The Term Structure of Interest Rates in a DSGE Model with Recursive Preferences

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Abstract

A dynamic stochastic general equilibrium (DSGE) model in which households have Epstein and Zin recursive preferences is solved with perturbation. The parameters governing preferences and technology are estimated by maximum likelihood using macroeconomic data and the term structure of interest rates. The estimates imply a large risk aversion, an elasticity of intertemporal substitution higher than one, and substantial adjustment costs. Furthermore, the paper identifies the tensions within the model by estimating it on subsets of these data. The analysis concludes by pointing out potential extensions that may improve the model’s fit.

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1. Introduction

Can a dynamic stochastic general equilibrium (DSGE) model in which households have Epstein and Zin (EZ) recursive preferences match both macroeconomic and yield curve data? After solving such a model using perturbation methods, its likelihood function is built with the particle filter to estimate the preference and technology parameters via maximum likelihood using macroeconomic and yield curve data. The model is also estimated on subsets of the data to illustrate how the parameters are identified. The estimated model cannot match the average slope of the term structure, nor its variation, even with extreme amounts of risk aversion, unless inflation in the model is allowed to be considerably more volatile than in the data.

The motivation for our exercise is that economists are paying increasing attention to recursive utility functions (see Hansen et al., 2008, for a survey of the literature). The key advantage of these preferences is that they allow separation between the intertemporal elasticity of substitution (IES) and risk aversion. This separation is attractive in several contexts. In asset pricing, researchers have argued that EZ preferences account for many patterns in the data, possibly in combination with other features such as long-run risk or stochastic volatility. Bansal and Yaron (2004) is a prime representative of this work. From a policy perspective, EZ preferences generate radically bigger welfare costs of the business cycle than those coming from standard expected utility (Tallarini, 2000) and change the trade-offs that policy makers face (Levin, López-Salido, and Yun, 2007). Finally, EZ preferences can be reinterpreted, under certain conditions, as a case of robust control (Hansen, Sargent, and Tallarini, 1999).

Our paper makes three contributions. First, it studies the role of EZ preferences in a DSGE production economy with endogenous capital and labor supply and their interaction with the yield curve. Production economies deliver additional insights over the endowment economies more commonly used in the analysis of EZ preferences. One, production economies can be used to conduct policy experiments, which cannot be done in endowment economies. An attractive promise of integrating macroeconomics and finance is to have, in the middle run, richer models for policy advice. Two, production economies allow us to link movements in the bond yield curve to macro variables such as capital and expected inflation. Such relationships have been studied largely in reduced-form empirical work, but not in a structural model. Three, considering production economies with labor supply is
quantitatively relevant. With EZ preferences, leisure affects asset pricing through the risk-adjusted expectation operator, even when leisure enters separately in the period utility function.

All these points are reflected in the consumption process that drives the stochastic discount factor. Most papers in asset pricing study endowment economies in which consumption follows an exogenous process. This is an important shortcoming. Production economies place tight restrictions on the comovements of consumption with other endogenous variables that exogenous consumption models are not forced to satisfy. Furthermore, in production economies, the consumption process itself is endogenous and therefore dependent on the parameters of the model, such as the IES and risk aversion. In comparison, by fixing the consumption process in endowment economies, a change in preferences implicitly translates to a change in the labor income process. This complicates the interpretation of estimated preference parameters and prevents us from performing policy experiments.

Working with EZ preferences is harder than working with expected utility because they generate necessary conditions that include the value function itself. Therefore, standard linearization techniques are hard to employ. The literature has resorted to either simplifying the problem by dealing with endowment economies or using computationally costly algorithms such as value function iteration (Croce, 2006) or projection methods (Campanale, Castro, and Clementi, 2010). The former solution suffers from the shortcomings of endowment economies described before. The latter solution makes likelihood or (simulated) moment estimation exceedingly challenging.

One can get around this obstacle by computing a third-order perturbation to the equilibrium dynamics of the economy. Our choice is motivated by three considerations. First, perturbation offers insights into the role of recursive preferences. A first-order approximation to the decision rules of our model with EZ preferences is equivalent to that of the model with standard utility and the same IES. The risk aversion parameter does not show up in this first-order approximation. Instead, risk aversion appears in the constant of the second-order approximation that captures precautionary behavior. This constant moves the ergodic distribution of the endogenous states, affecting, through this channel, allocations, prices, and welfare. By changing the mean of capital in the ergodic distribution, the risk

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1Epstein and Zin (1989) avoid this problem by showing that with access to the total wealth portfolio, one can derive a first-order condition in terms of observables that can be estimated using a method of moments. However, researchers cannot observe the total wealth portfolio because of the difficulties in measuring human capital, forcing the use of a proxy for the return on wealth. See, for instance, Lustig, Van Nieuwerburgh, and Verdelhan (2007).
aversion parameter influences the average level and the slope of the yield curve. Risk aversion also enters into the coefficients of the third-order approximation changing the slope of the response of the yield curves to variations in the state variables. Second, a third-order approximation guarantees a time-varying term premium, an important feature of the data. Third, in companion work, Caldara et al. (2012) document that this solution is highly accurate and fast enough for estimation.

Our second contribution is the estimation of a DSGE model with EZ preferences by maximum likelihood using the particle filter. In studying the asset pricing implications of equilibrium models, it is common to calibrate the parameters. This approach illuminates the main economic mechanism at work, but it might overlook some restrictions implied by the model. While various asset pricing models can explain the same set of moments, the economic mechanism generating the results, be it habits, long-run risks, or rare disasters, is quite different and implies diverse dynamics. In comparison, our likelihood-based inference imposes all cross-equation restrictions implied by the model.

The combination of 1) a non-linear solution to the equilibrium dynamics of the model; 2) the inclusion of endogenous capital; and 3) the likelihood-based estimation of the structural parameters pushes us to the frontiers of computational power. Thus, the paper is forced to compromise between theoretical detail and empirical relevance, such as in assuming an exogenous process for inflation. The effort is nevertheless worthwhile because, even with these compromises, there is much to learn about asset pricing in production economies with EZ preferences.

Third, our paper contributes to the fast-growing literature on term structure models. These models are successful in fitting the term structure of interest rates, but this is typically accomplished using latent variables.\(^2\) Even though some papers include macroeconomic or monetary policy variables, such variables still enter in a reduced-form way. Our approach imposes additional structure on such models, but the restrictions directly follow from optimality, preferences, and technology. Our approach obviously underperforms the statistical models along many dimensions, but it improves our understanding as to which preferences and technology processes induce a realistic term structure of interest rates. Also, as argued before, macroeconomists require a structural model to design and evaluate economic policies that might affect the term structure of interest rates.

The first step in this paper is to obtain a third-order approximation to the model given some parameter values, which can be done in a trivial amount of time. Given our goals, an additional advantage of our solution technique is that it does not limit ourselves to the case with unitary IES, as Campbell (1993), Tallarini (2000), and others do. There are two reasons why this flexibility is important. First, restricting the IES to one is hard to reconcile with empirical findings. Second, a value of the IES equal to one implies that the consumption-wealth ratio is constant over time. This implication of the model is hard to verify because total wealth, which includes human capital, is not directly observable. However, attempts at measurement, such as Lettau and Ludvigson (2001) or Lustig, van Nieuwerburgh, and Verdelhan (2007), reject the hypothesis that the ratio of consumption to wealth is constant.

Our second step is to use the particle filter to evaluate the likelihood function of the model (Fernández-Villaverde and Rubio-Ramírez, 2007). Evaluating the likelihood of a DSGE model is equivalent to keeping track of the conditional distribution of unobserved states of the model with respect to the data. Our perturbation approximation is inherently non-linear (otherwise, risk aversion would not show up in the likelihood). These non-linearities make the conditional distribution of states intractable and prevent the application of conventional methods, such as the Kalman filter. The particle filter is a sequential Monte Carlo method that replaces the conditional distribution of states by an empirical distribution of states drawn by simulation.

The model is estimated with US data on consumption growth, output growth, five bond yields, and inflation over the period 1953.Q1 to 2008.Q4. The point estimates reveal a high coefficient of risk aversion, an IES well above one, and substantial adjustment costs of capital. However, the model barely reproduces the average slope of the yield curve and substantially underestimates the volatility of bond yields. On the positive side, the model is able to reproduce the autocorrelation patterns in consumption growth, the 1-year bond yield, and inflation.

To better understand these shortcomings and parameter identification, the model is re-estimated based on subsets of our data. First, inflation is omitted from the observables. The new estimates imply an average slope of the yield curve that is comparable to the observed one. The model also reproduces the volatility of the bond yields. However, this “success” is explained by the fact that the estimated
volatility of inflation is too high. Second, the model is estimated using only bond yields. Our findings marginally improve with respect to the previous case, but now the fitted volatility of consumption and output growth is too high.

Our paper is related to Doh (2012), Doh (2011), and Rudebusch and Swanson (2012). Doh (2012) estimates an endowment economy with recursive preferences, persistent components in expected consumption growth and inflation (long-run risks), and stochastic volatility. He finds that time-varying volatility for inflation generates time variation in term premia. Doh (2011) estimates a DSGE model with habit formation preferences, and labor, but no endogenous capital, and finds that the variation in term premia generated by the model is too small, relative to the data (see Rudebusch and Swanson, 2011, for an alternative investigation of the yield curve using CRRA preferences).

Closer to our work is that of Rudebusch and Swanson (2012), who also use perturbation methods to solve a calibrated DSGE model with EZ preferences, endogenous inflation, and firm-specific capital to account for the dynamics of the yield curve. Their approach considers a model with endogenous inflation that allows them to obtain substantial term premium variation, something that had eluded the previous literature. There are three differences between Rudebusch and Swanson (2012) and our paper that are intriguing. First, we estimate the model parameters via maximum likelihood, whereas Rudebusch and Swanson calibrate their parameters. The estimation stage adds complexity to our problem, which forces us to simplify the model along several dimensions, but it helps us to understand the tensions that the model faces at matching the data by disciplining our selection of parameter values. Second, for simplicity, Rudebusch and Swanson (2012) do not consider endogenous capital: investment is fixed (after scaling with the trend process) and equal to depreciation. Hence, the elasticity of consumption to technology shocks depends on the level of technology, making consumption growth heteroscedastic. Thus, the term premium and its time variation go up substantially with risk aversion. This channel is also highlighted if one compares Doh (2008) with Doh (2011). While Doh (2012) (which considers a model with stochastic volatility) finds a significant term premium variation, Doh (2011) (without stochastic volatility) does not. In our model, capital is endogenous subject to adjustment costs. Even though our estimate is that these adjustment costs are fairly high, our data (which do not include either investment or capital) clearly favor a model where investment is not fixed. Hence, our
estimation forces us to move away from the amplifying effect of the technology dynamics allowed by fixed investment. Finally, Rudebusch and Swanson (2012) approximate the yields on bonds through a consol. Since Andreasen and Zabczyk (2010) show that a consol approximation of yields may introduce approximation biases, this paper solves for the nominal bond yield at each maturity. Because of these three differences, the lessons from Rudebusch and Swanson (2012) and this paper complement each other and that, in the middle run, it would be important to “merge” both approaches by estimating (as done here) a model with endogenous inflation (as they do).

The rest of the paper is organized as follows. Section 2 presents our model. Section 3 solves the model. Section 4 describes the likelihood-based estimation procedure. Section 5 reports the data and our empirical findings. Section 6 outlines several extensions and section 7 concludes.3

2. A production economy with recursive preferences

This section presents a simple production economy and uses it to price bonds at different maturities. The only deviation from the standard stochastic neoclassical growth model is that the model incorporates EZ preferences, instead of the traditional state-separable constant relative risk aversion (CRRA) ones. Also, there is an exogenous process for inflation that captures well the dynamics of price changes in the data and that will allow us to value nominal bonds.

2.1. Preferences

There is a representative household whose utility function over streams of consumption $c_t$ and leisure $1 - l_t$ is:

$$U_t = \left[ \left( c_t^\psi (1 - l_t)^{1 - \psi} \right)^{1 - \gamma} + \beta \left( \mathbb{E}_t U_{t+1}^{1 - \gamma} \right)^{\frac{1}{\psi}} \right]^{1/\gamma},$$ (1)

where $\gamma \geq 0$ is the parameter that controls risk aversion, $\psi \geq 0$ is the IES, and $\theta \equiv (1 - \gamma) / \left( 1 - \frac{1}{\psi} \right)$. The term $\left( \mathbb{E}_t U_{t+1}^{1 - \gamma} \right)^{\frac{1}{\psi}}$ is often called the risk-adjusted expectation operator. If $\gamma = \frac{1}{\psi}$, then $\theta = 1$ and the recursive preferences collapse to the CRRA case. The EZ framework implies that the household has preferences for the timing of the resolution of uncertainty. If $\gamma > \frac{1}{\psi}$, the household prefers an early

3 The interested reader is referred to an appendix, available on line through Science Direct, for further details on the paper.
resolution of uncertainty, and if \( \gamma < \frac{1}{\psi} \), a later resolution. The discount factor is \( \beta \) and one period corresponds to one quarter.

2.2. Technology

There is a representative firm with a production function \( y_t = k_t^\zeta (z_t l_t)^{1-\zeta} \), where output \( y_t \) requires capital, \( k_t \), labor, \( l_t \), and technology \( z_t \). Technology evolves as a random walk in logs with drift \( \lambda \):

\[
\log z_{t+1} = \lambda + \log z_t + \chi \sigma \varepsilon_{zt+1},
\]

where \( \varepsilon_{zt} \sim \mathcal{N}(0, 1) \). The perturbation parameter \( \chi \) scales the standard deviation of the productivity shock, \( \sigma \varepsilon \). This parameter will facilitate the presentation of our solution method later on. Tallarini (2000) shows that a unit root representation such as (2) matches the observed market price of risk in a model close to ours better than a specification with a deterministic trend. Similarly, Álvarez and Jermann (2005) calculate that most of the unconditional variation in the pricing kernel comes from the permanent component. As emphasized by Rouwenhorst (1995), period-by-period unit root shifts of the long-run growth path of the economy increase the variance of future paths of the endogenous variables and, hence, the utility cost of risk.

2.3. Budget and resource constraints

The budget constraint of the household is:

\[
c_t + i_t + \frac{b_{t+1}}{p_t} \frac{1}{R_t} = r_t k_t + w_t l_t + \frac{b_t}{p_t},
\]

where \( p_t \) is the price level at time \( t \), \( i_t \) is investment in period \( t \), \( k_t \) is capital in period \( t \), \( b_t \) is the number of one-period non-contingent bonds held in period \( t \) that pay one nominal unit in period \( t+1 \), \( R_t^{-1} \) is their unit price at time \( t \), \( w_t \) is the real wage at time \( t \), and \( r_t \) is the real rental price of capital at time \( t \), both measured in units of the final good. In the interest of clarity, the budget constraint includes only the one-period non-contingent bond just described. Using the pricing kernel, in section 2.6., we will write the set of equations that determine the prices of nominal bonds at any maturity.
Their price in equilibrium will be such that the representative agent will hold a zero amount of them.

The aggregate resource constraint is

\[ y_t = c_t + i_t. \] (4)

2.4. Dynamics of the capital stock

Capital depreciates at rate \( \delta \). Thus, the dynamics of the capital stock are given by:

\[ k_{t+1} = (1 - \delta) k_t + G \left( \frac{i_t}{k_t} \right) k_t, \] (5)

in which:

\[ G \left( \frac{i_t}{k_t} \right) = a_1 + \frac{a_2}{1 - \frac{1}{\tau}} \left( \frac{i_t}{k_t} \right)^{1 - \frac{1}{\tau}}, \] (6)

denotes the adjustment cost of capital as in Jermann (1998). The normalization \( a_1 = \frac{e^{\lambda} - 1 + \delta}{1 - \gamma} \) and \( a_2 = \left( e^{\lambda} - 1 + \delta \right)^{\frac{1}{\tau}} \) ensures that adjustment costs do not affect the steady state of the model.

2.5. Inflation dynamics

Our data will include nominal bond yields at different maturities as part of our observables. Hence, the model needs to take a stand on how inflation, \( \log \pi_t \), evolves over time. To keep the model as stylized as possible, inflation is assumed to be an exogenous process that does not affect allocations. Therefore, money is neutral in our economy. Also, the representative household has rational expectations about these inflation dynamics.

We specify \( \log \pi_t \equiv \log p_t - \log p_{t-1} \) as:

\[ \log \pi_{t+1} = \log \bar{\pi} + \rho (\log \pi_t - \log \bar{\pi}) + \chi \left( \sigma_{\omega} \omega_{t+1} + \kappa_0 \sigma_{\varepsilon} \varepsilon_{zt+1} \right) + \iota \left( \sigma_{\omega} \omega_t + \kappa_1 \sigma_{\varepsilon} \varepsilon_{zt} \right), \] (7)

where \( \omega_t \sim \mathcal{N}(0,1) \), \( \omega_t \perp \varepsilon_{zt} \). The parameters \( \kappa_0 \) and \( \kappa_1 \) capture the correlation of unexpected and expected inflation with innovations to technology, \( \varepsilon_{zt+1} \) and \( \varepsilon_{zt} \) respectively. As before, \( \chi \) is the perturbation parameter.

This specification accomplishes two objectives. First, it lets us consider a correlation between innovations to inflation expectations and innovations to the stochastic discount factor. Thus, bond
Pricing nominal bonds

The stochastic discount factor (SDF) for our economy is:

\[ M_{t+1} = \beta \left( \frac{c_{t+1}^{\nu} (1 - l_{t+1})^{1-\nu}}{c_t^{\nu} (1 - l_t)^{1-\nu}} \right)^{1-\gamma} \frac{c_t}{c_{t+1}} \left( \frac{V_t^{1-\gamma}}{\mathbb{E}_t V_{t+1}^{1-\gamma}} \right)^{1-\frac{1}{\gamma}}. \] (8)
where the value function $V_t$ is defined as $V_t = \max_{c_t, l_t, i_t} U_t$, subject to (4) and (5). The switch in notation to $V_t$ is convenient to distinguish between the utility function of the household, $U_t$, and the value function that solves the household’s problem $V_t$. Since the welfare theorems hold in our model, this is also the value function of the social planner, a result used in a couple of steps below. Nothing of substance depends on working with the social planner’s problem except that the notation is easier to handle.

Hence, the Euler equation for the one-period nominal bonds is $\mathbb{E}_t \left( M_{t+1} \frac{1}{\pi_{t+1}} \right) = \frac{1}{R_t}$, which can be written as:

$$
\mathbb{E}_t \left[ \beta \left( \frac{c_t^\gamma (1 - l_t+1)^{1-\gamma}}{c_t^\gamma (1 - l_t)^{1-\gamma}} \right) \frac{1-\gamma}{1-\gamma} \frac{1}{\pi_{t+1}} \right] = \frac{1}{R_t}.
$$

More importantly, bond prices can be computed recursively using:

$$
\mathbb{E}_t \left( M_{t+1} \frac{1}{\pi_{t+1} R_{t+1,t+m}} \right) = \frac{1}{R_{t,t+m}},
$$

with $R_{t,t+m}^{-1}$ being the time-$t$ price of an $m$-period nominal bond, $R_{t,t+1} = R_t$, and $R_{t+1,t+1} = 1$.

There is no analytic expression for the equilibrium dynamics of the model. The next two sections will explain, first, how to use perturbation methods to solve for these dynamics and, second, will show how to exploit the output of the perturbation to write a state-space representation of the model that can be employed to evaluate the associated likelihood function.

### 3. Solving the model using perturbation

Our economy is solved by perturbing the value function of the household plus the equilibrium conditions defined by optimality and feasibility. We find a third-order approximation to the value function and decision rules. Third-order terms are needed to allow for a time-varying risk premium, an important feature of the data. The accuracy of a third-order perturbation in terms of Euler equation errors is excellent even far away from the steady state, which suggests higher-order approximations are not needed (see Caldara et al., 2012, for details). The advantage of perturbation over value function iteration or projection is that it is sufficiently fast for likelihood estimation.
Our exposition uses a concise notation to illustrate the required steps. Otherwise, the algebra becomes too involved to be developed explicitly. In the application, the symbolic algebra is undertaken by Mathematica, which automatically generates Fortran 95 code that is evaluated numerically.

3.1. Approximating the value function and decision rules

First, the model is rendered stationary by rescaling the variables by $z_{t-1}$. For any variable $x_t$, $\tilde{x}_t = x_t/z_{t-1}$ is its normalized value. Also, remember that the stochastic processes are written in terms of a perturbation parameter $\chi$. The case $\chi = 1$ deals with the stochastic version of the model and the case $\chi = 0$ deals with the deterministic case with steady state $\tilde{k}_{ss}$ and $\log \tilde{z}_{ss} = \lambda$.

Thus, one can write the value function, $V(\tilde{k}_t, \log \tilde{z}_t; \chi)$, and the decision rules for any variable $var (\tilde{c}_t, \tilde{l}_t, \tilde{i}_t$, and $\tilde{k}_{t+1})$ as $var (\tilde{k}_t, \log \tilde{z}_t; \chi)$, as depending on the rescaled states, $\tilde{k}_t$ and $\log \tilde{z}_t$, and the perturbation parameter, $\chi$. Since money is neutral in this model, these functions do not depend on inflation. This allows to solve for them without considering inflation and, in a second step, to use them to solve for nominal bond prices that do depend on inflation.

Define $s_t = (\tilde{k}_t - \tilde{k}_{ss}, \log \tilde{z}_t - \log \tilde{z}_{ss}; 1)$ as the vector of states in differences with respect to the steady state, where $s_{it}$ is the $i - th$ component of this vector at time $t$ for $i \in \{1, 2, 3\}$. Under differentiability conditions, the third-order Taylor approximation of the value function, evaluated at $\chi = 1$, around the steady state is

$$V(\tilde{k}_t, \log \tilde{z}_t; 1) \approx V_{ss} + V_{i,ss} s_t^{i} + \frac{1}{2} V_{ij,ss} s_t^{i} s_t^{j} + \frac{1}{6} V_{ijkl,ss} s_t^{i} s_t^{j} s_t^{k},$$

(11)

where $V_{...,ss}$ is the derivative of the value function evaluated at the steady state, $(\tilde{k}_{ss}, \log \tilde{z}_{ss}; 0)$. Thus, $V_{i,ss} \equiv V_i (\tilde{k}_{ss}, \log \tilde{z}_{ss}; 0)$ for $i \in \{1, 2, 3\}$, $V_{ij,ss} \equiv V_{ij} (\tilde{k}_{ss}, \log \tilde{z}_{ss}; 0)$ for $i, j \in \{1, 2, 3\}$, and $V_{ijkl,ss} \equiv V_{ijkl} (\tilde{k}_{ss}, \log \tilde{z}_{ss}; 0)$ for $i, j, l \in \{1, 2, 3\}$. Note the use of the tensors $V_{i,ss} s_t^{i} = \sum_{i=1}^{3} V_{i,ss} s_t^{i}$, $V_{ij,ss} s_t^{i} s_t^{j} = \sum_{i=1}^{3} \sum_{j=1}^{3} V_{ij,ss} s_t^{i} s_t^{j}$, and $V_{ijkl,ss} s_t^{i} s_t^{j} s_t^{k} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} V_{ijkl,ss} s_t^{i} s_t^{j} s_t^{k}$, which eliminate the symbol $\sum_{i=1}^{3}$ when no confusion arises. Also, define $V_{ss} \equiv V (\tilde{k}_{ss}, \log \tilde{z}_{ss}; 0)$.

When we evaluate expression (11) at $(\tilde{k}_{ss}, \log \tilde{z}_{ss}; 1)$ (the steady state values and positive variance of shocks), all terms drop, except $V_{ss}$, $V_{3,ss}$, $V_{33,ss}$, and $V_{333,ss}$. Since all the terms in odd powers of $\chi$ (in this case, $V_{3,ss}$ and $V_{333,ss}$) are equal to zero, a third-order approximation of the value function
evaluated in \((\bar{k}_{ss}, \log \bar{z}_{ss}; 1)\) is \(V(\bar{k}_{ss}, \log \bar{z}_{ss}; 1) \approx V_{ss} + \frac{1}{2} V_{33, ss}\), where \(\frac{1}{2} V_{33, ss}\) is a measure of the welfare cost of the business cycle, that is, of how much utility changes when the variance of the productivity shocks is \(\sigma_{\varepsilon}^2\) instead of zero (as done in section 5, this welfare cost can easily be transformed into consumption equivalent units). Deriving this term is another advantage of perturbation.

Following the same notation, the decision rule for a variable \(\text{var}\) can be approximated as

\[
\text{var}(\bar{k}_t, \log \bar{z}_t; 1) \approx \text{var}_{ss} + \text{var}_{i, ss} s_t^i + \frac{1}{2} \text{var}_{ij, ss} s_t^i s_t^j + \frac{1}{6} \text{var}_{ijl, ss} s_t^i s_t^j s_t^l. \tag{12}
\]

A perturbation method finds the unknown derivatives \(V_{..., ss}\) and \(\text{var}_{..., ss}\) by taking derivatives of the equilibrium conditions of the model plus the value function (as enumerated in the appendix) and applying an implicit function theorem to solve for them.

### 3.2. Approximating nominal bond yields

To complete our computation, the yield of nominal bonds is approximated with the recursive bond price equation (10). First, define \(sa_t = (\bar{k}_t - \bar{k}_{ss}, \log \bar{z}_t - \log \bar{z}_{ss}, \log \pi_t - \log \bar{\pi}, \omega_t; 1)\), which is the state vector in deviations with respect to the steady state augmented with inflation and the inflation innovation (\(sa\) stands for states augmented).

In similar fashion to the value function and the decision rules, a third-order Taylor approximation to the yields is, for all \(m\):

\[
R_m(\bar{k}_t, \log \bar{z}_t, \log \pi_t, \omega_t; 1) \approx R_{m, ss} + R_{m, i, ss} sa_t + \frac{1}{2} R_{m, ij, ss} sa_t^i s_t^j + \frac{1}{6} R_{m, ijl, ss} sa_t^i s_t^j s_t^l, \tag{13}
\]

where \(R_{m, ss} \equiv R_{m, ss}(\bar{k}_{ss}, \log \bar{z}_{ss}, \log \bar{\pi}, 0; 0)\), \(R_{m, i, ss} \equiv R_{m, i}(\bar{k}_{ss}, \log \bar{z}_{ss}, \log \bar{\pi}, 0; 0)\) for \(i \in \{1, \ldots, 5\}\), \(R_{m, ij, ss} \equiv R_{m, ij}(\bar{k}_{ss}, \log \bar{z}_{ss}, \log \bar{\pi}, 0; 0)\) for \(i, j \in \{1, \ldots, 5\}\), and \(R_{m, ijl, ss} \equiv R_{m, ijl}(\bar{k}_{ss}, \log \bar{z}_{ss}, 0; 0)\) for \(i, j, l \in \{1, \ldots, 5\}\).

Since in our data set includes bond yields up to 20 quarters, we consider

\[
\mathbb{E}_t \left( M_{t+1} \frac{1}{\pi_{t+1} R_{t+1, t+m}} \right) = \frac{1}{R_{t, t+m}}, \tag{14}
\]

for \(m \in \{1, \ldots, 20\}\) and approximate these 20 first-order conditions as done before for the perturbation of the value function and decision rules (once the results from the approximation of the value function and decision rules are substituted in).
3.3. Role of $\gamma$

Direct inspection of the derivatives presented before (since the expressions are inordinately long, the paper cannot include them) reveals several points of interest. First, the constant terms $V_{ss}$, $var_{ss}$, or $R_{m,ss}$ do not depend on $\gamma$. Second, none of the terms in the first-order approximation, $V_{.,ss}$, $var_{.,ss}$, or $R_{m.,ss}$ depend on $\gamma$. Third, none of the terms in the second-order approximation, $V_{..,ss}$, $var_{..,ss}$, or $R_{m..,ss}$ depend on $\gamma$, except $V_{33,ss}$, $var_{33,ss}$, and $R_{m,33,ss}$. These last terms are constants that capture precautionary behavior caused by productivity shocks. Fourth, in the third-order approximation only the terms $V_{33,ss}$, $V_{3,3,ss}$, $V_{3,ss}$ and $var_{33,ss}$, $var_{3,3,ss}$, $var_{3,ss}$ and $R_{m,33,ss}$, $R_{m,3,3,ss}$, $R_{m,3,ss}$, $R_{m,33,ss}$, that is, terms involving $\chi^2$, depend on $\gamma$.

These observations tell us three important facts. First, a linear approximation to the decision rules is certainty equivalent: it does not depend on the parameter that controls risk aversion $\gamma$ or on the variance of the productivity shock. Therefore, researchers interested in recursive preferences must go, at least, to a second-order approximation. Second, given some fixed parameter values, the difference between the second-order approximation to the decision rules of a model with CRRA preferences and a model with recursive preferences is just a constant. This constant generates a second, indirect effect, because it changes the ergodic distribution of the state variables and, hence, the points where the decision rules are evaluated along the equilibrium path. In the third-order approximation, all of the terms on functions of $\chi^2$ depend on $\gamma$. Thus, they can be used to further identify the risk aversion parameter, which is only weakly identified in the second-order approximation as it shows up only in one term and is not identified at all in the first-order approximation. These arguments also demonstrate how perturbation methods provide analytic insights beyond computational advantages and help in understanding the numerical results in Tallarini (2000).\footnote{This characterization is also crucial because it is plausible that EZ preferences are not identified (as in the example built by Kocherlakota, 1990). Fortunately, higher-order terms circumvent this problem. This confirms previous, although somehow more limited, theoretical results. In a simpler environment, Wang (1993) shows that the parameters of EZ preferences are generically recoverable from the price of equity or from the price of bonds. Furthermore, equity and bond prices are generically unique and smooth with respect to parameters.}
4. Estimation

With the solution from the previous section, we can write a state-space representation of the dynamics of the states and observables. Our observables are per capita consumption growth, per capita output growth, the 1-, 2-, 3-, 4-, and 5-year nominal bond yields, and inflation. Per capita consumption growth and per capita output growth bring macro information. The price of the nominal bonds provides us with financial data.

Since the model has only two sources of uncertainty, the productivity shock and the inflation shock, measurement error is introduced to avoid stochastic singularity. It is common to have measurement error in term structure models because zero coupon bonds are not observed. Instead, researchers observe the market prices of bonds with coupons and some procedure is needed to back out the zero coupon bonds. This procedure induces measurement error. Similarly, National Income and Product Accounts (NIPA) can provide researchers only with approximated estimates of output and consumption. Hence, all the variables (except inflation) are assumed to be observed subject to a measurement error.\(^5\) An alternative interpretation of our measurement errors is as specification errors not modelled explicitly. The online appendix presents the detailed derivations of the state-space transition and measurement equation.

Next, this representation is exploited to evaluate the likelihood function of the model. Unfortunately, this task is difficult since there is no analytic expression for the state-space representation. The problem is tackled by using the particle filter, a sequential Monte Carlo, as shown by Fernández-Villaverde and Rubio-Ramírez (2007). The appendix explains this approach in detail.

Our paper emphasizes the likelihood-based estimation of DSGE models. In the interest of space, only results for the MLE are shown. The appendix comments briefly on how to find results for Bayesian estimation. Obtaining the MLE is complicated because the shape of the likelihood function may be rugged and multimodal. Moreover, the particle filter generates an approximation to the likelihood that is not differentiable with respect to the parameters, precluding the use of optimization algorithms based on derivatives. To circumvent these problems, our optimization routine is a procedure

\(^5\)Our exogenous process for inflation already has a linear additive innovation \(\omega_{t+1}\), which will make an additional measurement error difficult to identify.
known as covariance matrix adaptation evolutionary strategy, or CMA-ES. The CMA-ES is a powerful evolutionary optimization algorithm and has been applied in a fruitful way to many problems. See the online appendix for details and references.

The standard errors reported below come from the bootstrapping procedure described by Efron and Tibshirani (1993, chapter 6). The estimated model is used to generate 100 artificial samples of data. These artificial series are used to re-estimate the model 100 times and the standard errors get computed as the standard deviations of the MLE taken across these 100 replications. This bootstrapping procedure accounts for the finite-sample properties of the MLE and avoids the numerical instabilities that often appear while inverting the matrix of second derivatives of a likelihood function. These instabilities would be even more acute in our case since the particle filter delivers non-differentiable approximation of the likelihood function.

5. Data and main results

This section describes the data and presents the empirical results. To fully understand how the parameters are identified in our model, the model is estimated in three steps. First, the model is estimated using all data. Second, the model is estimated excluding inflation. Third, only bond yields are used. Studying which parameters change by changing information sets will uncover which moments pin down which parameters.

5.1. Data

Our sample is the period 1953.Q1 to 2008.Q4. Output and consumption data come from the Bureau of Economic Analysis NIPA. Nominal consumption is defined as the sum of personal consumption expenditures on non-durable goods and services. Per capita nominal output and consumption are defined as the ratio between our nominal output and consumption series and the civilian non-institutional population over 16. For inflation (and to transform nominal into real variables), we use the gross domestic product deflator. The data on bond yields are from CRSP Fama-Bliss discount bond files, which have fully taxable, non-callable, non-flower bonds. Fama and Bliss construct their data by interpolating observations from traded Treasuries. This procedure introduces measurement
error, possibly correlated across time and cross-sectionally (although in our estimation, and to reduce
the number of parameters to maximize over, these correlations are not allowed for).

Hours per capita are not included in the observables because the model cannot generate enough
fluctuations in hours. In any case, to put some restrictions on the behavior of the model-based hours,
hours worked are normalized to have mean 0.5 during the sample period (this normalization level is
per se irrelevant) and make $v$ a function of the rest of the parameters such that hours worked in the
steady state model are always 0.5 for any value of the rest of the parameters.

Table 1 reports the summary statistics from our data (means, standard deviations, and quartiles).
Key, well-known observations are as follows. First, the volatility of output growth is higher than the
volatility of consumption growth. Second, the yield curve is, on average, upward sloping. Third,
the volatilities of bond yields are downward sloping for maturities of one year and longer. The next
paragraphs study how the model scores along these dimensions.

5.2. Estimation results

Before proceeding with the estimation, a subset of the parameters is fixed. The reason is because
estimating a third-order approximation model, which, as argued before, is crucial for identification,
is extremely time consuming. Time constraints make it infeasible, in practice, to estimate the whole
set of parameters. Thus, in addition to the calibrated inflation parameters described above, we set
$\lambda = 0.0045$, $\zeta = 0.3$, and $\delta = 0.0294$. The value of $\lambda$ is chosen to match the average growth rate of per
capita output in our sample, the values of $\zeta$ to match the capital income share, and the value of $\delta$ to
match the investment-capital ratio. Finally, the standard deviation of the measurement error shocks
is set such that the model explains around 75 percent of the standard deviation in the data.

The first findings are reported in table 2. The table displays estimates of the parameters of the
model. The first column lists our estimated parameters. The second and third columns report the
MLEs and their standard errors when consumption growth, output growth, five bond yields, and inflation are used in the estimation. The fourth and fifth columns report the same statistics when inflation is excluded from the estimation. The last two columns contain the statistics when only the five bond yields are used in the estimation. Let us concentrate first on the result for the whole data set and explore the other columns in the subsections below.

5.3. Data set I: consumption, output, bond yields, and inflation

5.3.1. Preference Parameters

The discount factor, $\beta$, is estimated to be 0.997. This value, a standard result in the literature, allows us to match the nominal yield (the model includes inflation and growth, and both factors affect the nominal yield). The coefficient that controls risk aversion, $\gamma$, is estimated to be 65.78, which is high.$^6$ The risk aversion coefficient has a strong impact on the welfare calculations using the formula derived in section 3.1. Figure 1 plots the welfare costs as a share of consumption in the steady state. The horizontal axis displays the risk aversion coefficient and the vertical axis the fraction of consumption the agent is willing to give up to avoid uncertainty (conditional on all the other parameters being at their estimated or calibrated values). In this figure, one can see a substantial estimated welfare cost of the business cycle, close to 42 percent of consumption. The size of these losses is in the range of those reported by Tallarini (2000) for his random walk specification.

[Figure 1 Here]

The IES is estimated to be 1.73. An estimate higher than one resonates with the parameter values picked in the literature on long-run risks (see, for instance, Bansal and Yaron, 2004). A good way to understand our high MLE for the IES is as follows. A low IES means that the household dislikes changes in consumption over time. Thus, after a positive technology shock, consumption jumps at impact and it only grows slowly after the first period. A higher IES means that the household dislikes changes in consumption less. Thus, consumption will jump less at impact, allowing for more resources

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$^6$We need to be careful assessing this number, since our model includes leisure. It happens that, given the Cobb-Douglas specification of our utility aggregator of consumption and leisure, relative risk aversion is equal to $\gamma$. This does not need to be the case with other aggregators of consumption and leisure. See Swanson (2012) for details.
to be invested (to take advantage of the higher productivity) and generating more consumption in the
future. That is, a low IES makes the household reluctant to sacrifice current consumption in exchange
for future consumption generated by higher investment today, while a high IES does exactly the
opposite. A direct consequence of this willingness to substitute consumption today versus tomorrow
is the autocorrelation of consumption growth. A lower IES will trigger a very low (even negative
in a relatively standard parameterization of an RBC model) autocorrelation of consumption growth.
A higher IES, in comparison, will generate a larger autocorrelation. Since there is quite a bit of
autocorrelation of consumption growth in the data, it is not surprising that the likelihood peaks at a
large IES (below, the same argument will be revisited, but in the form of the slope of the yield curve,
and understand why the likelihood does not pick an even larger value for the IES). The combination
of the estimated values for the parameters controlling risk aversion and IES implies \( \theta = -153.64 \),
indicating very different attitudes toward intertemporal substitution and toward substitution across
states of nature. Moreover, given that in our point estimate \( \gamma \gg \frac{1}{\psi} \), our representative household has
a very strong preference for an early resolution of uncertainty.

5.3.2. Technology and Inflation Parameters

The adjustment cost parameter, \( \tau \), is estimated to be 0.101, which indicates substantial adjustment
costs and a too low volatility of investment growth (around 1 percent). The estimate of \( \tau \) comes about
because our data favor a situation in which capital cannot adjust easily to smooth consumption. When
this is the case, the SDF fluctuates more and it is easier to match both the premium and the volatility
of the yield curve. The volatility of the technology process, \( \sigma_\varepsilon \), is 0.008. This number is similar to
many estimates in the literature and allows us to nicely match output and consumption volatility.
Since the first-order approximation of our model behaves in the same way as the one from a simple
real business cycle model, and this one is also able to match output and consumption properties, this
finding is not a surprise.

The parameter controlling the MA component of the inflation process, \( \iota \), is well into negative
terms, \(-0.452\), and close to the value reported by Stock and Watson (2007), allowing us to capture
the negative first-order autocorrelation and the small higher-order autocorrelations of inflation growth
observed in the data. Since the nominal yield curve slopes up in the data, \( \kappa_0 \) and \( \kappa_1 \) are estimated such that the correlation between innovations to inflation expectations and innovations to the SDF expectations implies that increases in inflation are bad news for consumption growth, that is, such that
\[
(\rho \kappa_0 + \iota \kappa_1) \sigma_\varepsilon \text{ is negative (see, for a similar reason, the analysis of Piazzesi and Schneider, 2006).}
\]
The problem for that mechanism is that observed inflation volatility imposes a constraint on the maximum for the absolute value of \((\rho \kappa_0 + \iota \kappa_1) \sigma_\varepsilon \) and \( \sigma_\omega \) (estimated to be 0.002) and, hence, while inflation volatility can matched, the model is barely able to generate an upward-sloping term structure. We will come back to this point momentarily.

5.3.3. Volatilities and Autocorrelations

Table 3 displays means (panel A) and volatilities (panel B) of consumption growth, output growth, five bond yields, and inflation. In each panel, the first row displays the sample moments in the data. The second row corresponds to the moments of the model implied by the estimates when all available data is used. The third row uses the estimates based on consumption growth, output growth, and five bond yields, but omit inflation data in estimation. The last row uses the estimates obtained using only bond yields in estimation. To compute the moments of the estimated model, the model is simulated 5,000 times as burn-in plus the sample size of 224 observations at the MLE.

Table 3 tells us that the model that uses all the data does a fair job at matching the mean of consumption growth and the average yields. However, it has more problems with output growth and the average slope of the yields. The spread between the 5-year and 1-year yields amounts to 59 basis points in the data, whereas our model produces an average yield spread of only 11 basis points. This is not a novel result. Koijen \textit{et al.} (2010) show that it is hard to generate a realistic slope in the yield curve even in models of long-run risks. Furthermore, our estimated model does reasonably well with inflation volatility, but underestimates the volatility of bond yields by about a factor of two. Hence, our model has a difficult time jointly reproducing the salient features of consumption and output growth, the term structure of nominal interest rates, and inflation.

---

\(^7\)Stock and Watson (2007) split their sample into two parts: 1960:Q1 to 1983:Q4 and 1984:Q1 to 2004:Q4. Their estimated values for \( \iota \) are lower (in absolute value) for the first part and higher for the second. Our sample period 1953:Q1 to 2008:Q4 includes both parts and, as expected, our estimate is right in the middle of their two estimates.
Table 4 has a structure similar to table 3, but reports the autocorrelation of consumption growth (panel A), the 1-year bond yield (panel B), and inflation (panel C) for lag lengths from one to ten quarters. The model is able to generate the autocorrelation patterns of the 1-year yield and inflation remarkably well. It does worse for consumption growth, where it misses the slow decay of the autocorrelation function. This is an advantage of MLE, which tries to match the whole set of moments of the data, including the autocorrelations, instead of focusing on a limited set of moments.

5.3.4. Understanding the Mechanism

A key question to understand the results of the estimation is to determine why the likelihood function picks the combination of values of the IES and risk aversion reported before. To cast some light on this result, figure 2 plots a proxy for the slope of the yield curve, the mean spread between the 5- and the 1-year bonds, as a function of the IES for two risk aversion parameters $\gamma$, 80 (a bit higher than our MLE, dashed line) and 10 (a much lower value, closer to the range in other studies, solid line) when all the other parameters are fixed at their MLE.

Figure 2 tells us that the spread is growing both in the IES and in $\gamma$. However, even for high values of the IES and $\gamma$, the spread is much smaller than in the data. For example, for an IES of 2.5 and $\gamma = 80$, the spread is 11.5 basis points (against 59 in the data). For the lower value of $\gamma$ the situation is even more dire: the spread is negative for all values of the IES. Hence, the estimation wants to go to areas of high IES and high $\gamma$.

The next question is why our MLE does not choose an IES and $\gamma$ as big as possible. The answer is in figures 3 and 4. These figures plot the standard deviation of 1-year (figure 3) and 5-year bond yields (figure 4) as the IES and $\gamma$ are changed in the same way than in figure 2. Both standard deviations decrease as either IES or $\gamma$ increases. But the model is already underestimating the standard
deviations of the yields at the reported MLE. Hence, the MLE must compromise between generating a sufficiently large average slope of the yield curve and avoiding too small standard deviations of the yields. Interestingly, since the slope of the yield curve responds substantially to changes in $\gamma$ (the difference between the two lines in figure 2), but the standard deviations much less (see the difference in figures 3 and 4), the likelihood opts for a large IES and a much larger $\gamma$.

![Figure 3 Here]

![Figure 4 Here]

The intuition behind figures 2, 3, and 4 is as follows. A higher IES implies that the household has a stronger preference for an earlier resolution of the uncertainty and hence for the earlier resolution of payoffs implied by bonds with short maturity. This point is reinforced by the fact that consumption growth and inflation are negatively correlated as shown in table 5. Only a higher yield induces the household to hold long-maturity bonds. With respect to $\gamma$, as it increases, the inflation risk loading of long-term bonds becomes more costly in terms of welfare and the household must be compensated to take this risk. At the same time, the household responds to more risk by raising its precautionary savings. Higher savings mean more capital and, with it, a better ability to smooth consumption and avoid fluctuations in the SDF and, hence, in the volatility of the yields.

Our argument resurfaces in figure 5, where the real yield spreads (5-year bond minus 1-year real bond) are plotted as a function of the IES, again for two risk aversion parameters $\gamma$, 10 and 80. Figure 5 shows the trade-off between real and nominal yield data. As mentioned above, to generate substantial inflation risk premia, a large $\gamma$ is asked for. However, for a large $\gamma$, the real term structure is more downward sloping. By choosing a higher value of the IES, this effect of the downward sloping real yield curve is mitigated.8

![Figure 5 Here]

The tension between matching the slope of the yield curve and the standard deviations of the yields is a consequence of dealing with a production economy (through the endogenous responses of

8We thank one referee for the argument in this paragraph.
consumption and investment). Instead of considering this as a failure of the model, one could also interpret it as proof of the extra discipline imposed by production economies and of how the additional structure casts light on the economics of the problem.

5.4. Data set II: consumption, output, and bond yields

To gain further insight into why the model generates neither a substantial slope of the yield curve nor enough volatility of the yields, the model is re-estimated using only parts of the data. This subsection discusses the results when the observations on inflation are omitted. Column 4 of table 2 shows how omitting inflation leads to two main changes in the results. First, it changes the estimates of the inflation parameters. This is because these parameters are no longer disciplined by observed inflation. In particular, \( \kappa_0 \) and \( \kappa_1 \) are estimated such that the absolute values of \( (\rho \kappa_0 + \iota \kappa_1) \sigma_\varepsilon \) and \( \sigma_\omega \) are larger. Second, the new MLEs of the IES and \( \gamma \) are also larger. These two changes lead to a dramatic improvement in terms of fit of the average yield spread and yield volatilities. The model now replicates the observed average slope of the yield curve and does a better job matching the bond yield volatilities, at least for shorter maturities. This success of the model is accomplished at a cost, though. The volatility of inflation is now overestimated; it is 2.33 percent in the data, and the MLEs imply a volatility of inflation of 3.69 percent.

The reason why omitting inflation allows a better match of the yield data (both average slope and volatility of yields) is simple. When all the data were used, the model wanted to increase the IES and \( \gamma \) estimates to match the average slope of the yield curve. The fact that yield volatilities drop as the IES and \( \gamma \) increase puts a restriction on the slope of the yield curve the model can generate. However, if one is not constrained to fit the dynamics of inflation, its volatility can be made sufficiently high so as to generate a sizable volatility of the yields and the IES and \( \gamma \) large enough to also match the average slope. In addition, more volatility in inflation also helps because in our model, bad news about inflation is bad news about consumption growth, which makes longer-maturity bonds particularly risky and generates a larger spread of the yields. Omitting inflation data is inconsequential for matching the autocorrelation patterns in consumption growth, the 1-year bond yield, and inflation as shown in table 4.
This exercise illustrates the importance of a joint estimation of inflation and structural parameters. Without the constraint of having to jointly match inflation and the yield curve, the model is sufficiently flexible to capture selected aspects of the data. This is an example of why simple calibration exercises, where the focus is on a set of moments selected by the researcher, may be undesirable.

5.5. Data set III: bond yields

Finally, the model parameters are estimated using only information contained in bond yields. The main difference in point estimates is that an increase in the volatility of the shocks ($\sigma_\varepsilon = 0.011$ and $\sigma_\omega = 0.0045$) allows a reduction of $\gamma$ to 40.78 and of $\psi$ to 1.301. Also, $\kappa_1$ changes sign and it becomes 1.110. This new combination of MLEs, not surprisingly, allows the model to match well the means of the average slope of the yields and considerably improve the match of their volatilities. The model accomplishes that by reducing the IES and $\gamma$ (remember that a lower IES and $\gamma$ imply larger yield volatilities). To compensate for this reduction, and to match the average yield, the model increases the volatility of the shocks that increase the volatility of the SDF. The cost of doing that is that the volatility of consumption growth, output growth, and inflation implied by the estimated parameters is considerably at odds with the data. The ML procedure does that because it does not include these data in the estimation. This result can be read as indicating that yield data (slopes and volatilities) carry a large amount of information about structural parameters of the economy, including the discount factor, risk aversion, and the IES. This demonstrates the potentiality of incorporating finance data into the standard estimation of DSGE models as a key additional source of information.

5.6. Equity premium: implications of our estimates

We analyze now the implications of our estimates for the equity premium.\footnote{Because of computational limitations, it is rather hard to incorporate observed equity premia in our estimation.} The gross real return on equity (conditional on the state tomorrow) is identified with the marginal gross return on investment. The equity premium with our MLE from data set I is 1.88 at an annualized rate. When inflation is dropped from the observables (data set II), the equity premium rises to 2.53, and when only yield data is left (data set III) to 2.67. These three equity premia are sizeable (much bigger than in a standard
RBC model), but still far away from the observed one. This is not a surprise in a model such as ours  
with endogenous accumulation of capital. It was already understood, at least from Tallarini (2000),  
that non-state separable utility functions do not generate large equity premia in production economies  
even with large risk aversions because of the precautionary behavior of savings. Furthermore, in our  
model, a large estimated value of the IES makes the household more willing to substitute returns across  
time and, hence, to ask for a lower premium to invest in equity. This can be seen in data set III, when  
the IES falls from 1.729 to 1.301 while the equity premium increases to 2.67 despite a halving of the  
risk aversion MLE.

5.7. Correlations of inflation with consumption and output growth

Our final exercise is to compute the correlations of inflation with consumption and output growth  
in the data and in the model. This further clarifies the mechanism through which a positive term  
premium is generated. First, we compute the data statistics. The second row of table 5 reports  
the correlation in the data between inflation, consumption growth, and output growth (the model  
is expressed in growth rates, so it is more natural to have these units). The negative correlation of  
consumption growth and inflation is well known: increases in inflation are bad news for consumption  
growth. This correlation causes an inflation risk premium: precisely when the household want to have  
more resources to consume (consumption growth is small), higher inflation eats away the nominal  
returns. The correlation with output growth is similar.

[Table 5 Here]

The third row of the same table reports the model analogs for the case where the model is estimated  
using all the observables (data set I). The model has a hard time delivering the right level of correlation  
between consumption growth and inflation. This is one of the reasons the estimation searches for large  
risk aversions. The fourth row repeats the same exercise, except that now it concentrates on the  
conditional moments after a productivity shock (the only one that moves consumption in our model).  
The correlations improve but they are still far away from the data.

For completeness, the same exercise for data set II (fifth and sixth rows) and data set III (seventh  
and eighth rows) is repeated. When interpreting these conditional correlations, it is important to re-
member that, in a non-linear model, the sum of all the conditional correlations is not the unconditional one. The main lesson from these two alternative data sets is that, once the researcher does not need to worry about accounting for inflation, the model has more freedom to replicate the correlations with inflation, although the improvement is not large.

6. Extensions

Despite some empirical shortcomings, our previous estimation has shown that a rich DSGE model with production and EZ preferences can be successfully taken to the data. Thus, the paper opens the door to a large number of potential extensions. We discuss several that can be solved using our estimation procedure and that might improve the fit of the model to the data. They are left, though, for future work.

**Predictable technology growth** The model assumes technology growth is i.i.d., which might be too restrictive. The model can be extended to feature a predictable component in technology growth. Such a model is analyzed, for instance, in Croce (2006) and is related to the long-run risk literature (Bansal and Yaron, 2004).

**Habit formation** In our specification of recursive preferences, the period utility is of the CRRA type. The model could allow for habit formation in the period utility. Habit formation preferences have been successfully applied in asset pricing by, for instance, Constantinides (1990) and Campbell and Cochrane (1999).

**Variable rare disasters** Rietz (1998) and Barro (2006) showed that variable rare disasters might be a fruitful way to think about asset pricing. In the context of a production economy, Gabaix (2009) constructs a model in which the real business cycle properties of the model are unaffected relative to a standard model without rare disasters, but the asset pricing properties are improved substantially. This extension, however, would depend on our ability to have a perturbation method that can properly capture the effect of large, yet rare shocks.
Fiscal policy Finally, as pointed out by Rudebusch and Swanson (2012), fiscal policy and its variation over time may be important in understanding asset pricing, in particular within the context of DSGE models that can be used for applied policy analysis. Again, this extension is limited by the need to specify the stochastic structure of fiscal policy rules, something about which the profession has not reached yet a consensus.

7. Conclusions

Our investigation has methodological and substantive contributions. Methodologically, this paper has shown how a DSGE model in which the representative agent household has EZ preferences can be solved and estimated by maximum likelihood thanks to the combination of perturbation methods and the particle filter. This leads the way for a large set of future applications. Our substantive findings are that the data indicate large levels of risk aversion, high levels of the IES, and high adjustment costs. The cross-equation restrictions imposed by the equilibrium of the model, in particular by the endogenous physical capital accumulation, limit the ability of the model to jointly account for the slope of the nominal yield curve and the associated volatilities. However, we have pointed out a number of potential avenues of improvement that may solve this problem. All of them can be explored for the first time in the context of a likelihood-estimated DSGE model that can move toward the integration of macro and finance observations with the tools provided in this paper. F. Alvarez, U. Jermann, Using asset prices to measure the persistence of the marginal utility of wealth, Econometrica 73 (2005), 1977–2016.

References


Table 1: Summary Statistics of the Data

<table>
<thead>
<tr>
<th></th>
<th>Cons. gr.</th>
<th>Output gr.</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>Inflation</th>
<th>Hours</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean</strong></td>
<td>2.06%</td>
<td>1.67%</td>
<td>5.56%</td>
<td>5.76%</td>
<td>5.93%</td>
<td>6.06%</td>
<td>6.15%</td>
<td>3.43%</td>
<td>49.99%</td>
</tr>
<tr>
<td><strong>St.dev.</strong></td>
<td>1.96%</td>
<td>3.74%</td>
<td>2.91%</td>
<td>2.87%</td>
<td>2.80%</td>
<td>2.76%</td>
<td>2.72%</td>
<td>2.33%</td>
<td>1.12%</td>
</tr>
<tr>
<td><strong>25%</strong></td>
<td>0.98%</td>
<td>-0.22%</td>
<td>3.42%</td>
<td>3.63%</td>
<td>3.84%</td>
<td>3.99%</td>
<td>4.03%</td>
<td>1.79%</td>
<td>49.36%</td>
</tr>
<tr>
<td><strong>50%</strong></td>
<td>2.11%</td>
<td>1.84%</td>
<td>5.36%</td>
<td>5.45%</td>
<td>5.59%</td>
<td>5.65%</td>
<td>5.71%</td>
<td>2.76%</td>
<td>49.99%</td>
</tr>
<tr>
<td><strong>75%</strong></td>
<td>3.25%</td>
<td>3.82%</td>
<td>7.15%</td>
<td>7.31%</td>
<td>7.44%</td>
<td>7.57%</td>
<td>7.67%</td>
<td>4.46%</td>
<td>50.79%</td>
</tr>
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</table>

All statistics are expressed in annual terms. The sample period is 1953.Q1 to 2008.Q4.
### Table 2: Estimates, different observables

<table>
<thead>
<tr>
<th>Data</th>
<th>All data</th>
<th>All data, but no inflation</th>
<th>Only Yield</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>MLE</td>
<td>Std. Error</td>
<td>MLE</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.997</td>
<td>0.0003</td>
<td>0.997</td>
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<tr>
<td>$\gamma$</td>
<td>65.78</td>
<td>9.5745</td>
<td>84.78</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.729</td>
<td>0.7843</td>
<td>2.027</td>
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<tr>
<td>$\tau$</td>
<td>0.101</td>
<td>0.0312</td>
<td>0.103</td>
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<tr>
<td>$\kappa_0$</td>
<td>-0.115</td>
<td>0.0785</td>
<td>-0.075</td>
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<tr>
<td>$\iota$</td>
<td>-0.452</td>
<td>0.1367</td>
<td>-0.102</td>
</tr>
<tr>
<td>$\kappa_1$</td>
<td>-0.175</td>
<td>0.0134</td>
<td>0.358</td>
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<tr>
<td>$\sigma_z$</td>
<td>0.008</td>
<td>0.0006</td>
<td>0.008</td>
</tr>
<tr>
<td>$\sigma_\omega$</td>
<td>0.002</td>
<td>0.0003</td>
<td>0.003</td>
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</table>
Figure 1: Welfare cost of business cycle fluctuations
Table 3: Model and Data, means and volatilities

Panel A: Means

<table>
<thead>
<tr>
<th></th>
<th>Cons. gr.</th>
<th>Output gr.</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observed Data</td>
<td>2.06%</td>
<td>1.67%</td>
<td>5.56%</td>
<td>5.76%</td>
<td>5.93%</td>
<td>6.06%</td>
<td>6.15%</td>
<td>3.43%</td>
</tr>
<tr>
<td>Model, all data</td>
<td>2.12%</td>
<td>2.09%</td>
<td>5.94%</td>
<td>5.96%</td>
<td>6.00%</td>
<td>6.04%</td>
<td>6.05%</td>
<td>3.58%</td>
</tr>
<tr>
<td>Model, all data, but no inflation</td>
<td>2.11%</td>
<td>2.13%</td>
<td>5.56%</td>
<td>5.74%</td>
<td>5.89%</td>
<td>6.02%</td>
<td>6.13%</td>
<td>3.58%</td>
</tr>
<tr>
<td>Only Yields</td>
<td>2.10%</td>
<td>2.09%</td>
<td>5.53%</td>
<td>5.72%</td>
<td>5.91%</td>
<td>6.12%</td>
<td>6.22%</td>
<td>3.57%</td>
</tr>
</tbody>
</table>

Panel B: Volatilities

<table>
<thead>
<tr>
<th></th>
<th>Cons. gr.</th>
<th>Output gr.</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
<th>Inflation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>1.96%</td>
<td>3.74%</td>
<td>2.91%</td>
<td>2.87%</td>
<td>2.80%</td>
<td>2.76%</td>
<td>2.72%</td>
<td>2.33%</td>
</tr>
<tr>
<td>Model, all data</td>
<td>2.99%</td>
<td>3.01%</td>
<td>2.00%</td>
<td>1.83%</td>
<td>1.69%</td>
<td>1.56%</td>
<td>1.44%</td>
<td>2.31%</td>
</tr>
<tr>
<td>Model, all data, but no inflation</td>
<td>3.13%</td>
<td>2.93%</td>
<td>3.28%</td>
<td>3.00%</td>
<td>2.75%</td>
<td>2.53%</td>
<td>2.33%</td>
<td>3.69%</td>
</tr>
<tr>
<td>Only Yields</td>
<td>4.45%</td>
<td>4.06%</td>
<td>2.96%</td>
<td>2.91%</td>
<td>2.79%</td>
<td>2.57%</td>
<td>2.34%</td>
<td>3.94%</td>
</tr>
</tbody>
</table>
Table 4: Model and Data, autocorrelations

Panel A: Consumption growth

<table>
<thead>
<tr>
<th>Lag length</th>
<th>1Q</th>
<th>2Q</th>
<th>3Q</th>
<th>4Q</th>
<th>5Q</th>
<th>6Q</th>
<th>7Q</th>
<th>8Q</th>
<th>9Q</th>
<th>10Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.32</td>
<td>0.17</td>
<td>0.18</td>
<td>0.09</td>
<td>-0.03</td>
<td>0.06</td>
<td>0.01</td>
<td>-0.151</td>
<td>-0.04</td>
<td>0.01</td>
</tr>
<tr>
<td>All data</td>
<td>0.10</td>
<td>0.03</td>
<td>-0.01</td>
<td>-0.06</td>
<td>-0.06</td>
<td>-0.05</td>
<td>-0.05</td>
<td>-0.04</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>All data, but no inflation</td>
<td>0.11</td>
<td>0.03</td>
<td>-0.01</td>
<td>-0.08</td>
<td>-0.07</td>
<td>-0.06</td>
<td>-0.04</td>
<td>-0.05</td>
<td>-0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>Only Yield</td>
<td>0.10</td>
<td>0.03</td>
<td>-0.00</td>
<td>-0.07</td>
<td>-0.07</td>
<td>-0.06</td>
<td>-0.02</td>
<td>-0.04</td>
<td>-0.04</td>
<td>0.02</td>
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</table>

Panel B: 1-year bond yield

<table>
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<th>Lag length</th>
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<th>2Q</th>
<th>3Q</th>
<th>4Q</th>
<th>5Q</th>
<th>6Q</th>
<th>7Q</th>
<th>8Q</th>
<th>9Q</th>
<th>10Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.95</td>
<td>0.91</td>
<td>0.87</td>
<td>0.81</td>
<td>0.76</td>
<td>0.71</td>
<td>0.67</td>
<td>0.64</td>
<td>0.61</td>
<td>0.58</td>
</tr>
<tr>
<td>All data</td>
<td>0.95</td>
<td>0.93</td>
<td>0.89</td>
<td>0.84</td>
<td>0.78</td>
<td>0.73</td>
<td>0.72</td>
<td>0.67</td>
<td>0.62</td>
<td>0.59</td>
</tr>
<tr>
<td>All data, but no inflation</td>
<td>0.95</td>
<td>0.93</td>
<td>0.89</td>
<td>0.82</td>
<td>0.78</td>
<td>0.73</td>
<td>0.73</td>
<td>0.69</td>
<td>0.63</td>
<td>0.61</td>
</tr>
<tr>
<td>Only Yield</td>
<td>0.95</td>
<td>0.94</td>
<td>0.88</td>
<td>0.82</td>
<td>0.79</td>
<td>0.72</td>
<td>0.72</td>
<td>0.68</td>
<td>0.62</td>
<td>0.60</td>
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</table>

Panel C: Inflation

<table>
<thead>
<tr>
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<th>1Q</th>
<th>2Q</th>
<th>3Q</th>
<th>4Q</th>
<th>5Q</th>
<th>6Q</th>
<th>7Q</th>
<th>8Q</th>
<th>9Q</th>
<th>10Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>0.88</td>
<td>0.83</td>
<td>0.80</td>
<td>0.77</td>
<td>0.71</td>
<td>0.67</td>
<td>0.61</td>
<td>0.59</td>
<td>0.55</td>
<td>0.54</td>
</tr>
<tr>
<td>All data</td>
<td>0.84</td>
<td>0.79</td>
<td>0.75</td>
<td>0.76</td>
<td>0.69</td>
<td>0.66</td>
<td>0.61</td>
<td>0.59</td>
<td>0.56</td>
<td>0.54</td>
</tr>
<tr>
<td>All data, but no inflation</td>
<td>0.94</td>
<td>0.90</td>
<td>0.86</td>
<td>0.82</td>
<td>0.78</td>
<td>0.74</td>
<td>0.71</td>
<td>0.68</td>
<td>0.65</td>
<td>0.62</td>
</tr>
<tr>
<td>Only Yield</td>
<td>0.93</td>
<td>0.91</td>
<td>0.86</td>
<td>0.80</td>
<td>0.77</td>
<td>0.72</td>
<td>0.69</td>
<td>0.68</td>
<td>0.66</td>
<td>0.64</td>
</tr>
</tbody>
</table>
Table 5: Correlation with Inflation

<table>
<thead>
<tr>
<th></th>
<th>consumption growth</th>
<th>output growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>-0.3504</td>
<td>-0.2450</td>
</tr>
<tr>
<td>All data, unconditional</td>
<td>-0.0562</td>
<td>-0.0523</td>
</tr>
<tr>
<td>All data, conditional</td>
<td>-0.0945</td>
<td>-0.1052</td>
</tr>
<tr>
<td>All data, but no inflation, unconditional</td>
<td>-0.0591</td>
<td>-0.0587</td>
</tr>
<tr>
<td>All data, but no inflation, conditional</td>
<td>-0.1180</td>
<td>-0.1098</td>
</tr>
<tr>
<td>Only Yield, unconditional</td>
<td>-0.0674</td>
<td>-0.0693</td>
</tr>
<tr>
<td>Only Yield, conditional</td>
<td>-0.1463</td>
<td>-0.1394</td>
</tr>
</tbody>
</table>
Figure 2: Yield Spreads (5-year bond minus 1-year bond)
Figure 3: Std of 1-year Bond Yields
Figure 4: Std of 5-year Bond Yields