Journal of Econometrics 207 (2018) 71–91

Contents lists available at ScienceDirect

Journal of Econometrics

journal homepage: www.elsevier.com/locate/jeconom

Modeling and forecasting (un)reliable realized covariances for more reliable financial decisions

Tim Bollerslev [a,b], Andrew J. Patton [a,*], Rogier Quaedvlieg [c]

[a] *Department of Economics, Duke University, United States*
[b] *NBER and CREATES, Denmark*
[c] *Department of Finance, Erasmus University Rotterdam, Netherlands*

ARTICLE INFO

Article history:
Received 27 May 2017
Received in revised form 8 February 2018
Accepted 21 May 2018
Available online 24 July 2018

JEL classification:
C32
C58
G11
G32

Keywords:
Common risks
Realized covariances
Forecasting
Asset allocation
Portfolio construction

ABSTRACT

We propose a new framework for modeling and forecasting common financial risks based on (un)reliable realized covariance measures constructed from high-frequency intraday data. Our new approach explicitly incorporates the effect of measurement errors and time-varying attenuation biases into the covariance forecasts, by allowing the ex-ante predictions to respond more (less) aggressively to changes in the ex-post realized covariance measures when they are more (less) reliable. Applying the new procedures in the construction of minimum variance and minimum tracking error portfolios results in reduced turnover and statistically superior positions compared to existing procedures. Translating these statistical improvements into economic gains, we find that under empirically realistic assumptions a risk-averse investor would be willing to pay up to 170 basis points per year to shift to using the new class of forecasting models.

1. Introduction

The presence of common risk factors plays a crucial role in the theory and practice of finance. Common risks, and predictions thereof, are typically quantified through return covariances. While inherently unobservable, there is extensive empirical evidence to support the idea that covariances of asset returns vary through time. From a practical estimation and forecasting perspective, this naturally poses a trade-off between the use of long historical samples to accurately estimate the latent covariances, potentially biasing the forecasts if the risks are changing, or restricting the estimation to more recent observations to better capture the short-term dynamics, thereby potentially increasing the estimation and forecast errors. In response to this, we develop a new framework for more accurately forecasting dynamically varying covariances by explicitly incorporating the effects of (un)reliable past covariance measures into the forecasts. Applying the new procedures in asset allocation decisions, we show that these statistical improvements in the accuracy of the covariance forecasts translate into sizable economic gains for a representative risk averse investor seeking to minimize the overall risk or the tracking error of her portfolio.

Modeling and forecasting of dynamically varying covariances have received much attention in the literature, with numerous multivariate ARCH, GARCH and stochastic volatility specifications been proposed for the job. All of these procedures effectively treat the covariances as latent. More recently, however, the increased availability of reliable

* Correspondence to: Department of Economics, Duke University, 213 Social Sciences Building, Box 90097, Durham, NC 27708-0097, United States.
E-mail address: andrew.patton@duke.edu (A.J. Patton).

https://doi.org/10.1016/j.jeconom.2018.05.004

high-frequency intraday asset prices has spurred somewhat of a paradigm shift based on the idea of directly modeling and forecasting ex-post realized covariance measures constructed from intraday data (see, e.g., Andersen et al., 2013, for a discussion of both the earlier parametric models and the more recent realized volatility literature). The benefits of high-frequency-based realized volatility procedures for practical investment and portfolio allocation decisions have also been extensively documented in the literature (e.g., Fleming et al., 2003; Bandi et al., 2008; Pooter et al., 2008; Liu, 2009; Varneskov and Voev, 2013; Hautsch et al., 2015; Bollerslev et al., 2018, among others).

Even though the use of high-frequency intraday data generally allows for the construction of more accurate realized covariance measures than lower frequency (e.g., daily) data, they are still estimates and as such subject to estimation error. This naturally suggests the use of dynamic state space type representations for handling the inherent signal plus noise structure in the formulation of statistically coherent multivariate volatility forecasting models. This general theoretical framework is exemplified by the Conditional Autoregressive Wishart model of Golosnoy et al. (2012), and the HEAVY model of Opschoor et al. (2018), which assumes a conditional matrix-F distribution for the realized covariance matrices.

Rather than postulating a specific parametric distribution for the realized covariance matrix, or the measurement errors therein, we propose an alternative easy-to-implement reduced-form forecasting procedure. Our approach is guided by an errors-in-variables type interpretation. Intuitively, while measurement errors will generally result in an attenuation of the model parameters towards zero relative to a model based on the true (latent) covariances, if the measurement errors were homoskedastic, the parameter attenuation would be time invariant, and from a practical forecasting perspective inconsequential. If, however, the measurement errors are heteroskedastic, as we show is the case with realized covariance matrix estimates, all observations are "not equal", and the measurement errors can indeed have important consequences from a forecasting perspective. Our new class of multivariate realized covariance based forecasting models explicitly recognize this through the use of the asymptotic distribution theory for high-frequency realized covariance estimation along with measures of integrated quarticity (Barndorff-Nielsen and Shephard, 2004; Barndorff-Nielsen et al., 2011) to help guide the magnitude of the time-varying attenuation in the parameters: the parameters should be relatively large on days when the realized covariances are precisely estimated and more heavily attenuated on days when the measurement errors are large and the signals are weak. By contrast, the average attenuation bias implicit in conventional constant parameter models will be too large (small) on days when the realized covariances are (im)precisely estimated.

Our approach may be seen as a multivariate extension of the univariate HARQ class of volatility models recently proposed by Bollerslev et al. (2016).[1] While conceptually straightforward, moving from the univariate setting to empirically realistic implementable multivariate volatility models presents a number of formidable econometric challenges, not least of which involves appropriate procedures to control the profusion of variables subject to attenuation bias and the number of parameters to be estimated. Of course, moving to a multivariate setting also opens up a host of interesting practically relevant economic applications imbued with their own unique econometric complications, as directly illustrated by our application to portfolio choice.

There is, of course, a vast econometrics literature concerned with the use of generated regressors subject to measurement errors (as exemplified by the classic work of Pagan, 1984). The focus of that literature, however, primarily centers on the consistency and efficiency of parameter estimates, and correspondingly procedures designed to allow for valid inference. By contrast, the new dynamic attenuation models proposed here are explicitly designed for the purpose of improved volatility *forecasting*. The idea of attenuating the influence of past realized covariance matrices in the construction of covariance matrix forecasts based on the magnitude of the corresponding measurement errors also has an earlier precedent in the literature on rolling sample variance estimation and the design of "optimal" data-driven volatility filters (e.g., Foster and Nelson, 1996; Andreou and Ghysels, 2002; Meddahi, 2002). The unpublished work by Ghysels et al. (2006), in particular, explicitly advocates the use of realized volatilities from prior periods as a way to more accurately estimate current period volatility, by allowing the weights assigned to the past realized volatilities to depend in part on the estimation errors and measures of integrated quarticity. However, the main motivation behind that work is on accurate volatility measurement, as opposed to volatility forecasting. Also, in contrast to the present paper, all of these prior studies are exclusively focused on *univariate* volatility inference and estimation procedures.

Our actual empirical investigations are based on high-frequency intraday data for a sample of ten individual stocks. Using standard statistical tests, together with a variety of evaluation criteria and out-of-sample methods, we firstly show that the new carefully designed dynamic attenuation models systematically beat their constant attenuation benchmarks in the sense of providing covariance matrix forecasts that are significantly closer to the ex-post covariances. We further document that these improvements in forecast accuracy are not restricted to the attenuation of especially noisy covariance estimates, but occur in response to both reliable and unreliable covariances.

Next, in an effort to underscore the practical relevance of the forecast improvements afforded by the new class of models, we evaluate their performance in portfolio allocation decisions. The detrimental impact of measurement errors in the context portfolio construction has been widely studied in the literature, and it is well established that the use of ill conditioned and/or poorly estimated covariance matrices often leads to extreme positions far away from the ex-post

[1] In the univariate setting, a number of recent papers have attempted to delineate the attenuation bias attributed to measurement errors in the HARQ model from other non-linear dependencies (e.g. Buccheri and Corsi, 2017; Cipollini et al., 2017). In particular, Buccheri and Corsi (2017) rely on the aforementioned state-space approach to account for the signal plus noise structure of realized variance.

optimal portfolio weights (see, e.g., Li, 2015, and the many references therein). These extreme positions also typically result in excessively high turnover and transaction costs. One popular strategy to help mitigate these problems, and prevent the portfolio optimizers from "reading too much into the data", is to directly constrain the portfolio weights. Jagannathan and Ma (2003), for instance, advocate the use of no short-sale constraints, while DeMiguel et al. (2009a) and Brodie et al. (2009) propose L-norm constrained portfolios. Another popular strategy is to "shrink" the covariance matrix estimates to indirectly help control the portfolio weights. Ledoit and Wolf (2003, 2004a, b), in particular, recommend shrinking the unconditional sample covariance matrix estimate towards some "target" matrix based on a simple factor structure or some other pre-determined matrix.[2]

In contrast to these existing *exogenous* shrinking procedures, our new approach may be interpreted as a dynamic alternative in which the covariance matrix forecasts are *endogenously* shrunk away from the conditional to the unconditional covariance matrix with a time-varying shrinkage intensity that depends on the degree of (time-varying) measurement error. Consistent with the idea that the use of shrinkage techniques generally results in less extreme and more stable portfolio allocations, applying the new procedures in the construction of minimum variance portfolios using simulated data leads to positions that are systematically closer to the optimal positions than those implied by otherwise identical benchmark models without any dynamic shrinkage. Importantly, turnover is also reduced substantially compared to the positions for the benchmark models.

Our empirical analysis involves ten stocks from the Dow Jones Industrial Average. Relying on the utility-based framework of Fleming et al. (2001, 2003), we find that in the absence of any transaction costs, a risk averse investor seeking to minimize the daily variance of her portfolio would be willing to sacrifice up to 140 basis points annually to switch to the more accurate forecasts from our dynamic attenuation model compared to the forecasts obtained from the same model without any attenuation effects. Incorporating empirically realistic transaction costs results in *additional* gains of 20–70 basis points per year stemming from the reduced turnover. These same qualitative gains hold true for an investor seeking to minimize the tracking error of her portfolio relative to an S&P 500 benchmark portfolio. These utility gains remain intact after imposing no short-sales constraints. They also remain over longer weekly and monthly investment horizons, although the relative magnitudes of the gains tend to decrease with the horizon.

The utility benefits obtained from implementing the new models at the daily horizon also exceed the benefits obtained from implementing the models at the weekly and monthly frequencies, despite the increased turnover. In contrast, but consistent with past research, the utility from benchmark models that do not adjust for the time-varying estimation errors in the realized covariances are *not* generally higher at the daily level, underscoring the detrimental impact of unbalanced positions and "too much trading" stemming from the use of more conventional covariance forecasting procedures. Moreover, we show that our proposed new models compare favorably with existing portfolio allocation procedures, e.g. that of Ledoit and Wolf (2003), intended to counter the adverse effects of estimation errors discussed above.

The remainder of the paper is organized as follows. Section 2 sets up the notation and motivation for the new class of models. Section 3 defines the specific parametric models that we rely on our empirical applications. Section 4 discusses the data. Section 5 presents our estimates of the different models, along with various statistical in- and out-of-sample forecast comparisons. Section 6 discusses the metrics and utility-based approach that we rely on in our economic assessments of the new procedures. Our main empirical findings are presented in Section 7. Section 8 extends our main results to other forecast horizons and also compares the new dynamic attenuation models to other already existing static shrinkage procedures. Section 9 concludes. Additional technical details are deferred to four appendices.

2. Motivation and realized covariance estimation

Estimated realized covariances can be decomposed into the sum of two components: the latent true covariance and a measurement error term. Since the latter represents "noise", the time series of realized covariances will appear less persistent than they truly are. As a result the estimated parameters in conventional autoregressive models for the realized covariances are attenuated towards zero. When the measurement error is homoskedastic, the degree of attenuation is proportional to the measurement error variance, but when the measurement error is heteroskedastic, the parameter estimates will be attenuated based on the *average* magnitude of the measurement errors. Consequently, the attenuation is never optimal: it is too strong (weak) when the past realized covariances are (im)precisely estimated.

In most errors-in-variables settings, the distributions of the measurement errors are unknown. In the context of realized covariance estimation, however, the covariances of the measurement errors may be estimated on a day-to-day basis using theoretical developments in Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen et al. (2011). By directly incorporating an estimate of the magnitude of the errors into the formulation of realized covariance based forecasting models, it is possible to dynamically attenuate the parameters of the models, and in turn improve on the forecasts.

The discussion below briefly summarizes the basic theoretical arguments underlying this dynamic attenuation approach. We purposely keep the setup below as simple as possible to convey the main ideas and intuition.

[2] Numerous other combination and related Bayesian model averaging and learning approaches have also been proposed in the literature to help improve on the standard portfolio allocation procedures; see, e.g., Tu and Zhou (2011) and Anderson and Cheng (2016) and the many additional references therein.

2.1. Realized covariances

To set out the notation, consider the N-dimensional log-price process,

$$P(s) = \int_0^s \mu(u)du + \int_0^s \sigma(u)dW(u), \tag{1}$$

where $\mu(u)$ and $\sigma(u)$ denote the instantaneous drift and volatility processes, respectively, with corresponding spot covariance matrix $\Sigma(u) \equiv \sigma(u)\sigma(u)'$, $W(u)$ is an N-dimensional vector of independent Brownian motions, and the unit time interval is normalized to a day. We exclude jumps for simplicity, but the same ideas readily extend to price processes that contain jumps. Following advances in the realized volatility literature, we are interested in modeling and forecasting the daily ex-post covariation, or the daily integrated covariance (see, e.g., the discussion in Andersen et al., 2003). For the price process in Eq. (1) the integrated covariance for day t simply equals,

$$\Sigma_t = \int_{t-1}^t \Sigma(u)du. \tag{2}$$

The integrated covariance is not directly observable. However, it may be consistently estimated based on ever-finer sampled high-frequency intraday data.

In addition to the practical market microstructure complications that plague the estimation of integrated variances (see, e.g., the discussion in, Hansen and Lunde, 2006) the practical estimation of integrated covariances is further complicated by the Epps (1979) effect and non-synchronous price observations. A number of alternative estimators have been proposed in the literature to circumvent these complications. In our empirical analysis below, we rely on the Multivariate Kernel (MK) of Barndorff-Nielsen et al. (2011), for which the asymptotic variance of the measurement errors can easily be estimated; further details about the MK estimator and its asymptotic distribution are provided in Appendix A below. However, the same basic ideas carry over to any consistent estimator for Σ_t, say S_t.[3]

To facilitate the discussion, let $\varsigma_t \equiv \text{vech}\ \Sigma_t$ denote the $N^* = N(N+1)/2$ dimensional vectorized version of the true latent integrated covariance matrix of interest Σ_t. Similarly, let $s_t \equiv \text{vech}\ S_t$ denote the vectorized version of the S_t realized covariance matrix estimator. We will refer to the $N^* \times N^*$ covariance matrix for the corresponding measurement error vector $\eta_t = s_t - \varsigma_t$ by Π_t. The exact form of Π_t obviously depends on the specific estimator S_t. For the MK estimator that we rely on in the empirical analysis below, Π_t is proportional to the so-called integrated quarticity,

$$\Pi_t \ \propto \ \int_{t-1}^t \Sigma(u) \otimes \Sigma(u)du,$$

where $\otimes$ denotes the Kronecker product.[4] In parallel to the integrated covariance Σ_t, the integrated quarticity matrix may be consistently estimated using high-frequency intraday data. In the empirical results reported below, we rely on the specific estimator proposed by Barndorff-Nielsen et al. (2011). In the following we will refer to the resulting (up-to-scale) measurement error covariance matrix estimates as Π_t for short.

2.2. (Un)reliable covariances and attenuation biases

The effect of measurement errors in univariate autoregressive models has been extensively studied in the literature (see, e.g., Staudenmayer and Buonaccorsi, 2005). In the multivariate setting the problem is less well-developed and the literature have mostly focused on the effect of measurement errors for identification and hypothesis testing (see, e.g., Holtz-Eakin et al., 1988; Komunjer and Ng, 2014). Even so, it is well-established that the autoregressive parameter estimates are attenuated towards zero, both in the uni- and multivariate setting.

In order to provide some intuition for this result and our new dynamic attenuation approach, assume for simplicity that the true (latent) vectorized integrated covariance ς_t follows a VAR(1) model,

$$\varsigma_t = \Phi_0 + \Phi_1 \varsigma_{t-1} + u_t, \tag{3}$$

where Φ_0 is of dimension N^*, and the autoregressive parameter Φ_1 is of dimension $N^* \times N^*$. Even though this does not necessarily correspond to the exact discretization of the underlying continuous-time model in Eq. (1), a simple first-order VAR may still provide a good approximation in many situations (see also the related discussion in Andersen et al., 2004).

In practice, of course, ς_t is not observable, so the researcher must estimate any given model using realized covariances, s_t. Suppose that the researcher relies on the identical VAR(1) formulation,[5]

$$s_t = \Theta_0 + \Theta_1 s_{t-1} + \epsilon_t. \tag{4}$$

3 Other "noise" robust consistent covariance estimators include Aït-Sahalia et al. (2010), and Christensen et al. (2010).

4 A similar result holds true for other realized covariance matrix estimators. Instead of relying on explicit analytical asymptotic expressions for estimating Π_t, following Gonçalves and Meddahi (2009) the variance of the measurement errors may alternatively be estimated by bootstrap procedures.

5 In the univariate case the population parameters of an AR(p) model with homoskedastic measurement errors can easily be identified by estimating an ARMA(p, p) model. In the multivariate case the composite error term $\epsilon_t - \eta_t + \Theta_1 \eta_{t-1}$ is typically not a finite order moving average process, so a finite dimensional VARMA model will not identify the population parameters.

For simplicity, assume that the "structural" errors u_t and the measurement errors $\eta_t = s_t - \varsigma_t$ are both *i.i.d.* and uncorrelated. (This, of course, would not be the case for the realized covariances if the underlying $\sigma(u)$ process is time-varying.) Under these simplifying assumptions, it is straightforward to show that the OLS estimate of Θ_1 will be functionally related to the population value of Φ_1 by,

$$\begin{aligned}\widehat{\Theta}_1 &= (\mathbf{s}'\mathbf{s})^{-1}(\boldsymbol{\varsigma}'\boldsymbol{\varsigma})\Phi_1 \\ &= (\boldsymbol{\varsigma}'\boldsymbol{\varsigma} + \boldsymbol{\eta}'\boldsymbol{\eta})^{-1}(\boldsymbol{\varsigma}'\boldsymbol{\varsigma})\Phi_1,\end{aligned} \tag{5}$$

where $\mathbf{s} = (s_1, \ldots, s_{T-1})$ denotes the $T-1 \times N^*$ matrix of lagged realized covariances, with $\boldsymbol{\varsigma}$ and $\boldsymbol{\eta}$ defined analogously. The actual estimated parameter matrix therefore equals the population parameter matrix times the ratio of the variation of the true latent process divided by the variation of the estimated process.[6] This latter ratio is commonly referred to as the reliability ratio. As this relationship makes clear, depending on the covariance structure of η, certain parameters may be biased towards zero, while other parameters may be biased away from zero. However, regardless of the direction, the bias is always proportional to the magnitude of the measurement errors.

The bias implied by Eq. (5), could be removed based on an estimate for the reliability ratio to arrive at an unconditionally unbiased estimate for the "structural" Φ_1 parameters. Although these parameters may sometimes be of interest in their own right, from a practical forecasting perspective it is the Θ_1 parameters and the dynamic dependencies in the actually observed realized covariances s_t that matter. It follows from the expression in Eq. (5) that even if the Φ_1 parameters are time-invariant, as long as the measurement errors are heteroskedastic, the "optimal" Θ_1 parameters to be used for forecasting purposes should be dynamically attenuated to reflect the temporal variation in the reliability of the past realized covariance measures.

The specific expression in Eq. (5) is based on a number of simplifying assumptions and merely meant to illustrate the main idea. A number of important choices need to be made in the way in which the dynamic attenuation idea is actually implemented in practice. Firstly, a VAR(1) model is likely not the best specification for characterizing the dynamic dependencies in s_t. Secondly, reliable estimation of all the elements in the Π_t matrix on a period-by-period basis, not to mention the inverse, presents some formidable challenges, even for moderately large N. The next section discusses the dynamic attenuation models that we use in our empirical investigations.

3. Dynamic attenuation models

The specification of empirically realistic, yet practically feasible, multivariate realized volatility models raises a number of important practical issues. Motivated by the discussion in the previous section, we propose new parsimonious dynamic specifications for the high-frequency realized covariances, in which we allow the autoregressive parameters of the models to depend linearly on the measurement errors of covariance matrix estimates. We purposely restrict the parameterizations to relatively simple scalar formulations, while allowing each element of the covariance matrix to exhibit its own distinct dynamic dependencies as a function of the measurement errors variances.

We consider three popular baseline models: the vech HAR (Chiriac and Voev, 2010), the HAR-DRD model (Oh and Patton, 2016), and the HEAVY model (Noureldin et al., 2012). For comparison purposes, we also consider a simple Exponentially Weighted Moving Average (EWMA) filter, in which we allow the filter weights to vary with the estimation error. It is not our goal to run a horse-race between the various models. Instead, we seek to illustrate how the basic approach may be implemented quite generally, and in turn evaluate the performance of the models with dynamically attenuated parameters relative to the otherwise identical models with constant parameters.

3.1. HARQ models

The Heterogeneous AutoRegressive (HAR) model of Corsi (2009) has arguably emerged as the most widely used univariate realized volatility-based forecasting model. The model was first extended to a multivariate setting by Chiriac and Voev (2010). The scalar version of their "vech HAR" model is:

$$s_t = \theta_0 + \theta_1 s_{t-1} + \theta_2 s_{t-5|t-1} + \theta_3 s_{t-22|t-1} + \epsilon_t, \tag{6}$$

where $s_{t-h|t-1} = \frac{1}{h}\sum_{i=1}^{h} s_{t-i}$ denote the vectorized version the h-day realized covariance matrix. The intercept θ_0 is a $N^* \times 1$ dimensional vector, while the θ_1, θ_2 and θ_3 parameters are all assumed to be scalar. This simple specification is highly parsimonious and ensures that the covariance matrix forecasts are positive definite.

The standard vech HAR formulation in (6) does not take into account the fact that the lagged realized covariances are measured with error. Bollerslev et al. (2016) proposed a simple modification to accommodate this in the context of

[6] In the univariate case the relationship simplifies to the easier to interpret $\widehat{\Theta}_1 = (1 + Var(\eta_t)/Var(\varsigma_t))^{-1}\Phi_1$, where the estimated parameter equals the population parameter times a noise-to-signal ratio.

the univariate HAR model, by allowing θ_1 in (6) to depend on an estimate for the measurement error variance: $\theta_{1,t} = (\theta_1 + \theta_{1Q} RQ_{t-1}^{1/2})$, where RQ_t denotes the daily realized quarticity. Following the discussion in the previous section, the resulting "HARQ" model may be interpreted as a linear approximation to the inverse of (5) in the univariate setting.[7] This same basic idea is readily extended to the multivariate HAR model in (6) by allowing the θ_1 parameter in that model to depend on the magnitude of the measurement errors in the s_{t-1} vector. In its most general form this would entail the estimation of an additional $N^*(N^*+1)/2 = (N^4+2N^3+3N^2+2N)/8$ parameters associated with each of the unique elements in the Π_{t-1} covariance matrix. This is clearly not practically feasible for N much beyond 1, and some parsimony has to be invoked in the parameterization. Moreover, it generally also becomes increasingly more difficult to accurately estimate the Π_{t-1} matrix as the dimension N increases. Correspondingly, it is imperative to strike a careful balance between the difficulties in being able to accurately estimate the magnitude of the measurement errors and the way in which the dynamic attenuation of the autoregressive parameters is actually being implemented.

In particular, as for the covariance matrix s_t itself, it is generally easier to estimate the diagonal elements of Π_t than the off-diagonal covariance elements. Hence, we focus on $\pi_t \equiv \sqrt{diag(\Pi_t)}$, the vector of asymptotic standard deviations for each of the individual element in the S_t covariance matrix. The correspondingly modified vech HARQ model estimated below takes the form,[8]

$$\begin{aligned} s_t &= \theta_0 + \theta_{1,t} \circ s_{t-1} + \theta_2 s_{t-5|t-1} + \theta_3 s_{t-22|t-1} + \epsilon_t, \\ \theta_{1,t} &= \theta_1 \iota + \theta_{1Q} \pi_{t-1}, \end{aligned} \tag{7}$$

where $\circ$ denotes the Hadamard product, ι is an N^* dimensional vector of ones, and θ_1 and θ_{1Q} are scalar parameters. More general specifications in which the θ_2 and θ_3 parameters are similarly allowed to depend on the measurement errors in $s_{t-5|t-1}$ and $s_{t-22|t-1}$, respectively, could, of course, be implemented. However, the magnitude of the errors generally decreases with the horizon, so we purposely restrict the estimated attenuation effect to the most difficult to accurately estimate daily lagged s_{t-1} only.[9] Also, note that even though the θ_1 and θ_{1Q} parameters are both assumed to be scalar, the resulting $\theta_{1,t}$ parameter vector allows for different dynamics in each of the individual elements in the covariance matrix based on their own measurement error variances. Given the intuition underlying this model, we expect θ_1 to be larger than would be found in a conventional HAR model, and we expect θ_{1Q} to be negative, so that realized covariance matrices that are measured with greater error are given lower weight in the forecasting model. Of course, in a relative sense this effectively also assigns more weight to the weekly and monthly lags.

3.2. HARQ-DRD models

An alternative approach to possibly allow for more general dynamic dependencies, while still ensuring positive definite covariance matrix forecasts, is to model the variances and correlations separately. The HAR-DRD model of Oh and Patton (2016), is based on the decomposition of the covariance matrix into

$$S_t = D_t R_t D_t, \tag{8}$$

where D_t denotes the diagonal matrix of standard deviations, and R_t is the correlation matrix.[10] In the HAR-DRD model, the individual variances are modeled by univariate HAR models, and the conditional correlation matrix is modeled using a scalar multivariate HAR model. The HAR-DRD model is readily extended to allow for dynamic attenuation effects by incorporating the influence of measurement errors into the parameters in the models for the variances and/or correlations.

In the HARQ-DRD model analyzed below, we rely on the univariate HARQ model of Bollerslev et al. (2016), defined analogously to the multivariate model in Eq. (7), together with the scalar multivariate HAR in Eq. (6) for modeling the vectorized correlation matrix. It would of course be possible to also consider a HARQ specification for the correlations. However, the heteroskedasticity in the measurement errors for the correlations tend to be somewhat limited, likely outweighing the benefits of attenuating the autoregressive correlation parameters.[11]

The parameters in the vech HAR(Q) and the HAR(Q)-DRD models discussed above are easily estimated by standard OLS procedures. The estimation of the parameters in the HEAVY(Q) models, which we discuss next, require the use of more complicated (pseudo-)maximum likelihood techniques and non-linear optimization procedures.

7 The Q suffix refers to the use of the quarticity measure to guide the attenuation.

8 In all of the empirical results reported below, we demean the π_t vector to render the θ_1 coefficients directly interpretable as the value at the average measurement error level.

9 The analysis of the corresponding trade-offs in the univariate context in Bollerslev et al. (2016) also clearly suggests that the difficulties in accurately estimating the integrated quarticity easily outweigh the benefits of adjusting the weekly and monthly coefficients.

10 This same decomposition was originally applied in the context of multivariate GARCH models by Bollerslev (1990), who assumed the conditional correlation matrix R_t to be constant.

11 Following Barndorff-Nielsen and Shephard (2004) and assuming a constant spot volatility for simplicity, it is possible to show that the asymptotic standard deviation for the correlation coefficient ρ_t approximately equals $1 - \rho_t^2$, which invariably is limited by $|\rho_t| \leq 1$.

3.3. HEAVYQ models

The multivariate HEAVY class of models was originally introduced by Noureldin et al. (2012). In contrast to the HAR models discussed above, which are only meant to forecast the daily covariances, HEAVY models are designed to characterize the entire conditional daily return distribution, explicitly incorporating information in past realized covariances.

To set out the basic idea, let $V_t = E(r_t r_t' \mid \mathcal{F}_{t-1})$ denote the conditional covariance matrix for the daily returns r_t, where the time $t-1$ information set $\mathcal{F}_{t-1}$ includes all of the past realized covariances. Correspondingly, let $v_t = \text{vech}\, V_t$ denote the vectorized version of the daily conditional covariance matrix. The HEAVY model with covariance targeting that we rely on for modeling v_t below, may then be expressed as

$$v_t = (I_{d^*} - b - a\kappa)\lambda_V + bv_{t-1} + as_{t-1}, \tag{9}$$

where a and b are scalar parameters, and κ serves to adjust the expectation of the high-frequency intraday covariance matrix to match the unconditional expectation of the daily covariance matrix λ_V.[12] The model's parameters can be estimated using standard quasi-maximum likelihood techniques. To facilitate the practical implementation, we rely on the composite likelihood approach of Pakel et al. (2014), which is easy to implement in large dimensions; further details are provided in Appendix C.

The conventional HEAVY model in (9) does not account for estimation errors in the realized covariance estimates. Following the same approach used in the specification of the vech HARQ and HARQ-DRD models above, the HEAVY model may similarly be adapted to allow the impact of s_{t-1} to vary over time depending on the degree of measurement error,

$$\begin{aligned} v_t &= (I_{d^*} - b - a_t\kappa)\lambda_V + bv_{t-1} + a_t \circ s_{t-1}, \\ a_t &= a\iota + a_Q\pi_{t-1}, \end{aligned} \tag{10}$$

where again a_Q is imposed to be scalar, and is expected to be negative. We will refer to this specification as the HEAVYQ model below.

3.4. EWMAQ filters

Our last empirical approach is based on an Exponentially Weighted Moving Average (EWMA) filter. EWMA filters are widely used in practice as a simple and easy-to-implement procedure to accommodate time-varying variances and covariances. EWMA filters have traditionally been based on daily or lower frequency data, but they are equally applicable in the high-frequency realized volatility setting (see, e.g., Fleming et al., 2003, for an early application of an EWMA filter with realized covariances).

Let $v_t = \text{vech}\, V_t$ denote the vectorized daily covariance matrix of interest. The standard EWMA filter based on high-frequency realized covariances may then be expressed as

$$v_t = (1-\alpha)v_{t-1} + \alpha s_{t-1}, \tag{11}$$

where α defines the decay rate.[13] Standard choices for α are 0.03 and 0.06. When α is low the filter is persistent and the filtered V_t covariance matrices are fairly stable. When α is high more of the information in s_{t-1} is immediately incorporated into the filtered V_t. In the analysis reported on below, we estimate α based on the auxiliary assumption that the daily returns are conditionally normally distributed using the same composite likelihood approach described in Appendix C used in estimating the HEAVY(Q) models.

In parallel to the models described above, the EWMA filter is readily adapted to incorporate the effect of measurement errors in s_t by allowing the α parameter to vary with π_t. Specifically, we define the EWMAQ filter as,

$$\begin{aligned} v_t &= (1-\alpha_t) \circ v_{t-1} + \alpha_t \circ s_{t-1}, \\ \alpha_t &= \alpha\iota + \alpha_Q\pi_{t-1}. \end{aligned} \tag{12}$$

We would naturally expect the α parameter in the EWMAQ filter to be higher than the α parameter in the conventional EWMA filter, allowing for more immediate reactions to s_{t-1} on average. Correspondingly, the α_Q parameter should be negative so that the impact of highly uncertain s_{t-1} estimates is attenuated, shifting more of the weight towards the past filtered v_{t-1} estimates.

4. Data

Our empirical analysis is based on a set of ten Dow Jones stocks. The names and ticker symbols for each of the stocks are listed in Table 1. In addition, we use the SPY exchange traded fund in the construct of S&P 500 tracking portfolios. We rely on 5-minute returns retrieved from the TAQ database over the February 1993 to December 2013 sample period, for a total

[12] See Noureldin et al. (2012) for details.

[13] As long as the initial V_0 matrix is positive definite, the EWMA filter automatically ensures that the filtered V_t matrices are all positive definite.

Table 1
Descriptive statistics.

Name	Ticker	Variance		Quarticity		Correlations		Beta	
		Mean	StDev	Mean	StDev	Mean	StDev	Mean	StDev
Market	SPY	0.954	1.165	3.882	13.933				
American Express	AXP	3.906	55.941	3.694	12.797	0.315	1.120	1.043	0.634
Boeing	BA	2.782	16.425	1.714	5.192	0.271	0.934	0.821	0.584
Chevron	CVX	1.975	13.185	0.671	1.775	0.307	1.113	0.764	0.530
DuPont	DD	2.723	15.397	1.562	4.466	0.315	1.132	0.928	0.562
General Electric	GE	2.709	35.373	1.621	6.342	0.351	1.287	0.998	0.499
IBM	IBM	2.250	10.700	1.152	3.367	0.321	1.148	0.883	0.497
JPMorgan Chase	JPM	4.329	84.575	4.392	16.741	0.316	1.110	1.172	0.656
Coca-Cola	KO	1.740	6.757	0.605	1.797	0.274	0.922	0.640	0.447
Microsoft	MSFT	2.897	13.411	1.716	4.344	0.310	1.079	1.061	0.603
ExxonMobil	XOM	1.789	13.271	0.546	1.623	0.329	1.208	0.773	0.474

Note: The table reports averages and time series standard deviations for the variances of each individual asset, as well as their estimated measurement error variances (Quarticity). Next it depicts averages and standard deviations of the average correlations with the other assets and the final two columns show averages and standard deviations of the realized betas with respect to the SPY market index.

of 5,267 daily observations. To most directly highlight the benefits of the high-frequency-based procedures and the new dynamic attenuation models, we focus our analysis on the intraday realized covariances and corresponding open-to-close returns. This also mirrors a number of studies in the recent literature (see, e.g., Lunde et al., 2016; Hautsch et al., 2015; De Lira Salvatierra and Patton, 2015, among others).[14]

Table 1 provides summary statistics for the resulting daily MK covariance estimates. The first two columns report the averages and time series standard deviations of the realized variances for each of the individual asset. The following two columns report the mean and standard deviation of estimated measured errors, based on realized quarticity. The presence of variation in measurement error is the basis for all of the analysis in this paper, and we see from Table 1 that it is substantial: the coefficient of variation (the ratio the standard deviation to the mean) is well above one for all ten of these stocks. The last four columns summarize the linear dependence between returns on these assets, showing the average realized correlation of each asset with all other assets, as well as the average realized beta (defined as in Andersen et al., 2006) with respect to the SPY market portfolio. The average correlation among the assets is around 0.28, indicating important diversification benefits and potentially large gains from risk minimizing portfolios constructed on the basis of more accurate covariance matrix forecasts. The average realized beta equals 0.91. However, the betas vary importantly both across stocks and time, again suggesting potentially large gains from portfolios designed to track the market based on more accurate covariance matrix forecasts. We turn next to a discussion of the models that we rely on below to explore these conjectures.

5. Model estimates and covariance forecasts

We begin our empirical analysis by comparing and contrasting the estimated dynamic attenuation models with their constant counterparts based on conventional statistical criteria. Section 5.1 discusses the in-sample parameter estimates and quality of the model fits, while Section 5.2 presents the results from out-of-sample forecast comparisons.

5.1. In-sample estimates

The parameter estimates obtained for each of the different models are reported in Table 2, along with robust standard errors in parentheses. To conserve space, for the HAR(Q)-DRD models, we only report the averages of the parameter estimates and standard errors over each of the ten individually estimated HAR(Q) variance models.[15]

As expected, all of the estimated Q-coefficients are negative and statistically significant. This directly corroborates the basic idea and mechanics of the models, that as the measurement error variance π_t increases (decreases), the informativeness of the past covariance estimate decreases (increases), resulting in the models endogenously increasing (decreasing) the degree of attenuation. To put the magnitude of the attenuation in further perspective, the cross-sectional average of the composite $\theta_{1,t}$ parameter for the HARQ model varies substantially over the sample, from a low of 0.21 to a high of 0.87. Correspondingly, there is a "redistribution" of weight from the longer weekly and monthly lags to the daily lag when comparing the HARQ to HAR model. This effect is even more pronounced for the HAR-DRD(Q) models, in which the individual

[14] Following Andersen et al. (2011), who treat the overnight returns as "jumps", a separate model could be used to predict the overnight variation. However, a dynamic attenuation model is unlikely to provide any benefits in that context, as the variation of the overnight measurement errors cannot be accurately estimated. Alternatively, following Hansen and Lunde (2005) the intraday variation may be scaled up to reflect the variation for the whole day, akin to the κ adjustment term in the HEAVY model.

[15] All of the models are likely misspecified. Correspondingly, all of the reported parameter estimates (OLS for HAR, HAR-DRD and EWMA, and QMLE for HEAVY) are treated as estimating a pseudo-true parameter vector, and all standard errors are computed using methods that are robust to model misspecification.

Table 2
In-sample estimates.

	HAR	HARQ		HAR-DRD	HARQ-DRD		EWMA	EWMAQ		HEAVY	HEAVYQ
θ_1	0.247	0.541	θ_1	0.260	0.660	α	0.079	0.102	a	0.106	0.148
	(0.040)	(0.040)		(0.064)	(0.065)		(0.003)	(0.004)		(0.009)	(0.009)
θ_2	0.410	0.333	θ_2	0.395	0.223	α_Q		−0.004	b	0.876	0.825
	(0.038)	(0.038)		(0.016)	(0.022)			(0.001)		(0.004)	(0.004)
θ_3	0.244	0.113	θ_3	0.240	0.101				a_Q		−0.026
	(0.038)	(0.038)		(0.028)	(0.028)						(0.012)
θ_{1Q}		−0.043	θ_{1Q}		−0.067						
		(0.018)			(0.026)						
			θ_1^{Corr}	0.049	0.049						
				(0.007)	(0.007)						
			θ_2^{Corr}	0.159	0.159						
				(0.003)	(0.003)						
			θ_3^{Corr}	0.560	0.560						
				(0.003)	(0.003)						
Frob.	12.282	12.190		11.987	11.952		12.324	12.255		12.539	12.532
QLIKE	15.484	15.392		15.289	15.205		15.408	15.393		15.517	15.505

Note: The table reports in-sample parameter estimates and measures of fit for the different models. For the HAR(Q)-DRD models the reported parameter estimates and standard errors for the variance specification are the averages across the ten individually estimated univariate HAR(Q) models. The bottom panel reports the in-sample fit for the different models as measured by the Frobenius distance and the QLIKE loss.

variances have their own separate dynamics, and in turn their own individual sensitivity to the degree of measurement error. As such, the (average) estimated θ_{1Q} parameter is also greater (in absolute value) for the HARQ-DRD model than it is for the HARQ model.[16]

The same results carry over to the EWMA(Q) filters and HEAVY(Q) models. The estimated α for the EWMAQ filter exceeds the α for the standard EWMA filter, allowing for a greater immediate impact of the realized covariance on average. However, when the estimation error is large, the negative α_Q implies that the weight is shifted away from the current noisy estimate towards the long-run weighted average. Similarly, the estimated a parameter is higher for the HEAVYQ model than for the standard HEAVY model, together with a negative a_Q estimate for the HEAVYQ model.

In addition to the parameter estimates, the last two rows in Table 2 report the Frobenius distance and QLIKE loss for the fitted covariance matrices, denoted below as H_t, with respect to the ex-post realized covariances S_t. The Frobenius norm is commonly used to measure the distance between two matrices,

$$L_t^{Frobenius} = \sqrt{Tr[(H_t - S_t)(H_t - S_t)']}. \tag{13}$$

The quasi-likelihood (QLIKE) measure is based on the negative of the log-likelihood of a multivariate normal,

$$L_t^{QLIKE} = \log|H_t| + Tr(H_t^{-1}S_t). \tag{14}$$

The results reported in the table are obtained by summing $L_t^{Frobenius}$ and L_t^{QLIKE} over the full sample. Both functions measure "loss", so that lower values are preferable.[17] Across all four models and both loss functions, the dynamic attenuation models result in lower losses than their constant counterparts. The HARQ-DRD model results in the lowest in-sample loss overall. The next section investigates how the models compare on an out-of-sample basis using these same statistical loss functions.

5.2. Out-of-sample forecasts

Our out-of-sample forecast comparisons are based on the one-day-ahead forecasts for the same ten-dimensional covariance matrix and set of models analyzed above. Again, it is not our goal to run a horse-race between the different models to find a single "winner". Rather, we seek to understand how the forecasts from the models that allow for dynamically attenuated parameters perform relative to the otherwise identical models with constant parameters. All of the models are re-estimated every day based on a rolling sample window of the past 1000 days.[18] The forecasts are evaluated based on the same Frobenius distance and QLIKE loss defined in Eqs. (13) and (14), respectively, where the in-sample fitted value H_t are replaced with the forecast from the relevant model.

16 This in itself represents an attenuation effect, as the measurement error in π_t aggregates in the cross-section for the HARQ model, rendering the dynamic θ_{1Q} parameter less effective.

17 Despite the use of an ex-post estimate in place of the true covariance matrix, both of the loss functions provide consistent model rankings, as further discussed in Patton (2011) and Laurent et al. (2013).

18 Due to the difficulties in estimating the integrated quarticity matrix, the HARQ and HARQ-DRD models occasionally produce negative definite forecasts. If this occurs, we apply an "insanity filter", and replace the negative definite covariance matrix forecast with the simple average realized covariance over the relevant estimation sample. This only happens for one or two forecasts per series over the entire sample.

Table 3
Out-of-sample forecast results.

	Full sample		Lower 95% $\|\pi_t\|$		Upper 5% $\|\pi_t\|$	
	Frob.	QLIKE	Frob.	QLIKE	Frob.	QLIKE
HAR	12.305	14.382	9.274	13.275	**69.723**	32.651
HARQ	**12.107***	14.159*	9.161*	13.206*	**69.703**	**32.190***
HAR-DRD	**12.134**	14.140	**8.940**	**13.022**	70.467	32.603
HARQ-DRD	**11.976***	**13.896***	**8.888***	**12.990***	**70.056***	**31.050***
EWMA	12.379	14.105	9.303	13.109	71.104	32.686
EWMAQ	**12.178***	**14.091***	9.172*	13.122	70.633*	32.643*
HEAVY	12.473	**14.051**	9.299	13.041	71.789	32.263
HEAYVQ	**12.161***	**14.004***	**9.093***	13.050	**70.170***	32.258

Note: The table reports out-of-sample forecast loss for the different models. The first two columns are based on the full-sample. The last four columns split the sample into days when the measurement error variance is low, and $\|\pi_t\|$ is below the 95% quantile, and days when the estimation error is high, and $\|\pi_t\|$ is above the 5% quantile. Entries in boldface indicate models that are part of the 90% model confidence set (MCS) for the relevant column. Q-models that significantly improve on their non-Q benchmark are indicated by an asterisk.

Consistent with the in-sample results, the full out-of-sample results reported in the first two columns of Table 3 show that our new dynamic attenuation models systematically improve on their non-attenuated counterparts. If anything, the out-of-sample improvements are even bigger. This holds true for both loss functions.[19] To help understand where these forecast improvements are coming from, the last four columns of the table split the sample into days with the 95% lowest and 5% highest measurement error variances, respectively, as defined by the Frobenius norm of π_t. As the table shows, the dynamic attenuation models offer improvements in both situations, by either increasing the responsiveness when the realized covariances are precisely estimated, or increasing the attenuation when the estimates are poor. Interestingly, however, the largest percentage improvements stem from the increased responsiveness when the estimates are precise.

In order to formally determine whether the quality of the forecasts differ significantly across the different models, we apply the Model Confidence Set (MCS) of Hansen et al. (2011). This approach identifies the (sub)set of models that contains the best forecasting model with 90% confidence. For each of the two loss functions and three sample-splits we determine the subset of models that comprise the MCS (members of the set indicated in boldface). Additionally, we separately compare each of the Q-models with their non-attenuated counterpart using the Diebold and Mariano (1995) test (significance indicated by *).

Looking first at the pairwise comparisons, the Diebold and Mariano (1995) test confirms that each of the Q-models significantly outperforms their respective benchmark over the full sample. This is true irrespective of the loss function used in comparing the models. The Q-models generally also beat their non-Q counterparts over each of the two split samples, but the differences are not always significant. Jointly comparing all of the eight models, the HARQ-DRD model stands out as always being included in the MCS, regardless of the loss function and/or sample-split underlying the comparisons. The EWMAQ filter and HEAVYQ model are also both included the MCS for the full sample based on either loss function. The HARQ model, on the other hand, appears to work particularly well when the covariances are imprecisely estimated. Taken as whole, the results clearly corroborate the superior statistical performance of the dynamic attenuation models.

The next section seeks to address whether these statistical improvements translate into economic gains by evaluating the new dynamic attenuation models from a practical portfolio allocation perspective.

6. Minimum variance and tracking error portfolios

Our economic evaluations of the different models are based on their use in the construction of Global Minimum Variance (GMV) portfolios and portfolios designed to track the aggregate market. The GMV and tracking portfolio weights only depend on return covariances, and as such provide an especially clean framework for assessing the merits of the different covariance forecasting models. Most other portfolio allocation decisions depend on forecasts for the expected returns as well, and these are notoriously difficult to accurately estimate. As reported by Jagannathan and Ma (2003) and DeMiguel et al. (2009a), mean–variance optimized portfolios typically do not perform as well as GMV portfolios in terms of out-of-sample Sharpe ratios, as the estimation error in the expected returns tends to distort the positions. In order to most directly highlight the advantages of the new dynamic attenuation models, we initially focus on a daily investment horizon. Daily re-balancing schemes have also previously been employed in a number of studies concerned with volatility-timing strategies (see, e.g., Fleming et al., 2003; Fan et al., 2012, among others). In Section 8, we also consider weekly and monthly investment horizons.

6.1. Practical implementation and utility comparisons

Consider a risk-averse investor who allocates her funds into N risky assets based on the forecasts for the daily covariance matrix of the returns on the assets, $H_{t|t-1}$. To minimize the conditional volatility, the investor solves the global minimum

[19] Unreported results show that all of the models produce approximately unbiased forecasts, and that the apparent improvements for the Q-models mainly stem from a reduction in the forecast error variance.

variance portfolio problem,

$$\begin{aligned} w_t &= \arg\min w_t' H_{t|t-1} w_t \\ &\quad s.t. \quad w_t'\iota = 1, \end{aligned} \tag{15}$$

where ι is a $N \times 1$ vector of ones, resulting in the optimal portfolio allocation vector,

$$w_t = \frac{H_{t|t-1}^{-1}\iota}{\iota' H_{t|t-1}^{-1}\iota}. \tag{16}$$

In the sequel, we will denote the nth element of w_t, corresponding to the allocation to the nth asset, by $w_t^{(n)}$. Correspondingly, we will denote the return on the nth asset by $r_t^{(n)}$.

One potentially important feature of the dynamic attenuation models is that they result in more stable covariance matrix forecasts, and therefore less turnover than conventional procedures. Importantly, this should make trading strategies based on the new models cheaper to implement. To evaluate this, we assume that the investor faces fixed transaction costs c proportional to the turnover rates in her portfolios. Specifically, following standard arguments (see, e.g., the discussion in Han, 2006; Liu, 2009; DeMiguel et al., 2014), the total portfolio turnover from day t to day $t+1$ is readily measured by,

$$TO_t = \sum_{n=1}^{N} \left| w_{t+1}^{(n)} - w_t^{(n)} \frac{1 + r_t^{(n)}}{1 + w_t' r_t} \right|. \tag{17}$$

With proportional transaction costs cTO_t, the portfolio excess return net of transaction cost is therefore,

$$r_{pt} = w_t' r_t - cTO_t. \tag{18}$$

In the results below, we consider values of c ranging from 0 to 2%, in line with values employed in earlier studies, see Fleming et al. (2003) and Brown and Smith (2011) for example.

The more stable and less susceptible to estimation error covariance matrix forecasts from the dynamic attenuation models should also result in less extreme portfolio allocations. To assess this, we also report the portfolio concentrations,

$$CO_t = \left(\sum_{n=1}^{N} w_t^{(n)2} \right)^{1/2}, \tag{19}$$

and the total portfolio short positions,

$$SP_t = \sum_{n=1}^{N} w_t^{(n)} \mathbb{I}_{\{w_t^{(n)} < 0\}}. \tag{20}$$

Again, less extreme and fewer short positions are likely to facilitate the practical implementation of the portfolios, and help further mitigate transaction costs in situations when the costs are not simply proportional to the turnover.

In addition to the GMV portfolios defined in (16), we also consider portfolios designed to track the SPY market portfolio. Many institutional investors are evaluated based on their performance relative to a benchmark, thus the ability to closely track a particular portfolio is often of great import. Our construction of the tracking portfolios is based on minimizing the variance of the tracking error, or equivalently the GMV for the returns on each of the assets in excess of the SPY return. Since the assets correlate to varying degrees with the market, the covariances among the returns net of the market return are typically more dispersed than the covariances of the returns themselves. As such, this renders the construction of the minimum tracking error portfolios more challenging and potentially also more prone to extreme positions than the GMV portfolios.

To evaluate the economic significance of the different forecasting models, we consider the utility-based framework of Fleming et al. (2001, 2003). In particular, assuming that the investor has quadratic utility with risk aversion γ, the realized daily utility generated by the portfolio based on the covariance forecasts from model k, may be expressed as

$$U(r_{pt}^k, \gamma) = (1 + r_{pt}^k) - \frac{\gamma}{2(1+\gamma)}(1 + r_{pt}^k)^2. \tag{21}$$

The economic value of the different models may therefore be determined by solving for Δ_γ in

$$\sum_{t=1}^{T} U(r_{pt}^k, \gamma) = \sum_{t=1}^{T} U(r_{pt}^l - \Delta_\gamma, \gamma), \tag{22}$$

where Δ_γ can be interpreted as the return an investor with risk aversion γ would be willing to sacrifice to switch from using model k to using model l. In order to determine whether the estimated values of Δ_γ are significantly different from zero we use the Reality Check of White (2000), based on the stationary bootstrap of Politis and Romano (1994), with 999 bootstrap samples and an average block length of 22 days.

Table 4
Fundamental versus spurious turnover.

M	∞		390		78		26	
	Optimal	HAR	HAR	HARQ	HAR	HARQ	HAR	HARQ
Turnover	0.2206	0.1878	0.2379	0.2071	0.2830	0.2257	0.2999	0.2360
StDev	0.1650	0.1829	0.1844	0.1837	0.1879	0.1853	0.1920	0.1887
Distance to optimal	0.0000	0.0446	0.0471	0.0457	0.0531	0.0488	0.0601	0.0457
Distance to HAR$_\infty$	0.0446	0.0000	0.0104	0.0062	0.0198	0.0102	0.0231	0.0141

Note: The table reports the results from a simulation study pertaining to the out-of-sample GMV portfolios. The covariance matrix forecasts are based on the HAR and HARQ models and the MK estimates with M intraday observations. The $M = \infty$ column shows the results for the HAR model applied to population covariance matrix as well as the optimal results based on ex-post estimates. Turnover is defined in Eq. (17). StDev refers to the portfolio standard deviation based on the population covariance. Distance reports the distance, as defined in the main text, between the portfolio weights and the weights based on the true covariance matrix Σ_t, or the weights based on the HAR_∞ model.

7. Dynamic attenuation models in action

In parallel to the statistical model comparisons in Section 5.2, in our economic evaluations we rely on a one-day-ahead out-of-sample forecasting scheme, in which we re-estimate all of the models based on a rolling estimation window of 1000 days, re-balancing the resulting portfolios daily. In addition to the basic features of the daily GMV and tracking error portfolios, we are particularly interested in whether accurate and stable covariance matrix forecasts from the dynamic attenuation models manifest in systematically lower portfolio turnover and reduced transaction costs. We begin by considering the results from an empirically realistic simulation study explicitly designed to highlight these features. We then present our main empirical findings.

7.1. Fundamental and spurious turnover

The statistically more accurate covariance forecasts from the dynamic attenuation models should result in portfolio allocations closer to the optimal weights implied by the true covariance matrix. The more stable forecasts from the attenuation models should also result in more stable portfolio allocations and a reduction in turnover. Comparing the estimated GMV portfolios with the optimal GMV portfolio based on the true covariance matrix within a controlled simulation setting, allows us to directly assess these conjectures and dissect the workings of the models.

To keep the simulations manageable, we restrict the analysis to $N = 5$ and focus on the easy-to-implement vech HAR and HARQ models. We begin by simulating 2000 days of one-second returns; a more detailed description of the simulation setup is given in Appendix B. We then aggregate the one-second returns to 1, 5 and 15-minute returns ($M = 390, 78, 26$), and estimate the models on rolling windows of the corresponding 1000 daily MK estimates, resulting in a total of 1000 out-of-sample portfolio decisions for each of the different models and procedures. As a baseline, we also consider a HAR model for the true population covariances Σ_t, so that there are no measurement errors, only forecast errors. We will refer to this model as HAR$_\infty$. We report the portfolio turnover defined in (17), as well as the portfolio standard deviation based on the true covariances, $\sqrt{\hat{w}_t \Sigma_t \hat{w}_t'}$. To assess how far the estimated weights are from the "fundamental" weights, we also report the distance to the weights based on the true covariances, $\sqrt{\sum_n (\hat{w}_t^{(n)} - w_t^{(n)})^2}$. Finally, we report the distance of the portfolio weights to the weights implied by the HAR$_\infty$ model based on the true integrated covariances.

Table 4 details the findings from the simulations. To begin, note that even if the true covariances are observed, there are still non-trivial forecast errors, as evidenced by the results for the HAR$_\infty$ model. This provides a useful benchmark for the other models. Comparing the results for the HAR models across the different sampling frequencies clearly shows the detrimental impact of unreliable realized covariances, as manifested in higher turnover, portfolio standard deviation, and distance to the true Σ_t weights for lower values of M and increasingly inaccurate realized covariance matrices. By contrast, the performance of the HARQ model is much more stable across the different values of M, and also closer to the HAR$_\infty$ benchmark model. Compared to the HAR model, the HARQ model reduces the spurious turnover induced by estimation error by more than half when $M = 26$, and even more for $M = 78$ and $M = 390$.[20] In addition to the reduced excess turnover, the HARQ portfolios also achieve lower ex-post standard deviations than the HAR portfolios, and portfolio weights that are systematically closer to the true Σ_t weights.

7.2. Empirical intuition

Before presenting our main empirical results, it is instructive to visualize the key features of the dynamic attenuation models behind our findings. To do so, we consider an illustrative example based on just two stocks, Boeing (BA) and JP

20 The greater improvements relative to the HAR model for higher M stem from the fact that the degree of measurement error also needs to be estimated, which similarly becomes more accurate for higher values of M.

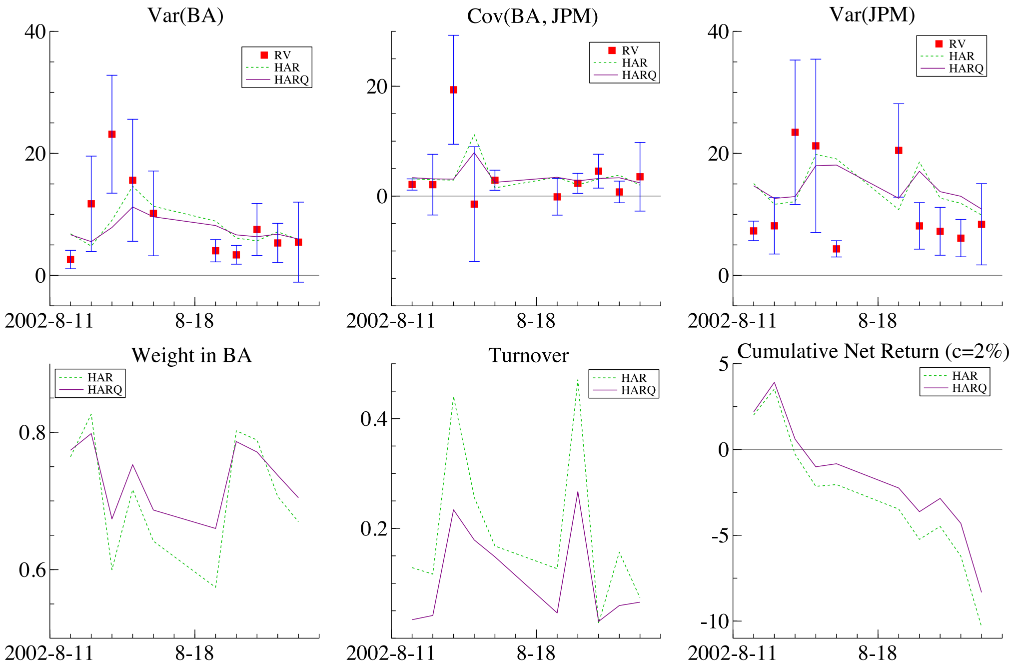

Fig. 1. Dynamic shrinkage. *Note:* The top row of panels shows the estimated (co)variance for BA and JPM, along with the forecasted values from the HAR and HARQ models. The bottom left panel shows the GMV weights for BA based on the HAR and HARQ models, the middle panel shows the turnover, and the right panel the cumulative portfolio return net of transaction costs.

Morgan (JPM), over the two-week period August 11–25, 2002. Following the simulations discussed in the previous section, we focus on the vech HAR and HARQ models, and the use of these models in the construction of daily GMV portfolios.[21]

The top row in Fig. 1 shows the ex-post estimated realized (co)variances and corresponding 95% confidence bounds, along with the one-day-ahead HAR and HARQ model forecasts. Evidently, the HARQ forecasts respond less aggressively to the lagged (co)variances than the HAR-based forecasts when the measurement errors are large. Since large uncertainty is often accompanied by transitory spikes in volatility, the HARQ-based forecasts appear more stable over time. Importantly, this translates into more stable portfolio allocations, as evidenced by the first two panels in the bottom row, which display the portfolio weight for BA and the portfolio turnover. Although the allocations for both of the models average around 70% to BA, the variability around this weight is obviously lower for the HARQ model. This in turn results in uniformly lower turnover for the HARQ model, and a reduction in transaction costs, as seen in the bottom right panel, which displays the cumulative net return over the two-week period based on transaction costs $c = 2\%$. While these two-week cumulative differences may appear numerically small, the results in the next section show that they add up to substantial improvements on an annual basis.

7.3. *Dynamically attenuated portfolio allocations*

Our main empirical results are reported in Tables 5 and 6. The first table gives the results related to the GMV portfolios, while the second reports the results for the portfolios designed to track the market. Looking first at the summary numbers in the top panel of Table 5, the HARQ-DRD model yields the lowest portfolio variance overall, and using the Model Confidence Set (MCS) approach of Hansen et al. (2011), we find the ex-post portfolio variances for the EWMAQ filter and the HEAVYQ model are not significantly larger. All other models generate significantly larger portfolio variances. The variation in the turnover observed across the different classes of models is quite large. Interestingly, however, even in the absence of transaction costs the models with the highest turnover do not result in the highest Sharpe ratios and/or lowest portfolio variance. On the contrary, all of the dynamic attenuation models result in lower turnover than their conventional benchmarks, while attaining a reduction in variance, an increase in average returns, and a higher Sharpe ratio.[22,23]

21 A similar figure for the same two stocks over the last three weeks of the sample is available in the Supplemental Appendix.

22 This finding is in line with the low volatility anomaly documented in Chan et al. (1999), Jagannathan and Ma (2003) and Baker et al. (2011), among others.

23 To help isolate the impact of expected returns, the Supplemental Appendix presents tables analogous to those here but with all of the returns set to zero. This eliminates the impact of the mean on Sharpe ratios and management fees. The resulting Sharpe ratios are lower, as expected, but the rankings of the models are unchanged. The same applies for the estimated management fees.

Table 5
Minimum variance portfolios.

		HAR	HARQ	HAR-DRD	HARQ-DRD	EWMA	EWMAQ	HEAVY	HEAVYQ
TO		0.522	0.385	0.391	0.339	0.135	0.096	0.172	0.122
CO		0.513	0.517	0.487	0.497	0.505	0.506	0.497	0.497
SP		−0.105	−0.109	−0.070	−0.082	−0.095	−0.096	−0.089	−0.089
Mean ret		3.079	3.156	3.612	4.371	3.455	3.867	3.847	4.124
StDev ret		14.912	14.781	15.016	**14.566**	14.972	**14.637**	14.922	**14.614**
$c = 0\%$	Sharpe	0.206	0.214	0.241	0.300	0.231	0.264	0.258	0.282
	Δ_1		9.7		82.6*		46.2		32.2
	Δ_{10}		27.3		142.6*		90.9*		73.3*
$c = 1\%$	Sharpe	0.118	0.148	0.175	0.241	0.208	0.248	0.229	0.261
	Δ_1		44.3*		95.6*		55.9*		44.9
	Δ_{10}		61.9*		155.6*		100.7*		86.0*
$c = 2\%$	Sharpe	0.030	0.082	0.109	0.183	0.185	0.231	0.200	0.240
	Δ_1		78.8*		108.6*		65.7*		57.6*
	Δ_{10}		96.4*		168.7*		110.4*		98.7*

Note: The table shows the results for the global minimum variance portfolio (GMV). We report turnover (TO), portfolio concentration (CO), and short positions (SP), as well as the average annualized return and standard deviation. Standard deviations in bold indicate the models that belong to the 90% model confidence set (MCS) of lowest ex-post daily volatility. The table also reports the economic gains of switching from the conventional model to the Q-model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

Table 6
Minimum tracking error portfolios.

		HAR	HARQ	HAR-DRD	HARQ-DRD	EWMA	EWMAQ	HEAVY	HEAVYQ
TO		0.173	0.102	0.134	0.114	0.063	0.045	0.080	0.057
CO		0.339	0.338	0.339	0.342	0.340	0.340	0.338	0.338
SP		−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	−0.000	−0.000
StDev TE		6.618	**6.489**	6.610	**6.521**	6.636	**6.513**	6.637	**6.506**
$c = 0\%$	Δ_1		29.1		27.8		33.6		32.5
	Δ_{10}		39.1		35.0*		43.4		42.8
$c = 1\%$	Δ_1		46.9*		42.8*		38.2		38.5
	Δ_{10}		56.9*		50.0*		47.9*		48.8
$c = 2\%$	Δ_1		64.6*		47.8*		42.7*		44.5*
	Δ_{10}		74.6*		55.0*		52.5*		54.8*

Note: The table shows the results for the minimum tracking error portfolio. We report turnover (TO), portfolio concentration (CO), and short positions (SP), as well as the standard deviation of the tracking error (TE). Standard deviations in bold indicate the models that belong to the 90% model confidence set (MCS) of lowest ex-post daily volatility. The table also reports the economic gains of switching from the conventional model to the Q-model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

The bottom panel of Table 5 shows the economic gains of switching from each of the conventional models to their dynamic attenuation counterparts. The gains effectively come from two separate sources: improved forecasts accuracy and reduced turnover. A simple gauge of the relative magnitude of the two effects is available by comparing the results for $c = 0\%$ (zero transaction, or turnover, costs) with those for $c = 1\%$ and $c = 2\%$. In the former case, reducing turnover leads to no gains in performance, while in the latter two cases it does. For $c = 0\%$ the total gains range between 10 and 140 basis points annually, depending on the model and coefficient of risk aversion. Since all of the Q-models result in lower turnover, these gains rise to 60 to 170 basis points when the transaction costs rise from $c = 0\%$ to $c = 2\%$.[24]

Turning to Table 6 and the results for the minimum tracking error portfolios tell a similar story. The standard deviation of the tracking error is significantly reduced by the Q-models, which jointly comprise the model confidence set of minimum TE models. The percentage reductions in turnover also closely mirror those of the GMV portfolios. Meanwhile, as the turnover for the tracking portfolios tends to be somewhat lower than for the GMV portfolios, the economic gains are similarly reduced. Nonetheless, the gains are consistently positive ranging from between 30 to 70 basis points annually. Again, these are non-trivial improvements from a practical investment perspective.

As indicated by the summary statistics in the top part of Table 5, many of the portfolios in Table 5, and the GMV portfolios in particular, involve negative or short positions. Short positions are generally more costly to implement than long positions. Many financial institutions are also explicitly prevented from holding short positions. Hence, in an effort to investigate the sensitivity of our results to the imposition of no-short-sale constraints, we repeat the previous analysis by adding the constraint that $w_t^{(n)} \geq 0, \forall n$ to the optimization problem in (15). While this constrained optimization problem no longer allows for an explicit closed form solution to the optimal portfolio weights, the constrained minimum variance portfolios may easily be solved for numerically using standard quadratic programming tools (see, e.g., Lawrence and Tits, 2001).

24 The turnover is reduced by a minimum of 14% for the HARQ-DRD model up to almost 30% for the HEAVYQ model.

Table 7
No short-sale minimum variance portfolios.

		HAR	HARQ	HAR-DRD	HARQ-DRD	EWMA	EWMAQ	HEAVY	HEAVYQ
TO		0.390	0.284	0.322	0.280	0.101	0.071	0.131	0.093
CO		0.477	0.479	0.466	0.471	0.475	0.475	0.469	0.469
Mean ret		3.755	3.927	3.922	4.633	3.925	4.355	4.229	4.668
StDev ret		15.365	14.977	15.296	14.855	15.260	**14.821**	15.249	**14.784**
$c = 0\%$	Sharpe	0.244	0.262	0.256	0.312	0.257	0.294	0.277	0.316
	Δ_1		23.1		77.8*		62.3		64.7
	Δ_{10}		76.1		137.6*		114.7*		120.0*
$c = 1\%$	Sharpe	0.180	0.214	0.203	0.264	0.241	0.282	0.256	0.300
	Δ_1		49.7		88.4		57.0*		60.6
	Δ_{10}		102.8*		148.3*		116.4*		123.6*
$c = 2\%$	Sharpe	0.116	0.167	0.150	0.217	0.224	0.270	0.234	0.284
	Δ_1		76.4*		99.0*		64.3*		70.3*
	Δ_{10}		129.4*		158.9*		123.9*		133.2*

Note: The table reports the results for minimum variance portfolios that do not allow for short positions. The table shows the portfolio turnover (TO), portfolio concentration (CO), together with the average annualized return and standard deviation. Standard deviations in bold indicate the models that belong to the 90% model confidence set (MCS) of lowest ex-post daily volatility. The table also reports the economic gains of switching from the conventional model to the Q-model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

The results pertaining to these constrained minimum variance portfolios not allowing for short-sales are reported in Table 7.[25] Interestingly, and consistent with the prior empirical evidence in Jagannathan and Ma (2003), the no short-sale restriction generally leads to higher ex-post Sharpe ratios than for the unconstrained GMV portfolios.[26] The no-short-sale constraint also systematically reduces turnover. As such, the economic gains from the dynamic attenuation models are slightly lower compared to the gains for the unconstrained GMV portfolios in Table 5. Nonetheless, the benefits of switching to the new Q-models remain economically large and statistically significant, ranging up to nearly 160 basis points per year in some situations.

8. Longer horizons and alternative procedures

Dynamic portfolio decisions naturally present a trade-off between the horizon over which the pertinent risks are assumed to be constant versus the accuracy with which the risks can be measured and the costs of implementing the investment decisions. In the results reported above, we relied on a daily horizon for both estimating the new models and re-balancing the portfolios. This section presents additional results pertaining to longer weekly and monthly investment horizons. In addition to the use of coarser re-balancing schemes, alternative shrinkage type procedures have previously been advocated in the literature to help mitigate the impact of estimation errors and excessive turnover. Below we also compare and contrast the new dynamic attenuation procedures developed here with some of these alternative shrinkage type procedures, in which the covariance matrix forecasts are shrunk towards some pre-determined target. We focus our comparisons on the HARQ-DRD model, which performed the best in the daily results discussed above.

8.1. Weekly and monthly rebalancing

The daily horizon underlying the results in the previous section might be expected to favor the new dynamic attenuation models, compared to the results obtained over longer weekly and monthly horizons, where the estimation errors in the realized covariances play a less important role. On the other hand, daily portfolio construction likely results in higher transaction costs than less frequent re-balancing. Correspondingly, some investors might be reluctant to change their positions on a daily basis, preferring instead a less frequent weekly or monthly re-balancing scheme. To investigate the robustness of our findings to the use of longer holding periods, we present the results in which we rely on the HAR(Q)-DRD models implemented at weekly and monthly frequencies.

To implement the models over these longer forecast horizons, we replace the one-day-ahead covariance matrix with the realized covariance matrix over the relevant horizon of interest.[27] Specifically, for the HAR formulation in Eq. (6),

$$s_{t|t+h} = \theta_0 + \theta_1 s_{t-1} + \theta_2 s_{t-5|t-1} + \theta_3 s_{t-22|t-1} + \epsilon_t, \tag{23}$$

25 Since the unconstrained tracking portfolios involve much fewer short positions, the results for the constrained tracking portfolios are much closer to the results in Table 5. We defer these results to the Supplemental Appendix to conserve space.

26 As discussed by Jagannathan and Ma (2003), restricting the portfolio weights to be non-negative may be seen as a way to limit the impact of estimation errors, akin to the main idea behind the new dynamic attenuation models, and the more traditional shrinkage type procedures discussed further below.

27 In the forecasting literature, this is commonly referred to as “direct” as opposed to “iterated” forecasting; see, e.g., Marcellino et al. (2006). In theory, if the daily model is correctly specified the iterated forecasts constructed from that model should be the most efficient. However, there is ample empirical evidence that even minor model mis-specifications tend to get amplified in iterated volatility forecasts, and as a result the direct forecast procedures often work better in practice; see, e.g., Andersen et al. (2003) and (Sizova, 2011).

Table 8
Longer horizon portfolio allocations.

		Weekly		Monthly	
		HAR-DRD	HARQ-DRD	HAR-DRD	HARQ-DRD
TO		0.112	0.116	0.031	0.031
CO		0.487	0.497	0.488	0.497
SP		−0.070	−0.082	−0.069	−0.081
Mean ret		3.122	3.258	3.244	3.364
StDev ret		15.366	14.734	15.856	15.540
$c = 0\%$	Sharpe	0.203	0.221	0.205	0.217
	Δ_1		23.1		16.9
	Δ_{10}		108.9*		61.6*
$c = 1\%$	Sharpe	0.185	0.201	0.200	0.211
	Δ_1		22.0		16.8
	Δ_{10}		107.8*		61.5*
$c = 2\%$	Sharpe	0.166	0.181	0.195	0.206
	Δ_1		20.9		16.8
	Δ_{10}		106.6*		61.5*

Note: The table reports the long-horizon GMV portfolio results. The portfolios are re-balanced weekly or monthly based on the relevant weekly and monthly covariance matrix forecasts. Each panel shows turnover (TO), portfolio concentration (CO), and short positions (SP). The top panel also shows the average annualized return and standard deviation, while the bottom panel reports the standard deviation of the tracking error (TE). The table also reports the economic gains of switching from the standard HAR-DRD model to the HARQ-DRD model in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

Table 9
Daily versus weekly and monthly portfolio allocations.

		Weekly HARQ-DRD		Monthly HARQ-DRD	
		HAR-DRD	HARQ-DRD	HAR-DRD	HARQ-DRD
$c = 0\%$	Δ_1	31.2	113.7*	92.4*	174.9
	Δ_{10}	−6.6	136.0*	152.7*	295.4*
$c = 1\%$	Δ_1	−38.1	57.5*	1.7	97.3*
	Δ_{10}	−75.9*	79.7*	61.9*	217.7*
$c = 2\%$	Δ_1	−107.3*	1.3	−88.9*	61.1*
	Δ_{10}	−145.2*	23.5	−28.8	140.0*

Note: The table reports the economic gains of switching from weekly or monthly re-balanced GMV portfolios based on HARQ-DRD forecasts to a daily strategy based on HAR(Q)-DRD forecasts in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

where $h = 4$ and 21 for the weekly and monthly forecasts, respectively. We continue to rely on (overlapping) daily data in actually estimating the models. However, when forecasting the weekly (monthly) covariances, the weekly (monthly) lag tends to receive a larger weight compared with the daily forecasting model. Correspondingly, we explicitly tailor the HARQ formulations to the specific forecast horizons, and only attenuate the relevant autoregressive parameters. That is, for the weekly HARQ model we keep θ_1 and θ_3 constant and replace θ_2 with $\theta_{2,t} = (\theta_2\iota + \theta_{2Q}\pi_{t-1|t-5})$, where $\pi_{t-1|t-h} \equiv (\frac{1}{h}\sum_{i=1}^{h}\pi_{t-i})^{1/2}$, while for the monthly model we keep θ_1 and θ_2 constant and replace θ_3 with $\theta_{3,t} = (\theta_3\iota + \theta_{3Q}\pi_{t-1|t-22})$. In parallel to the previous analysis, we assume that the investor relies on the resulting weekly $\Sigma_{t|t+4}$ and monthly $\Sigma_{t|t+21}$ covariance forecasts to construct GMV portfolios, re-balancing her positions at a weekly or monthly frequency, respectively.

The results for these longer horizon forecasting models and corresponding GMV portfolios averaged across all possible weekly (i.e., Monday-to-Monday, Tuesday-to-Tuesday, etc.) and monthly (i.e., 1st day-of-the-month, 2nd day-of-the-month, etc.) horizons are reported in Table 8. Consistent with the basic intuition underlying our approach, and the fact that it is easier to accurately estimate the realized covariance matrix over longer horizons, the estimated economic gains for the HARQ-DRD models relative to their conventional counterparts tend to be somewhat lower at the weekly horizon compared to the daily results reported in Table 5, and lower still at the monthly horizon. Nonetheless, all of the Δ_γs remain positive and statistically significant for all of the more risk averse cases.

To help further illuminate the trade-offs between the use of faster, but more costly-to-implement daily models, versus slower and cheaper-to-implement weekly and monthly models, Table 9 reports the estimated economic gains from shifting from the use of a daily HAR(Q)-DRD model to a weekly or monthly HARQ-DRD model. A positive Δ_γ in this table represents evidence in favor of daily modeling and re-balancing, while a negative Δ_γ supports the corresponding weekly or monthly model and less frequent re-balancing. Looking first at the results for the benchmark daily HAR-DRD portfolios, we see that in the absence of transaction costs, daily re-balancing generally improves on the weekly and monthly re-balanced HARQ-DRD portfolios, as reflected by the fact that all but one of the Δ_γs for $c = 0\%$ are positive. However, the benefits of the daily HAR-DRD model over the weekly HARQ-DRD model are systematically wiped out when transaction costs are incorporated into the comparisons. This supports the common choice of coarser weekly or monthly re-balancing schemes as a simple way to help mitigate trading costs, when using simple forecasting models. In sharp contrast to this, *all* of the Δ_γs for the our proposed daily HARQ-DRD model are positive, and all but two are statistically significant. This shows that by accounting for the time-varying measurement errors in the daily realized covariance estimates and obtaining more disciplined forecasts, the new dynamic attenuation models make daily re-balancing economically beneficial, even in the presence of transaction costs.

8.2. Alternative ex-post shrinkage procedures

The dynamic attenuation models endogenously shrink the influence of past realized covariances based on dynamically varying weights determined by an estimate of the reliability of the realized covariances. Alternative exogenous shrinkage type procedures, in which the forecasts themselves are shrunk to some target to help mitigate the impact of estimation errors in the context of portfolio construction, have previously been proposed in the literature, notably the work by Ledoit and Wolf (2003, 2004a, b). In this section we compare the effectiveness of these ex-post shrinkage procedures to the new dynamic attenuation models, contrasting the use of shrunk forecasts from the HAR-DRD model to the forecast from the new HARQ-DRD model in the construction of daily GMV portfolios.[28]

Our implementation of the shrinkage procedures, is based on the traditional lower frequency estimators and theoretical set-up of Ledoit and Wolf (2003). Specifically,

$$\widetilde{H}_{t|t-1} = a_{t|t-1}F_{t|t-1} + (1 - a_{t|t-1})H_{t|t-1}, \tag{24}$$

where $a_{t|t-1}$ and $F_{t|t-1}$ denote the shrinkage intensity and shrinkage target, respectively. For $a_{t|t-1} \equiv 0$ this obviously reduces to the forecast provided by the underlying dynamic forecasting model, while for $a_{t|t-1} \equiv 1$ the covariance matrix forecast is identically equal to the target matrix $F_{t|t-1}$. In the results reported on below, we rely on the estimate for the "optimal" shrinkage intensity formally derived by Ledoit and Wolf (2003, 2004a) to determine the amount of shrinkage away from the forecast. Further details are provided in Appendix D.

Several different choices have been proposed for the shrinkage target $F_{t|t-1}$. We consider three. First, motivated by the single-factor model advocated in Ledoit and Wolf (2003), we consider a high-frequency-based realized equivalent. In particular, we equate the diagonal elements of $F_{t|t-1}$ to the realized variances for each of the stocks, while the off-diagonal elements are set equal to $h^{(mkt)}_{t|t-1} b_{t|t-1} b'_{t|t-1}$, where $h^{(mkt)}_{t|t-1}$ and $b_{t|t-1}$ denote the forecasted realized variance of the market and the forecasted realized market betas for each of the stocks, respectively. This one-factor structure summarizes the correlation among the stocks using their exposure to the common market factor. Second, following Voev (2008) among others, we rely on an equicorrelation structure based on the decomposition in Eq. (8), in which we restrict the correlations among all of the stocks to be the same.[29] That is we set the target matrix to $D_{t|t-1}\bar{R}_{t|t-1}D_{t|t-1}$, where the off-diagonal elements in the $\bar{R}_{t|t-1}$ matrix are fixed at the average correlation among all of the stocks $\bar{\rho}_{t|t-1} = \frac{1}{N(N-1)/2}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\hat{\rho}_{ij,t|t-1}$. Finally, following Ledoit and Wolf (2004b), we consider the identity matrix as an all-purpose shrinkage target. For the target intensity $a_{t|t-1} \equiv 1$ this reduces to an equal weighted portfolio of all the stocks. As shown by DeMiguel et al. (2009b), this $1/N$ portfolio is often difficult to beat, especially in large dimensions, and we include it as an additional competitor. For comparison purposes, we also include the results for no short-sale HAR-DRD portfolios previously reported in Table 7.[30]

The results from using these different shrinkage-based forecasts in the construction of daily GMV portfolios are reported in Table 10.[31] As a reference, the first two columns report the results based on the HARQ-DRD and HAR-DRD model respectively, as previously reported in Table 5. The following columns report the results for the various shrinkage targets and the equally-weighted $1/N$ portfolio. A comparison of the results for the different shrinkage portfolios, to the portfolios based on the HAR-DRD forecasts, reveals a significant reduction in the turnover across shrinkage targets, and a reduction in the portfolio standard deviation for most targets.[32]

Abstracting from transaction costs, the Sharpe ratios for the factor- and equicorrelation-based shrinkage portfolios, as well as the no short-sale portfolios, improve on the standard HAR-DRD-based portfolios, and shrink the gap towards the HARQ-DRD-based portfolios. However, the economic gains of shifting from any one of the shrinkage procedures to the HARQ-DRD model, as measured by the Δ_γs, remain positive, and statistically significant. Moreover, including transaction cost into the comparisons results in large economic gains of more than two percent per year relative to the ex-post shrunk portfolios. These results clearly underscore the advantages of dynamically incorporating the effect of measurement errors in the high-frequency-based realized covariances into the construction of covariance matrix forecasts and financial decisions, rather than shrinking the covariance forecasts underlying the financial decisions to some naive target.

28 Appendix D contains some additional discussion related to the recent work of Voev (2008) and Hautsch et al. (2015) and shrunk random walk type realized covariance matrix forecasts, relying on the asymptotic distribution of the MKernel estimator to obtain the shrinkage intensities. The Supplemental Appendix further details results in which we apply the ex-post shrinkage procedure to HARQ-DRD forecasts themselves. As these results show, the HARQ-DRD forecasts are clearly superior to the more traditionally shrunk random walk type forecasts, while the additional "unconditional" shrinkage of the HARQ-DRD forecasts does not really help nor hurt.

29 This same idea also underlies the DECO model of Engle and Kelly (2012).

30 As demonstrated in Jagannathan and Ma (2003), the no short-sale restriction may be interpreted as a special case of shrinkage.

31 The results for the minimum tracking error portfolios are again deferred to the Supplemental Appendix.

32 The poor performance of the $1/N$ portfolio reflects the fact that with "only" ten stocks, it is still possible to forecast the covariance matrix sufficiently precisely to beat a completely uninformative forecast.

Table 10
Ex-post shrinkage HAR-DRD forecasts.

		HARQ-DRD	HAR-DRD	Shrunk HAR-DRD				1/N
				Factor	Equicorr	Identity	No short	
TO		0.339	0.391	0.369	0.361	0.303	0.322	0.009
CO		0.497	0.487	0.481	0.480	0.433	0.466	0.316
SP		−0.082	−0.070	−0.061	−0.060	−0.039	0.000	0.000
Mean ret		4.371	3.612	3.700	3.723	2.753	3.922	1.044
Std Ret		14.566	15.016	14.985	14.978	15.074	15.296	18.579
$c = 0\%$	Sharpe	0.300	0.241	0.247	0.249	0.183	0.256	0.056
	Δ_1		82.6*	73.3*	70.9*	169.3*	51.7*	399.2*
	Δ_{10}		146.0*	129.0*	125.8*	237.2*	150.0*	997.1*
$c = 1\%$	Sharpe	0.241	0.175	0.185	0.188	0.132	0.203	0.055
	Δ_1		95.6*	80.9*	76.5*	160.3*	47.4*	316.2*
	Δ_{10}		155.6*	136.7*	131.4*	228.1*	145.7*	914.1*
$c = 2\%$	Sharpe	0.183	0.109	0.123	0.127	0.081	0.150	0.054
	Δ_1		108.6*	88.6*	82.1*	151.3*	43.1*	233.1*
	Δ_{10}		168.7*	144.4*	137.0*	219.1*	141.5*	931.1*

Note: The table reports global minimum variance portfolios formed based on exogenously shrunk HAR-DRD forecasts. The top panel shows turnover (TO), portfolio concentration (CO), short position(SP), as well as the average annualized portfolio volatility. Each bottom panel shows the economic gains of switching from the alternatives to the HARQ-DRD model forecasts in annual basis points, Δ_γ, for various transaction cost levels c and risk aversion coefficients γ. Asterisks denote Δ_γ significantly different from zero at the 5% level.

9. Conclusion

Fusing ideas from the econometrics literature on errors-in-variables with more recent results from the finance literature on the use of high-frequency intraday data and the estimation of realized risk measures, we provide a new and broadly applicable framework for more accurately forecasting financial risks. The basic idea behind the new approach is simple and intuitive: when current risks are measured with a high (low) degree of uncertainty, they should receive relatively low (high) weights in forecasts of future risks. Adapting various state-of-the-art realized covariance-based models to accommodate this feature and dynamically attenuate the influence of the past covariances not only results in on average more accurate, but importantly also more stable and from a practical perspective cheaper to implement, covariance forecasts. These improvements in turn translate into sizable economic gains for a risk-averse investor seeking to minimize the variance of her equity portfolio or track the aggregate market.

The practical implementation of a host of other key concepts in risk management and asset pricing finance similarly depend on the ability to accurately forecast common risks as measured by the covariances of individual asset returns, or the covariances with appropriately defined benchmark portfolios. The new approach and simple-to-implement covariance forecasting models developed here thus hold the promise of empirically more accurate pricing models and improved financial decision making generally.

Acknowledgments

The research was partially supported by a grant from Inquire Europe. Bollerslev also gratefully acknowledges the support from CREATES funded by the Danish National Research Foundation (DNRF78). Quaedvlieg was financially supported by Netherlands Organisation for Scientific Research (NWO) Grant 407-11-042. We would also like to thank Oliver Linton (the Editor), two anonymous referees, Sébastien Laurent, Peter Schotman, George Tauchen, Rasmus Varneskov, as well as seminar and conference participants at Aix-Marseille University, EDHEC Singapore, Erasmus University Rotterdam, Northwestern University, Nova SBE Lisbon, Warwick Business School, the 2016 Toulouse Financial Econometrics Conference, the 2016 BI-SHOF Conference in Stockholm, the 2016 AP2 Conference in Gothenburg, the 2016 IAAE meetings in Milan, the 2016 CFE Network Conference in Seville, the 2017 VieCo Conference in Vienna, and the 2017 Volatility Institute and 10th Annual SoFiE Conferences at NYU Stern for helpful comments. Bingzhi Zhao kindly provided us with the cleaned high-frequency data underlying our empirical investigations.

Appendix A. Multivariate Kernel theory and estimation

Let the N-dimensional vector of discrete-time returns for the ith intraday time-interval on day t be denoted by $r_{i,t} \equiv P_{t-1+i\Delta} - P_{t-1+(i-1)\Delta}$. Further assume that we observe $M = 1/\Delta$ synchronized intra-daily returns for each of the N assets. The Multivariate Kernel (MK) estimator of Barndorff-Nielsen et al. (2011) (henceforth BHLS) is then defined as

$$\mathrm{MK}_t = \sum_{h=-M}^{M} k(h/H)\Gamma_h, \tag{A.1}$$

where $\Gamma_h = \sum_{i=h+1}^{M} r_{i,t} r'_{i-h,t}$, $\Gamma_h = \Gamma'_{-h}$, and $k(.)$ denotes an appropriate kernel function with bandwidth H. In the implementation here we use the Parzen kernel. The asymptotic theory of the MK further requires "jittering", as discussed in more detail in the BHLS paper. According to Theorem 2 of BHLS, it then follows that

$$M^{1/5}(\text{vech MK}_t - \text{vech } \Sigma_t) \rightarrow_{Ls} MN(\omega_t, \Pi_t), \tag{A.2}$$

where $\Pi_t = 3.777 IQ_t$, and IQ_t denotes the Integrated Quarticity. The bias term ω_t is negligible, and we ignore it in our implementation. We estimate Π_t based on pre-averaged data using the estimator of Barndorff-Nielsen and Shephard (2004). Specifically, defining $x_{i,t} \equiv \text{vech}(\bar{r}_{i,t}\bar{r}'_{i,t})$, where $\bar{r}_{i,t}$ refers to the pre-averaged data,

$$\widehat{IQ_t} = M \sum_{i=1}^{M-H} x_{i,t} x'_{i,t} - \frac{M}{2} \sum_{j=1}^{M-H-1} (x_{i,t} x'_{i+1,t} + x_{i+1,t}, x'_{i,t}). \tag{A.3}$$

defines our consistent estimator for IQ_t.

Appendix B. Simulation design

In order to simulate empirically realistic sample paths of daily varying covariances, we draw from the empirical distribution of our data. To do so, we randomly select five of the ten stocks to obtain an MK-based covariance sample path of 2000 consecutive days. For each day within that sample, we then simulate one-second returns with integrated covariance matrix equal to the MK estimate for that day. We do not allow for within day stochastic variation in the covariances, but we do include a diurnal pattern. The intraday volatility pattern is modeled by means of a diurnal U-shape function, $\sigma_d(u)$, by simulating from the following process using a standard Euler discretization scheme,

$$dP(u) = S(u)^{1/2} dW(u), \tag{B.1}$$

$$S(u) = \sigma_d(u) S \tag{B.2}$$

$$\sigma_d(u) = C + Ae^{-au} + Be^{-b(1-u)}, \tag{B.3}$$

where, following Andersen et al. (2012), we set the periodicity parameters to $A = 0.75$, $B = 0.25$, $C = 0.88929198$, and $a = b = 10$, respectively.

Appendix C. Composite likelihood estimators

The use of composite likelihoods in the estimation of large dimensional volatility models was recently popularized by Pakel et al. (2014). To define the approach, let V_t denote the $N \times N$ conditional covariance matrix for the $N \times 1$ vector of returns r_t. The standard quasi-likelihood obtained under the auxiliary assumption of conditional normality then takes the form,

$$\log L(\psi; r) = \sum_{t=1}^{T} \ell(\psi; r_t), \tag{C.1}$$

where,

$$\ell(\psi; r_t) = -\frac{1}{2} \log |V_t| - \frac{1}{2} r'_t V_t^{-1} r_t. \tag{C.2}$$

If N is of large dimension, this can be difficult and time-consuming to implement and also result in numerically unstable V_t^{-1} matrices. The composite likelihood approach sidesteps these problems by instead approximating the likelihood with a number of lower dimensional marginal densities, so that the dimension of the problem is reduced from N to 2. In the implementation adopted here, we rely on contiguous pairs, $Y_{1t} = (r_{1,t}, r_{2,t})$, $Y_{2t} = (r_{2,t}, r_{3,t})$, ..., $Y_{d-1,t} = (r_{d-1,t}, r_{d,t})$, resulting in the composite likelihood,

$$CL(\psi) = \frac{1}{T(N-1)} \sum_{j=1}^{N-1} \sum_{t=1}^{T} \ell_{jt}(\psi; Y_{jt}). \tag{C.3}$$

Maximizing this much easier-to-implement $CL(\psi)$ function yields a consistent and asymptotically normal estimate for ψ.

Appendix D. Ex-post shrinkage details

Ledoit and Wolf (2003) suggest choosing the unconditional shrinkage intensity α, by minimizing the quadratic loss function,

$$L(\alpha) = \|\alpha F + (1-\alpha)S - \Sigma\|^2, \tag{D.1}$$

where F refers to an estimate of the shrinkage target Φ. The resulting expected loss may be expressed as,

$$\begin{aligned}\mathbb{E}(L(\alpha)) &= \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbb{E}(\alpha f_{ij} + (1-\alpha)s_{ij} - \sigma_{ij}) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha^2 Var(f_{ij}) + (1-\alpha)^2 Var(s_{ij}) + 2\alpha(1-\alpha)Cov(f_{ij}, s_{ij}) + \alpha^2(\phi_{ij} - \sigma_{ij})^2.\end{aligned} \tag{D.2}$$

Optimizing this expression with respect to α, results in the optimal shrinkage intensity,

$$\alpha^* = \frac{\sum_{i=1}^{N}\sum_{j=1}^{N} Var(s_{ij}) - Cov(f_{ij}, s_{ij})}{\sum_{i=1}^{N}\sum_{j=1}^{N} Var(f_{ij} - s_{ij}) + (\phi_{ij} - \sigma_{ij})^2}. \tag{D.3}$$

The term $Var(f_{ij} - s_{ij})$ is of lower order than the other terms and may be ignored. The $Var(s_{ij})$ and $Cov(f_{ij}, s_{ij})$ terms can typically be deduced from the underlying distribution theory, while $(\phi_{ij} - \sigma_{ij})^2$ can simply be estimated based on sample values.

When the identity matrix is the target, estimating α^* is simple as $Cov(f_{ij}, s_{ij}) = 0$. For non-deterministic shrinkage targets the covariance between the target and the estimate s_{ij} needs to be taken into account. Since the target is often a function of the forecasted matrix itself, the Delta method can be used to obtain information on their joint distribution. Ledoit and Wolf (2003) provide consistent estimators in the case of the single factor target, while Ledoit and Wolf (2004a) provide an estimator for α^* when the target is the equicorrelation matrix. The shrinkage intensities used in Section 8.2 are based on the corresponding forecast error variances. To obtain the forecast error variance, we simply rely on the unconditional sample covariance matrix for the residuals over the estimation window.

In addition to the results presented in the main part of the paper based on shrinkage of the HAR-DRD forecasts, the Supplemental Appendix reports the results from applying the same ex-post shrinkage procedures to the HARQ-DRD forecasts themselves. As these results show, this additional shrinkage of the forecasts from the dynamically attenuated HARQ-DRD model offers at best very minor economic, and mostly statistically insignificant, improvements.

Finally, following Voev (2008) we also consider the case which relies on the asymptotic distribution theory for the high-frequency realized covariance matrix estimators in the construction of optimally shrunk random walk forecasts. This same approach has also recently been implemented by Hautsch et al. (2015), and we refer to these studies for additional details. The corresponding results comparing the dynamically shrunk random walk type forecasts with the portfolios based on the HARQ-DRD forecasts, reported in the Supplemental Appendix, reveal a significant reduction in both the turnover and the portfolio standard deviation for the shrunk random walk forecast. As such, this confirms the idea that the use of the historical realized covariance matrix leads to poor portfolio allocations, and that some regularization in the form of shrinkage generally improves the performance. Abstracting from transaction costs, the Sharpe ratios for the factor- and equicorrelation-based shrinkage portfolios are fairly close to the Sharpe ratio for the HARQ-DRD-based portfolios. At the same time, however, the economic gains of shifting from any one of the shrinkage procedures to the HARQ-DRD model, as measured by the Δ_γs, are always positive, and statistically significant in all but two cases. Moreover, including transaction cost into the comparisons results in massive economic gains of up to eight percentage points per year for the HARQ-DRD model compared to the more traditional shrunk random walk-based portfolios.

Appendix E. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jeconom.2018.05.004.

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