises of (7) the case $x_i > s_i$ ($s[R_i] < \psi_i(R)$), never happens for the egalitarian transfer solution; the case $s_i > x_i$ only happens if $i \in S_-(s,d)$. If s'_i is another peak such that $s_i > s'_i > x_i$, we saw in Case 1 of Step 2 that $x'_i = x_i$, as required by Invariance. This is still true for $s'_i = x_i$ by an easy continuity argument (omitted fro brevity). If $s'_i > s_i > x_i$, then the GE decomposition is unchanged at (s',d). By the recursive definition of the egalitarian solution in $S_-(s,d)$, agent i receives μ^k, one of the parameters generating the partitions $C^1, C^2, \cdots$, and $T^1, T^2, \cdots$, of D_+ and S_-, and it is clear that the same partitions and parameters $\mu^1, \mu^2 \cdots$, define the same egalitarian solution in $S_-(s',d)$.

8.5 Characterization Theorem

The egalitarian transfer rule $\mathcal{E}$ is characterized by Pareto optimality, strategyproofness, voluntary trade, and equal treatment of equals.

The egalitarian transfer rule selects by construction $\mathcal{E}(R) \in \mathcal{PO}^*(G,s,d)$ for all R, and $\mathcal{E}_i(R) \leq s_i, \mathcal{E}_j(R) \leq d_j$ ensure voluntary trade. The other properties are proven in Propositions 4,5. Conversely we fix a rule ψ meeting the four properties listed above.

Step 1 If the rule ψ satisfies Pareto optimality, strategyproofness, and voluntary trade, then $\psi(R) \in \mathcal{PO}^*(G,s,d)$ for all $R \in \mathcal{R}^{S\cup D}$.

By Proposition 1 this amounts to show that $\psi_i(R) > s_i$ is impossible for $i \in S_+(s,d)$, and $\psi_j(R) > d_j$ is impossible for $j \in D_+(s,d)$. Say $\psi_i(R) > s_i$ and choose $R'_i \in \mathcal{R}$ such that $s[R'_i] = s_i$ and $0P'_i\psi_i(R)$. Recall from Lemma 3 and the comments immediately before that ψ is *own-peak-only*, in particular

$\psi_i(R) = \psi_i(R'_i, R_{-i})$. Now $0P'_i\psi_i(R'_i, R_{-i})$ is a contradiction of voluntary trade. As usual the proof of the other statement is similar.

Step 2 It remains to prove that for all R the projections of $\psi(R)$ on $S_-(s,d)$ and $D_-(s,d)$ coincide with that of $\mathcal{E}$. We focus on $S_-(s,d)$, omitting the similar argument for $D_-(s,d)$. The property $\psi(R) \in \mathcal{PO}^*(G,s,d)$ imposes the following restrictions on the projection $\psi_{[S_-(s,d)]}(R)$, denoted x for simplicity

$$x_T \leq d_{f(T)} \text{ for all } T \subset S_-(s,d) \text{ and } x_{S_-(s,d)} = d_{D_+(s,d)} \tag{17}$$

$$x_i \leq s_i \text{ for all } i \in S_-(s,d) \tag{18}$$

For the submodular cooperative game $(S_-(s,d), v)$ with $v(T) = d_{f(T)}$ the system (17) means that x is in the core; and (18) captures voluntariness of trade.

Step 2.1 In this step we consider a profile R in which all suppliers have the same peak, $s[R_i] = s[R_{i'}]$ for all $i, i' \in S$ (there are no constraints on the preferences of demanders). For simplicity we write S_- instead of $S_-(s,d)$. We use ETE to show that $x = \psi_{[S_-]}(R)$ is precisely $\overline{x} = \mathcal{E}_{[S_-]}(R)$, the Lorenz dominant transfer profile within the set defined by the system (17),(18).

Claim 1. Pick an agent 1 in S_- such that $x_1 = x^{*\alpha}$, where $\alpha = |S_-|$ (so $x_1 = \max_{S_-} x_i$). Then:

$$x_1 = \overline{x}_1 = x^{*\alpha} = \overline{x}^{*\alpha} \tag{19}$$

As $\overline{x}$ is Lorenz dominant we have $x^{*\alpha} \geq \overline{x}^{*\alpha}$. If $x_i = x^{*\alpha}$ for all $i \in S_-$, then $x = \overline{x}$ (because $x_{S_-} = \overline{x}_{S_-}$) and we are done. Suppose next there is at least one $i \in S_-$ such that $x_i < x^{*\alpha}$. We show that if $x_i < s_i$, there exists a coalition $S(i) \subset S_-$ containing i but not 1, such that $x_{S(i)} = v(S(i))$. Suppose, on the contrary, $x_T < v(T)$ for all $T \subset S_-$ containing i but not 1. A Pigou Dalton transfer from x_1 to x_i transforms x into x' such that $x'_1 = x_1 - \varepsilon$, $x'_i = x_i + \varepsilon$, $x_j = x'_j$ elsewhere. If ε is small enough, we have $x'_T < v(T)$ for all $T \subset S_-$ and $x'_i < s_i$, therefore x' satisfies (17),(18). This is a contradiction of ETE.

We set $S^* = \cup_{i: x_i < x^{*\alpha}, s_i} S(i)$. By submodularity of v we have $x_{S^*} = v(S^*)$. By construction for all $i \in N \diagdown S^*$, x_i is $x^{*\alpha}$ or s_i, hence $x_i \geq \overline{x}_i$; moreover $N \diagdown S^*$ contains 1. On the other hand we have

$$\overline{x}_{S^*} \leq v(S^*) = x_{S^*} \Rightarrow \overline{x}_{N \diagdown S^*} \geq x_{N \diagdown S^*} \tag{20}$$

Combining this with $x_i \geq \overline{x}_i$ on $N \diagdown S^*$ gives (19).

Claim 2 Pick agent 2 in S_-, $2 \neq 1$, such that $x_2 = x^{*(\alpha-1)}$. Then

$$x_2 = \overline{x}_2 = x^{*(\alpha-1)} = \overline{x}^{*(\alpha-1)} \tag{21}$$

As $\overline{x}$ Lorenz dominates x, we have $x^{*\alpha} + x^{*(\alpha-1)} \geq \overline{x}^{*\alpha} + \overline{x}^{*(\alpha-1)} \Rightarrow x^{*(\alpha-1)} \geq \overline{x}^{*(\alpha-1)}$. If $x_i = x^{*(\alpha-1)}$ for all $i \in S_- \diagdown \{1\}$, then $x = \overline{x}$ and we are done. Suppose now there is at least one $i \in S_- \diagdown \{1\}$ such that $x_i < x^{*(\alpha-1)}$. By the same argument as above, if $x_i < s_i$ there exists a coalition $S(i) \subset S_-$ containing i but not 2, such that $x_{S(i)} = v(S(i))$ (else we can construct a Pigou-Dalton transfer from 2 to i, contradicting ETE). Set $S^* = \cup_{i:x_i<x^{*(\alpha-1)},s_i} S(i)$, then $x_{S^*} = v^+(S^*)$ by submodularity of v. Moreover for all i in $N \diagdown (S^* \cup \{1\})$, x_i is $x^{*(\alpha-1)}$ or s_i, in particular $x_i \geq \overline{x}_i$. Combining this with $x_1 = \overline{x}_1$, and $\overline{x}_{N\diagdown S^*} \geq x_{N\diagdown S^*}$ (proven by (20)), we see that x and $\overline{x}$ coincide in $N \diagdown S^*$, that contains 2. Property (21) follows.

The inductive argument establishing $x = \overline{x}$ is now clear.

Step 2.2 We just proved that ψ and $\mathcal{E}$ coincide on S when all suppliers have the same peak. We use another induction argument, inspired by Ching (1993), to establish this equality for an arbitrary profile R. We use the following notation: for $R, \widetilde{R} \in \mathcal{R}^{S\cup D}$ and $T \subset S$, $(R_{[T]}, \widetilde{R}_{[(S\diagdown T)\cup D]})$ is the profile equal to R for agents in T and to $\widetilde{R}$ elsewhere.

Fix a profile $\widetilde{R}$ where all suppliers have identical preferences, an integer $n, 0 \leq n \leq |S| - 1$ and consider the following subset of preference profiles

$$R \in \mathcal{B}(\widetilde{R}, n) \overset{def}{\Leftrightarrow}$$

for some $T \subset S, |T| \leq n$: $R_{[(S\diagdown T)\cup D]} = \widetilde{R}_{[(S\diagdown T)\cup D]}$ and $s[\widetilde{R}_i] \geq s[R_i]$ if $i \in S$

We prove by induction on n the following property $\mathcal{H}^+(n)$: for all $R \in \mathcal{R}^{S\cup D}$ and all $T \subset S$

$$R \in \mathcal{B}(\widetilde{R}, n) \Rightarrow \psi_i(R) = \mathcal{E}_i(R) \text{ for all } i \in S$$

Step 2.1 establishes $\mathcal{H}^+(0)$. Assume now $\mathcal{H}^+(n-1)$ is true, and fix $R \in \mathcal{B}(\widetilde{R}, n)$ with $R[(S\diagdown T) \cup D] = \widetilde{R}[(S\diagdown T) \cup D]$ and $|T| = n$.

We prove first $\psi_i(R) = \mathcal{E}_i(R)$ for $i \in T$. Pick such an agent and set $R' = (R_{[T\diagdown i]}, \widetilde{R}_{[(S\diagdown T)\cup\{i\}\cup D]}) \in \mathcal{B}(\widetilde{R}, n-1)$. By the inductive assumption $\psi_i(R') = \mathcal{E}_i(R') = x'_i$; by Pareto optimality and the definition of $\mathcal{B}(\widetilde{R}, n)$

$$s[\widetilde{R}_i] \geq s[R_i] \geq \psi_i(R), \mathcal{E}_i(R)$$

If $s[R_i] \geq \mathcal{E}_i(R) > \psi_i(R)$, Monotonicity implies $\mathcal{E}_i(R') \geq \mathcal{E}_i(R)$, and Invariance gives $\psi_i(R') = \psi_i(R)$, hence a contradiction. If $s[R_i] \geq \psi_i(R) > \mathcal{E}_i(R)$, we have a similar contradiction from $\psi_i(R') \geq \psi_i(R)$ (Monotonicity), and $\mathcal{E}_i(R') = \mathcal{E}_i(R)$ (Invariance).

It remains to check $\psi_i(R) = \mathcal{E}_i(R)$ for $i \in S \diagdown T$. This is clear in $S \diagdown S_-(R)$, so we check it for $S_-(R) \diagdown T$. Write $S_- = S_-(R)$ and $x = \psi_{[S_-]}(R)$, $\overline{x} = \mathcal{E}_{[S_-]}(R)$ as in step 2.1. Consider the set

$$\mathcal{C}(R) = \{z \in \mathbb{R}_+^{S_- \diagdown T} | (z, \overline{x}_{[S_- \cap T]}) \text{ satisfies system (17),(18)}\}$$

Clearly $\overline{x}_{[S_- \diagdown T]}$ is still Lorenz dominant in $\mathcal{C}(R)$, hence we can mimic the proof of Step 2.1 to show that ETE and Pareto optimality imply $x = \overline{x}$ in $S_- \diagdown T$. Indeed $\widetilde{R}_{[S_- \diagdown T]}$ consists of preferences with identical peaks, therefore we can apply ETE to any pair of agents in $S_- \diagdown T$. Moreover $\mathcal{C}(R)$ is defined by the system

$$z_{T'} \leq \widetilde{v}(T') = v(T' \cup [T \cap S_-]) - \overline{x}_{[T \cap S_-]} \text{ for all } T' \subset S_- \diagdown T, \text{ and } z_{S_- \diagdown T} = \overline{x}_{S_- \diagdown T}$$

$$z_i \leq s_i \text{ for all } i \in S_- \diagdown T$$

Then the proof proceeds exactly as in step 2.1. We omit the details.

We have proved that $\mathcal{H}^+(|S| - 1)$ for any choice of $\widetilde{R}$. Now consider an arbitrary profile R and choose i in S and such that $s[R_i] \geq s[R_{i'}]$ for all $i' \in S$. Choosing for $\widetilde{R}$ the profile of preferences $\widetilde{R}_{i'} = R_i$ for all $i' \in S, \widetilde{R}_j = R_j$ for all $j \in D$, we have $R \in \mathcal{B}(\widetilde{R}, |S| - 1)$ and the proof is complete.