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Name: ____________________________

**Instructions:** Relax! This test consists of 4 questions and is out of 100 points. You will have 75 minutes to complete the exam. Be sure to show all of your work and to clearly mark your answers.
1. (30 points) Indicate whether each of the following expressions is (always) TRUE or FALSE, and briefly justify your answer in each case. (No credit will be given unless a justification is provided.)

(a) (6 points) A firm that is maximizing profits is also minimizing costs.

**Solution:** TRUE. A profit maximizing firm will produce the optimal amount at the lowest possible cost. Additionally, the tangency conditions from the two problems are the same which imply that the profit maximizing quantity will be produced at the lowest possible cost.

(b) (6 points) In an edgeworth box (think two-person endowment/exchange economy for example), a general equilibrium allocation will occur on the contract curve.

**Solution:** TRUE. The contract curve is the elements of the Pareto set where the individuals are no worse off than the initial endowment. By the First Welfare Theorem and utility maximization, it must be that the general equilibrium lies on the contract curve.

(c) (6 points) A perfectly competitive market where each firm has constant returns to scale technology will result in a flat long-run industry supply curve.

**Solution:** FALSE. There may be pecuniary externalities which would cause the industry’s long-run supply function to be upward sloping.

(d) (6 points) For any pre-specified economy, every Pareto efficient allocation is a general equilibrium.

**Solution:** FALSE. The second welfare theorem states that any Pareto efficient allocation can be supported as a general equilibrium if there are appropriate initial transfers. Without any transfers, there will be many points in the Pareto set that are not general equilibrium allocations given the initial endowment.

(e) (6 points) A firm with capital and labor inputs will choose to shut down in the short run if its short run revenue doesn’t cover its cost of labor.

**Solution:** TRUE. In this case, the variable costs are simply the labor costs. The shutdown rule for a firm in the short run is that it must cover its variable costs. Since the labor costs are the variable costs, if a firm cannot cover its labor costs in the short run then it will shut down.
2. (30 points) Consider a market consisting of 5 firms and 10 consumers. Let the consumers have identical preferences over gizmos \( (q) \) and a composite good \( (c) \) represented by the utility function

\[
U(q, c) = c - \frac{5}{3}(3 - q)^2.
\]

Normalize the price of the composite good \( c \) to be 1 and let \( p \) be the price of gizmos \( (q) \). Additionally, assume that each individual has an income of 35.

(a) (5 points) Derive the aggregate market demand for gizmos, \( Q_D(p) \).

**Solution:** The Lagrangian for an individual’s problem is

\[
\max_{(q,c,\lambda)} c - \frac{5}{3}(3 - q)^2 + \lambda(35 - pq - x).
\]

Taking the FOCs and rearranging we get that the individual demand for gizmos is

\[
q^*(p) = 3 - \frac{3p}{10}.
\]

Since we have identical consumers, all we need to do is multiply \( q^*(p) \) by the number of consumers in order to derive the aggregate market demand for gizmos. This implies,

\[
Q_D(p) = 10 \cdot q^*(p) = 30 - 3p.
\]

Suppose the production for gizmos obeys the following technology

\[
q = 3l + 4k.
\]

Let the wage rate be \( w = 2 \) and the rental rate of capital be \( v = 4 \).

(b) (5 points) Derive an individual firm’s long-run cost function, \( C(q) \).

**Solution:** Since we have perfect substitutes technology, we need to compare the MRTS with the price ratio in order to determine which corner solution the firm will be at (i.e. will they only hire labor or only hire capital). In this case,

\[
MRTS = \frac{3}{4} > \frac{2}{4} = \frac{w}{v}.
\]

This implies that the firm will only hire labor.

If the firm only hires labor, then in order to produce an amount, \( q \), the firm must hire \( l = \frac{q}{3} \) units of labor. Therefore a firm’s long run cost function is given by

\[
C(q) = w \cdot l = \frac{2q}{3}.
\]

(c) (5 points) Find the long-run equilibrium price, \( p^* \), and quantity, \( Q^* \), of gizmos.

**Solution:** Since an individual firm’s marginal cost curve is flat at the 2/3, and there are no pecuniary externalities, we know that the market supply curve will be perfectly elastic at a price of \( p^* = \frac{2}{3} \). If we plug this price into either \( Q_D(p) \) or \( Q_S(p) \) we will get the equilibrium quantity of

\[
Q^* = Q_D(p) = Q_S(p) = 28.
\]
(d) (5 points) In an attempt to finance social security, suppose the government implements a $2 payroll tax (i.e. the government charges each firm $2 for each unit of labor they hire). Calculate the new long-run cost function of an individual gizmo firm.

**Solution:** This effectively raises the wage rate to $w' = 4$. Comparing the MRTS to the new price ratio, we find that

$$\text{MRTS} = \frac{3}{4} < \frac{4}{4} = \frac{w'}{v}.$$  

This implies that the firm will only be employing capital for the production of gizmos. This yields a long-run cost function of

$$C(q) = q.$$

(e) (5 points) Calculate the new equilibrium price, $p^{**}$, and quantity, $Q^{**}$, under this payroll tax.

**Solution:** As we readily see from above, the new marginal cost for each firm is 1. This implies the aggregate supply is perfectly elastic at the price $p^{**} = 1$. The equilibrium quantity can then be found by plugging this price into the aggregate demand function. This yields

$$Q^{**} = Q_D(1) = 30 - 3 = 27.$$

(f) (2 points) How much tax revenue is generated by the payroll tax?

**Solution:** Since the firms don’t hire any labor, the government doesn’t generate any additional tax revenue from the payroll tax.

(g) (3 points) Suppose the government is still deciding whether to implement the payroll tax (i.e. for this part they have not yet implemented it) or to implement a sales tax of $t$ per gizmo sold. What is the highest amount of sales tax, $t^{*}$, the consumers would be willing to tolerate to avoid a $2 payroll tax in the long run?

**Solution:** Adding a sales tax of $t$ would raise the long run equilibrium price to $p + t$. This means that the consumers would be willing to bear up to a sales tax of $t^{*} = \frac{1}{3}$ to avoid the $2 payroll tax. This can be seen from the fact that at $t^{*}$, the long run equilibrium prices under the two scenarios would be equal.
3. (40 points) Robinson Crusoe finds himself alone on a tropical island with limited resources. Specifically, there are only 32 tree branches that he can use to make tools for farming pineapples (good $x$) or for fishing (let fish be good $y$). Let’s assume that one branch can be made into either one farming tool, $k_x$, or one fishing pole, $k_y$. The number of tools and fishing poles directly affects his ability to produce pineapples ($x$) and fish ($y$). Specifically, he has the following two production functions

\[ x = 2\sqrt{k_x}, \quad y = \sqrt{k_y}. \]

Suppose Robinson only cares about the number of pineapples and fish he consumes and his preferences are represented by the following utility function

\[ U(x, y) = \min\{x, 2y\}. \]

(a) (4 points) Derive Robinson’s PPF, $y^*(x)$.

**Solution:** Inverting the production function for pineapples yields the capital requirement of

\[ k_x = \frac{x^2}{4}. \]

This implies that $k_y = 32 - k_x = 32 - x^2/4$. Plugging this into the production function for fish yields

\[ y^*(x) = \sqrt{32 - \frac{x^2}{4}}. \]

(b) (4 points) Find the Pareto efficient allocation of pineapples and fish, $(x^*, y^*)$.

**Solution:** Here we are maximizing Robinson’s utility subject to the PPF, or the production constraint. Since Robinson’s utility function is perfect complements, we know that he will optimally consume pineapples and fish according to $x = 2y$. Plugging this relationship into the PPF yields

\[ \frac{1}{2}x = \sqrt{32 - \frac{x^2}{4}} \]
\[ \Rightarrow x^2 = 64 \quad \Rightarrow x^* = 8 \]
\[ \Rightarrow y^* = 4. \]

Now that we have found the Pareto optimal allocation, let’s view this problem where Robinson is both the producer and the consumer (i.e. a General Equilibrium type problem). From now on, normalize the rental rate of capital to 1 ($v=1$). Additionally, let $p_x$ and $p_y$ be the prices of pineapples and fish respectively.

(c) (4 points) For Robinson the producer, find the cost function for the pineapple and fish markets, $C_x(x)$ and $C_y(y)$ respectively.

**Solution:** Inverting the production functions for each market yields

\[ \frac{x^2}{4} = k_x \Rightarrow C_x(x) = \frac{x^2}{4} \]
\[ y^2 = k_y \Rightarrow C_y(y) = y^2 \]
(d) (4 points) Calculate the supply that Robinson the producer will supply of pineapples, $x_S$, and fish, $y_S$.

**Solution:** Setting marginal revenue equal to marginal cost in each market yields

\[ p_x = \frac{x}{2} \Rightarrow x_S = 2p_x \]
\[ p_y = 2y \Rightarrow y_S = \frac{p_y}{2} \]

(e) (4 points) Calculate the profits from each market, $\pi_x$ and $\pi_y$.

**Solution:**

\[ \pi_x = 2p_x^2 - p_x^2 = p_x^2 \]
\[ \pi_y = \frac{p_y^2}{2} - \frac{p_y^2}{4} = \frac{p_y^2}{4} \]

(f) (4 points) Write down the budget constraint of Robinson the consumer.

**Solution:**

\[ p_xx + p_yy = 32 + p_x^2 + \frac{p_y^2}{4} \]

(g) (4 points) Find Robinson the consumer’s demand for pineapples, $x^{**}$, and his demand for fish, $y^{**}$ in terms of the prices for pineapples and fish.

**Solution:** Here we simply need to maximize Robinson’s utility subject to his budget constraint in part (f). We know he will be consuming according to

\[ x = 2y. \]

Plugging this into the budget constraint and solving for $x^{**}$ and $y^{**}$ yields

\[ x^{**} = \frac{2 \left( 32 + p_x^2 + \frac{p_y^2}{4} \right)}{2p_x + p_y} \]
\[ y^{**} = \frac{\left( 32 + p_x^2 + \frac{p_y^2}{4} \right)}{2p_x + p_y} \]
(h) (2 points) Set up, BUT DO NOT SOLVE, the two equations needed to find the equilibrium prices, \( p_x^* \) and \( p_y^* \), and the equilibrium allocation \((x^{**}, y^{**})\).

**Solution:** The markets simply need to clear (i.e. we equate our answers from parts (d) and (g)). This gives us the following two equations

\[
2p_x = \frac{2 \left( 32 + p_x^2 + \frac{p_y^2}{4} \right)}{2p_x + p_y} \\
\frac{p_y}{2} = \frac{1}{2} \left( 32 + p_x^2 + \frac{p_y^2}{4} \right)
\]

Suppose a major tropical storm hits Robinson’s island and destroys his capital stock. All he is left with is his store of 30 pineapples. The storm also brought the shipwrecked captain Jack Sparrow to Robinson’s island. Jack was able to save 30 preserved fish from his sinking ship. Let Jack’s preferences over pineapples and fish be represented by the utility function

\[ U_J(x, y) = \min\{2x, y\} \]

Normalize the price of fish to 1 \((p_y = 1)\). Assume Robinson and Jack can trade freely amongst themselves.

(i) (10 points) Find the competitive equilibrium allocation, \((x^{***}, y^{***})\), and price of pineapples, \( p_x^{***} \), of this newly created exchange economy.

**Solution:** In order to find the general equilibrium, we first need to solve each consumer’s utility maximization and then find the price that will clear the market.

Robinson’s problem is

\[
\max_{x,y} \min\{x, 2y\} \\
\text{s.t. } p_x x + y = 30p_x
\]

Here the utility function is one the we cannot differentiate. However, we know that optimally, Robinson will consume along the ray \( x = 2y \). Plugging this relationship into the budget constraint and solving yields

\[
y^*_R = \frac{30p_x}{2p_x + 1} \\
x^*_R = \frac{60p_x}{2p_x + 1}
\]

Setting up and solving Jack’s utility maximization problem (the utility function and budget constraint will be slightly different, but the process is the same) we find that

\[
y^*_J = \frac{60}{p_x + 2} \\
x^*_J = \frac{30}{p_x + 2}
\]

By Walras’ Law we only need to clear one of the markets and we know that the other will clear as well. Clearing the x market means that

\[ x^*_R + x^*_J = \bar{x} = 30 \]
Plugging in our expressions for $x_R^*$ and $x_J^*$ yields
\[ \frac{60p_x}{2p_x + 1} + \frac{30}{p_x + 2} = 30. \]
Dividing both sides by 30 and then multiplying by the common denominator leaves us with
\[ 2p_x(p_x + 2) + 2p_x + 1 = (2p_x + 1)(p_x + 2). \]
Solving this for $p_x$ gives us the market clearing price of
\[ p_x = 1. \]