Exploiting the Errors: A Simple Approach for Improved Volatility Forecasting

Tim Bollerslev Andrew J. Patton Rogier Quaedvlieg

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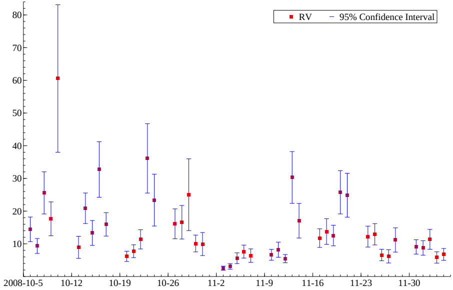
- Realized Volatility (RV) based modeling and forecasting procedures have become very popular
  - ARFIMA (Andersen, Bollerslev, Diebold and Labys, 2003)
  - HAR-RV (Corsi, 2009)
  - ...
- Generally superior to standard GARCH and Stochastic Volatility (SV) models
  - And much easier to estimate and implement ...
- Effective way of incorporating (the relevant) information in high-frequency data into the forecasts
  - Only minor efficiency loss (Andersen, Bollerslev and Meddahi, 2004)
  - Built-in robustness (Sizova, 2011)

- Most realized volatility-based forecasting models effectively treat the volatility as directly observable
  - But, RV still subject to non-trivial measurement errors ...



- The quality of the forecasts need to be properly interpreted
  - R^2 adjustments (Andersen, Bollerslev and Meddahi, 2005)
  - Ranking criteria (Patton, 2011)

- Instead of adjusting the measures of the quality of the forecasts, we adjust the forecasts themselves

- Our adjustments rely on the asymptotic distribution theory for RV and the heteroskedasticity of the measurement errors (Barndorff-Nielsen and Shephard, 2002; Meddahi, 2002)

- Simple and easy-to-implement modification of the popular HAR-RV model: HARQ

- The HARQ model performs well in simulations

- The HARQ model beats a challenging set of benchmark models when applied to the S&P 500 and a large set of individual stocks

- Log-price process:

$$d\log(P_t) = \mu_t dt + \sigma_t dW_t$$

  - Could easily allow for jumps

- Daily latent Integrated Volatility (IV) of interest:

$$IV_t \equiv \int_{t-1}^{t} \sigma_s^2 ds$$

- Consistently ($\Delta \to 0$) estimated by Realized Volatility (RV):

$$RV_t \equiv \sum_{i=1}^{1/\Delta} r_{t,i}^2$$

where $r_{t,i} \equiv log(P_{t-1+i\Delta}) - log(P_{t-1+(i-1)\Delta})$

- Suppose daily IV follows a simple AR(1):

$$IV_t = \phi_0 + \phi_1 IV_{t-1} + \epsilon_t$$

  - By standard arguments $RV_t \equiv IV_t + \eta_t$ follows ARMA(1,1)

- Suppose the researcher estimates an AR(1) for RV:

$$IV_t + \eta_t = \beta_0 + \beta_1 (IV_{t-1} + \eta_{t-1}) + \epsilon_t$$

- Attenuation bias with *i.i.d.* measurement errors:

$$\beta_1 = \phi_1 \left(1 + \frac{\sigma_\eta^2}{Var(IV_t)}\right)^{-1}$$

- But, the measurement errors are heteroskedastic:

$$RV_t = IV_t + \eta_t, \qquad \eta_t \sim N(0, \sigma^2_{\eta,t})$$

where $\sigma^2_{\eta,t} = 2\Delta IQ_t$ and $IQ_t \equiv \int_{t-1}^{t} \sigma^4_s ds$

- Allow the AR(1) coefficient to vary with $\sigma^2_{\eta,t}$:

$$\beta_{1,t} = \phi_1 \left(1 + \frac{\sigma^2_{\eta,t}}{Var(IV_t)}\right)^{-1}$$

- IQ_t (and $\sigma^2_{\eta,t}$) latent, but consistently ($\Delta \to 0$) estimated by:

$$RQ_t \equiv \frac{1}{3\Delta} \sum_{i=1}^{1/\Delta} r^4_{t,i}$$

- AR(1) coefficient depends on the inverse of *IQ*
  - Non-linear estimation

- *RQ* consistent but inaccurate estimate of *IQ*
  - Inverse even more problematic

- Simple linear time-varying parameter ARQ model:

$$RV_t = \beta_0 + \beta_{1,t} RV_{t-1} + \epsilon_t$$
$$\beta_{1,t} = \beta_1 + \beta_{1Q} RQ_{t-1}^{1/2}$$

  - Very easy to estimate and implement
  - Built-in robustness
  - Taylor series approximation

- IV (and RV) strongly persistent
    - AR(1) model too simplistic

- Heterogeneous AutoRegressive (HAR) model (Corsi, 2009):

$$RV_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \epsilon_t$$

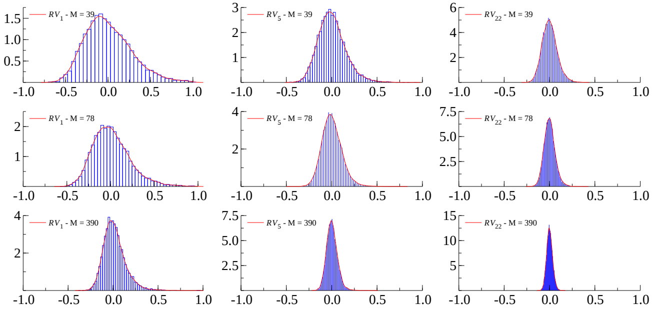
    - Poor man's long-memory model

- HARQ-F (Full) model:

$$RV_t = \beta_0 + \underbrace{\left(\beta_1 + \beta_{1Q} RQ_{t-1}^{1/2}\right)}_{\beta_{1,t}} RV_{t-1} + \underbrace{\left(\beta_2 + \beta_{2Q} RQ_{t-1|t-5}^{1/2}\right)}_{\beta_{2,t}} RV_{t-1|t-5}$$

$$+ \underbrace{\left(\beta_3 + \beta_{3Q} RQ_{t-1|t-22}^{1/2}\right)}_{\beta_{3,t}} RV_{t-1|t-22} + \epsilon_t$$

- Measurement errors in $RV_{t-1|t-h}$ decrease with horizon ($h = 1, 5, 22$) and sampling frequency ($1/\Delta = 39, 78, 390$):



- Basic HARQ model:

$$RV_t = \beta_0 + \underbrace{\left(\beta_1 + \beta_{1Q} RQ_{t-1}^{1/2}\right)}_{\beta_{1,t}} RV_{t-1} + \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \epsilon_t$$

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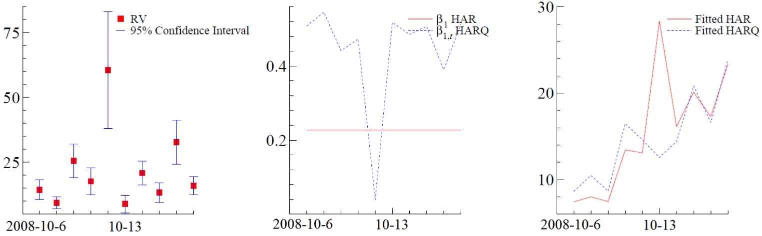
- HARQ vs HAR model estimates, S&P 500, October 2008:



Simulation Setup

- Two-factor stochastic volatility model as in Huang and Tauchen (2005):

$$d \log P_t = \mu dt + \sigma_{ut}\nu_t \left(\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1-\rho_1^2-\rho_2^2} dW_{3t} \right)$$

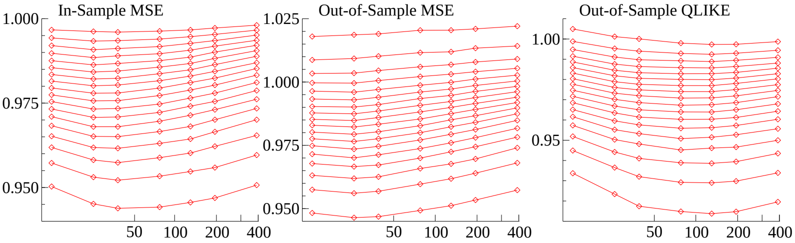
- Diurnal effects as in Andersen, Dobrev and Schaumburg (2012)
- Microstructure "noise" $P_t^* = P_t + u_t$, $u_t \sim N(0, 0.001 IV_t)$, as in Barndorff-Nielssen, Hansen, Lunde and Shephard (2008)
- Returns aggregated to 1-, 5- and 10-"minutes" $(1/\Delta = 390, 78, 39)$
- 1,000 replications

$$\text{MSE} \quad : \quad L(RV_t, X_t) = (RV_t - X_t)^2$$

$$\text{QLIKE} \quad : \quad L(RV_t, X_t) = \frac{RV_t}{X_t} - log\left(\frac{RV_t}{X_t}\right) - 1$$

	AR	HAR	ARQ	HARQ	HARQ-F
M			In-Sample MSE		
39	1.0291	1.0000	0.9980	0.9773	0.9718
78	1.0285	1.0000	0.9996	0.9791	0.9735
390	1.0277	1.0000	1.0064	0.9851	0.9793
			Out-of-Sample MSE		
39	1.0438	1.0000	1.0166	0.9878	0.9900
78	1.0425	1.0000	1.0188	0.9901	0.9920
390	1.0413	1.0000	1.0268	0.9968	0.9985
			Out-of-Sample QLIKE		
39	1.0893	1.0000	1.0258	0.9680	0.9850
78	1.0881	1.0000	1.0186	0.9644	0.9821
390	1.0841	1.0000	1.0187	0.9678	0.9859

HARQ/HAR loss ratios



$$\text{Persistence} \quad : \quad \beta_1 + \beta_2 + \beta_3$$

$$\text{Mean Lag} \quad : \quad 22\frac{\sum_{i=1}^{22} i \cdot b_i}{\sum_{i=1}^{22} b_i}$$

	AR	HAR	ARQ	HARQ	HARQ-F
			Persistence		
39	0.4303	0.6593	0.6552	0.8132	0.9200
78	0.4568	0.6736	0.6876	0.8328	0.9449
390	0.4739	0.6803	0.6913	0.8297	0.9621
			Mean Lag		
39		5.6598		4.2410	4.6956
78		5.4963		4.1026	4.5968
390		5.3685		4.1196	4.6530

Empirical Analysis

- S&P 500 futures, Tick Data Inc.
  - April 21, 1997 - August 30, 2013
- Dow Jones individual stocks (27), TAQ
  - April 21, 1997 - December, 2013
- Five-minute *RV* and *RQ*
  - Ameliorates the effect of market microstructure “noise”
  - Other *RV* and *RQ* estimators
- Out-of-sample forecast comparisons
  - Rolling estimation Windows (RW) (1,000 days)
  - Increasing estimation Windows (IW) (1,000-3,201 days)
  - RW and IW based on same samples

Company	Symbol	Min	Mean	Median	Max	AR	AR-IV	ARQ
S&P 500	SP500	0.043	1.175	0.629	60.563	0.651	0.953	0.983
Microsoft	MSFT	0.166	3.087	1.824	59.164	0.718	0.952	0.889
Coca-Cola	KO	0.049	2.011	1.154	54.883	0.618	0.949	0.834
DuPont	DD	0.093	3.327	2.165	81.721	0.707	0.950	0.956
ExxonMobil	XOM	0.114	2.348	1.476	130.667	0.668	0.947	0.997
General Electric	GE	0.131	3.440	1.794	173.223	0.681	0.915	0.987
IBM	IBM	0.115	2.464	1.340	72.789	0.657	0.959	0.890
Chevron	CVX	0.105	2.286	1.483	139.984	0.653	0.966	0.954
United Technologies	UTX	0.126	2.793	1.658	92.105	0.648	0.943	0.883
Procter & Gamble	PG	0.085	2.007	1.064	80.124	0.587	0.866	0.786
Caterpillar	CAT	0.207	3.810	2.401	127.119	0.727	0.954	0.896
Boeing	BA	0.167	3.371	2.147	79.760	0.630	0.936	0.822
Pfizer	PFE	0.176	2.822	1.809	60.302	0.570	0.933	0.837
Johnson & Johnson	JNJ	0.062	1.680	0.999	58.338	0.613	0.946	0.933
3M	MMM	0.140	2.278	1.358	123.197	0.495	0.952	0.748
Merck	MRK	0.127	2.758	1.718	223.723	0.372	0.913	0.708
Walt Disney	DIS	0.135	3.641	2.030	129.661	0.629	0.916	0.772
McDonald's	MCD	0.090	2.678	1.680	130.103	0.390	0.937	0.672
JPMorgan Chase	JPM	0.114	5.420	2.552	261.459	0.716	0.832	0.940
Wal-Mart	WMT	0.148	2.761	1.443	114.639	0.611	0.925	0.810
Nike	NKE	0.192	3.431	1.980	84.338	0.581	0.943	0.785
American Express	AXP	0.088	4.603	2.184	290.338	0.602	0.948	0.949
Intel	INTC	0.208	4.654	2.674	89.735	0.731	0.949	0.968
Travelers	TRV	0.098	3.579	1.637	273.579	0.646	0.933	0.915
Verizon	VZ	0.145	2.788	1.637	99.821	0.646	0.952	0.859
The Home Depot	HD	0.171	3.798	2.161	133.855	0.633	0.946	0.992
Cisco Systems	CSCO	0.234	5.120	2.742	96.212	0.715	0.939	0.942
UnitedHealth Group	UNH	0.222	4.145	2.304	169.815	0.616	0.920	0.846

In-Sample Estimates

	AR	HAR	AR-IV	ARQ	HARQ	HARQ-F
β_0	0.4109	0.1123		0.0892	-0.0098	-0.0187
	(0.1045)	(0.0615)		(0.0666)	(0.0617)	(0.0573)
β_1	0.6508	0.2273	0.9529	0.9830	0.6021	0.5725
	(0.1018)	(0.1104)	(0.0073)	(0.0782)	(0.0851)	(0.0775)
β_2		0.4903			0.3586	0.4368
		(0.1352)			(0.1284)	(0.1755)
β_3		0.1864			0.0976	0.0509
		(0.1100)			(0.1052)	(0.1447)
β_{1Q}				-0.5139	-0.3602	-0.3390
				(0.0708)	(0.0637)	(0.0730)
β_{2Q}						-0.1406
						(0.3301)
β_{3Q}						0.0856
						(0.3416)
R^2	0.4235	0.5224	0.3323	0.5263	0.5624	0.5628
MSE	3.1049	2.5722	3.5964	2.5512	2.3570	2.3546
QLIKE	0.2111	0.1438	0.1586	0.1530	0.1358	0.1380
$\overline{R}^2$ Stocks	0.3975	0.4852	0.2935	0.4676	0.5090	0.5139
$\overline{MSE}$ Stocks	17.4559	14.9845	20.0886	15.2782	14.1702	14.0154
$\overline{QLIKE}$ Stocks	0.2095	0.1496	0.1759	0.1804	0.1470	0.1547

Extended set of benchmark models

- HAR with Jumps (HAR-J) (Andersen, Bollerslev and Diebold, 2007):

$$RV_t = \beta_0 + \beta_1 RV_{t-1} + \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \beta_J J_{t-1} + \epsilon_t$$

- Continuous HAR (CHAR) (Andersen, Bollerslev and Diebold, 2007):

$$RV_t = \beta_0 + \beta_1 BPV_{t-1} + \beta_2 BPV_{t-1|t-5} + \beta_3 BPV_{t-1|t-22} + \epsilon_t$$

- Semi-variance HAR (SHAR) (Patton and Sheppard, 2015):

$$RV_t = \beta_0 + \beta_1^+ RV_{t-1}^+ + \beta_1^- RV_{t-1}^- + \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \epsilon_t$$

Out-of-Sample Forecast Comparison

			AR	HAR	HAR-J	CHAR	SHAR	ARQ	HARQ	HARQ-F
						S&P 500				
MSE	RW		0.9166	1.0000	0.9176	0.9583	0.8375	**0.8115**	0.8266	0.9750
	IW		1.2315	1.0000	0.9676	0.9707	0.9012	0.9587	**0.8944**	0.9312
QLIKE	RW		1.4559	1.0000	1.0062	1.0124	**0.9375**	0.9570	0.9464	0.9934
	IW		1.7216	1.0000	0.9716	0.9829	0.8718	1.1845	0.8809	**0.8686**
						Individual Stocks				
MSE	RW	Avg	1.1505	1.0000	1.0151	1.0080	1.0083	0.9659	**0.9349**	1.0149
		Med	1.1730	1.0000	1.0115	1.0158	1.0020	0.9864	**0.9418**	1.0263
	IW	Avg	1.2130	1.0000	1.0040	1.0013	0.9947	1.0371	**0.9525**	1.0071
		Med	1.2161	1.0000	1.0028	1.0010	0.9968	1.0396	**0.9525**	0.9660
QLIKE	RW	Avg	1.4204	1.0000	1.0018	0.9999	0.9902	1.1498	**0.9902**	1.1516
		Med	1.4044	1.0000	0.9976	1.0025	0.9941	1.1781	**0.9916**	1.1051
	IW	Avg	1.5803	1.0000	0.9930	1.0148	0.9829	1.2024	**0.9487**	0.9843
		Med	1.5565	1.0000	0.9959	1.0163	0.9887	1.1732	**0.9550**	0.9630

Weekly Forecast Horizon

			AR	HAR	HAR-J	CHAR	SHAR	ARQ	HARQ	HARQ-F	HARQ-h
						S&P500					
MSE	RW		1.1450	1.0000	1.4030	0.9919	**0.9018**	1.0798	0.9475	1.2138	0.8884
	IW		1.3509	1.0000	1.1549	0.9673	**0.8365**	1.0861	0.9031	0.9171	0.9232
QLIKE	RW		1.5589	1.0000	1.3047	1.0417	0.9350	1.1892	**0.9159**	1.2529	0.9491
	IW		1.8801	1.0000	1.0898	0.9870	0.8735	1.3717	0.8537	**0.7540**	0.7996
						Individual Stocks					
MSE	RW	Avg	1.2902	1.0000	1.0580	0.9960	0.9864	1.0985	0.9838	1.0234	**0.9765**
		Med	1.2859	1.0000	1.0504	0.9948	0.9904	1.1109	0.9806	1.0051	**0.9517**
	IW	Avg	1.4259	1.0000	1.0500	1.0003	0.9955	1.2126	0.9627	0.9601	**0.9477**
		Med	1.4322	1.0000	1.0435	1.0005	0.9922	1.2110	0.9596	0.9378	**0.9311**
QLIKE	RW	Aveg	1.6564	1.0000	1.1034	1.0124	0.9820	1.2111	**0.9309**	1.0665	0.9873
		Med	1.6554	1.0000	1.0980	1.0156	0.9827	1.2010	**0.9422**	1.0673	0.9869
	IW	Avg	1.9062	1.0000	1.0894	1.0279	0.9770	1.4147	**0.9066**	0.8529	0.8530
		Med	1.8762	1.0000	1.0721	1.0265	0.9781	1.4044	**0.9186**	0.8420	0.8579

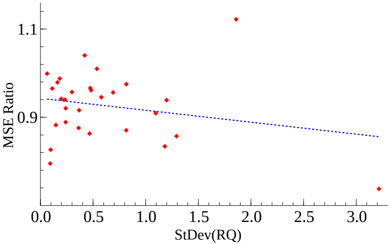
Monthly Forecast Horizon

			AR	HAR	HAR-J	CHAR	SHAR	ARQ	HARQ	HARQ-F	HARQ-h
						S&P500					
MSE	RW		1.1407	1.0000	0.9841	0.9642	**0.9558**	1.0964	1.0708	1.3485	1.2191
	IW		1.2411	1.0000	1.0312	1.0107	1.0119	1.1456	0.9667	**0.9339**	0.9832
QLIKE	RW		1.2455	1.0000	1.0552	0.9919	**0.9532**	1.0518	0.9808	1.1150	1.0450
	IW		1.4159	1.0000	1.0773	0.9937	0.9842	1.2144	0.9368	**0.8448**	0.8843
						Individual Stocks					
MSE	RW	Avg	1.2246	1.0000	1.0173	1.0159	0.9924	1.0969	0.9953	1.0198	**0.9756**
		Med	1.2613	1.0000	1.0118	1.0105	0.9949	1.1005	0.9965	0.9963	**0.9619**
	IW	Avg	1.4127	1.0000	1.0181	1.0123	0.9907	1.2366	0.9770	**0.9723**	0.9815
		Med	1.4052	1.0000	1.0172	1.0145	0.9927	1.2182	0.9692	**0.9480**	0.9705
QLIKE	RW	Avg	1.4125	1.0000	1.0385	1.0143	0.9909	1.1335	0.9485	0.9127	**0.8804**
		Med	1.4300	1.0000	1.0367	1.0125	0.9928	1.1228	0.9481	0.8778	**0.8635**
	IW	Avg	1.6612	1.0000	1.0360	1.0257	0.9885	1.3519	0.9371	**0.8185**	0.8278
		Med	1.6294	1.0000	1.0224	1.0296	0.9912	1.3619	0.9442	**0.8245**	0.8442

Persistence or Jumps

			AR	HAR	HAR-J	CHAR	SHAR	ARQ	HARQ	HARQ-F
						Bottom 95% RQ_t				
MSE	RW	Avg	1.2081	1.0000	0.9939	1.0044	0.9869	1.0324	**0.9557**	0.9928
		Med	1.1987	1.0000	0.9957	1.0034	0.9915	1.0459	**0.9693**	0.9802
	IW	Avg	1.2823	1.0000	0.9944	1.0053	0.9865	1.0814	**0.9633**	0.9581
		Med	1.2114	1.0000	0.9971	1.0067	0.9937	1.0675	**0.9697**	0.9684
QLIKE	RW	Avg	1.4379	1.0000	0.9962	1.0067	0.9888	1.1309	**0.9787**	1.1212
		Med	1.4399	1.0000	0.9958	1.0127	0.9934	1.1180	**0.9820**	1.0673
	IW	Avg	1.6204	1.0000	0.9979	1.0257	0.9787	1.1920	**0.9353**	0.9639
		Med	1.5941	1.0000	0.9995	1.0258	0.9833	1.1616	**0.9395**	0.9532
						Top 5% RQ_t				
MSE	RW	Avg	1.1426	1.0000	1.0228	1.0112	1.0163	0.9461	**0.9268**	1.0222
		Med	1.1591	1.0000	1.0212	1.0175	1.0073	0.9614	**0.9217**	1.0383
	IW	Avg	1.1933	1.0000	1.0053	0.9981	0.9983	1.0243	**0.9476**	1.0124
		Med	1.1984	1.0000	1.0052	1.0007	0.9979	1.0417	**0.9479**	0.9677
QLIKE	RW	Avg	1.3380	1.0000	1.0308	**0.9633**	1.0056	1.3161	1.0916	1.3535
		Med	1.3408	1.0000	0.9998	**0.9377**	1.0097	1.3052	1.0846	1.3250
	IW	Avg	1.3112	1.0000	0.9564	**0.9350**	1.0130	1.2864	1.0464	1.1301
		Med	1.3049	1.0000	0.9636	**0.9371**	1.0111	1.2186	1.0180	1.0045

HARQ/HAR Loss Ratios



- Previous HARQ results based on simple five-minute *RV*
  - Ameliorates the effect of market microstructure “noise”
- Many alternative “robust” estimators allowing for the use of higher frequency data
  - Subsampled five-minute *RV* (SS-RV) (Aït-Sahalia, Mykland and Zhang, 2005)
  - Two-scale *RV* (TS-RV) (Zhang, Mykland and Aït-Sahalia, 2005)
  - Realized kernel *RV* (RK) (Barndorff-Nielsen, Hansen, Lunde and Shephard, 2008)
  - Pre-averaged *RV* (PA-RV) (Jacod, Li, Mykland, Podolskij and Vetter, 2009)
- We implement these alternative estimators using one-minute returns

- HAR based on alternative one-minute *RVs* versus HARQ based on simple five-minute *RV*:

			RV	SS-RV	TS-RV	RK	PA-RV
					S&P 500		
MSE	RW		1.2574	1.0801	1.3472	1.3443	1.3521
	IW		1.1290	1.1882	1.2468	1.1769	1.1604
QLIKE	RW		1.0025	1.0149	1.1476	1.0493	1.0331
	IW		1.1487	1.1424	1.2637	1.4182	1.3608
					Individual Stocks		
MSE	RW	Ave	1.0714	1.0523	1.1641	1.0446	1.0476
		Med	1.0628	1.0430	1.1665	1.0604	1.0542
	IW	Ave	1.0531	1.0502	1.1319	1.0603	1.0579
		Med	1.0499	1.0544	1.1206	1.0695	1.0479
QLIKE	RW	Ave	1.0100	1.0438	1.0890	1.0615	1.1107
		Med	1.0064	1.0457	1.0923	1.0587	1.0988
	IW	Ave	1.0552	1.0535	1.1297	1.1557	1.1425
		Med	1.0471	1.0491	1.1240	1.1520	1.1458

- HARQ versus HAR based on same *RV*s:

			RV	SS-RV	TS-RV	RK	PA-RV
					S&P 500		
MSE	RW		0.7953	1.0059	0.8606	0.7873	0.7896
	IW		0.8857	0.8837	0.9749	0.8956	0.8952
QLIKE	RW		0.9975	1.0585	0.9776	0.9711	1.0504
	IW		0.8705	0.9195	0.9317	0.8971	0.9262
					Individual Stocks		
MSE	RW	Ave	0.9333	0.9496	0.9547	1.0072	0.9698
		Med	0.9409	0.9593	0.9521	1.0012	0.9719
	IW	Ave	0.9496	0.9582	0.9723	0.9755	0.9700
		Med	0.9525	0.9590	0.9730	0.9710	0.9771
QLIKE	RW	Ave	0.9901	0.9462	0.9804	1.0874	0.9748
		Med	0.9936	0.9600	0.9831	0.9973	0.9793
	IW	Ave	0.9477	0.9474	0.9665	0.9492	0.9443
		Med	0.9550	0.9447	0.9624	0.9431	0.9445

- Previous HARQ results based on simple five-minute *RQ*
  - *IQ* is difficult to accurately estimate
- Many alternative *IQ* estimators available
  - Tri-Power Quarticity (TPQ) (Barndorff-Nielsen and Shephard, 2006)
  - Median *RQ* (MedRQ) (Andersen, Dobrev and Schaumburg, 2012)
  - Truncated *RQ* (TrRQ) (Mancini, 2009)
  - Coarser sampled *RQ* (RQ_{15min}) (Bandi and Russell, 2008)
  - Bootstrapped variance of *RV* (Bootstrap) (Gonçalves and Meddahi, 2009)
- We implement the HARQ model with these alternative *RQ* estimators

- HARQ based on alternative *RQ* estimators versus HARQ based on five-minute *RQ*:

IQ-estimator			RQ	TPQ	MedRQ	TrRQ	RQ_{15min}	Bootstrap
					S&P500			
MSE	RW		1.0000	1.0497	1.0254	1.0208	1.0590	**0.9925**
	IW		1.0000	1.1635	1.0328	**0.9948**	0.9805	0.9981
QLIKE	RW		1.0000	1.0933	1.1227	0.9971	1.0231	**0.9933**
	IW		1.0000	**0.9814**	1.0548	1.2060	1.0414	0.9998
					Average and Median Ratio across stocks			
MSE	RW	Avg	1.0000	1.0403	1.0139	1.0531	1.0586	**0.9936**
		Med	**1.0000**	1.0497	1.0201	1.0378	1.0238	1.0003
	IW	Avg	1.0000	1.0211	1.0191	1.0491	1.0229	**0.9994**
		Med	1.0000	1.0220	1.0259	1.0558	1.0125	**0.9998**
QLIKE	RW	Avg	1.0000	1.0040	1.0089	1.0541	1.0254	**0.9995**
		Med	1.0000	0.9980	**0.9968**	1.0376	1.0015	0.9992
	IW	Avg	**1.0000**	1.0050	1.0047	1.0390	1.0014	1.0001
		Med	**1.0000**	1.0040	1.0065	1.0303	0.9989	1.0001

- The simple and easy-to-implement HARQ model relies on $RQ^{1/2}$ to adjust for the time-varying attenuation bias:

$$RV_t = \beta_0 + \underbrace{(\beta_1 + \beta_{1Q} RQ_{t-1}^{1/2})}_{\beta_{1,t}} RV_{t-1} + \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \epsilon_t$$

  - AR(1) suggests using nonlinear function of RQ^{-1}
  - Built-in robustness
  - Taylor series approximation

- We implement the HARQ model with different functions of RQ in place of $RQ^{1/2}$
  - RQ, $RQ^{-1/2}$, RQ^{-1}, $log(RQ)$
  - Higher-order terms

- Different HARQ specifications versus HARQ based on $RQ^{1/2}$:

			RQ_t	$RQ_t^{1/2}$	$RQ_t^{-1/2}$	RQ_t^{-1}	$Log(RQ_t)$
				S&P500			
MSE	RW		1.0037	**1.0000**	1.2123	1.2334	1.3313
	IW		1.0344	**1.0000**	1.1166	1.1357	1.0736
QLIKE	RW		**0.9484**	1.0000	1.0952	1.0950	1.8104
	IW		1.0222	**1.0000**	1.1327	1.3217	2.0107
				Average and Median across stocks			
MSE	RW	Avg	1.0108	**1.0000**	1.0808	1.0931	1.0329
		Med	1.0112	**1.0000**	1.0577	1.0664	1.0336
	IW	Avg	1.0189	**1.0000**	1.0495	1.0644	1.0143
		Med	1.0198	**1.0000**	1.0403	1.0598	1.0082
QLIKE	RW	Avg	**0.9973**	1.0000	1.0678	1.0814	1.3723
		Med	**0.9847**	1.0000	1.0458	1.0579	1.3324
	IW	Avg	1.0263	**1.0000**	1.0961	1.1155	1.2903
		Med	1.0241	**1.0000**	1.0778	1.0886	1.2084

- All of the benchmark models may be similarly adjusted for measurement errors

  - HARQ-J:

$$\begin{aligned} RV_t = \beta_0 &+ (\beta_1 + \beta_{1Q}\sqrt{RQ_{t-1}})RV_{t-1} \\ &+ \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \beta_J J_{t-1} + \epsilon_t \end{aligned}$$

  - CHARQ:

$$\begin{aligned} RV_t = \beta_0 &+ (\beta_1 + \beta_{1Q}\sqrt{TPQ_{t-1}})BPV_{t-1} \\ &+ \beta_2 BPV_{t-1|t-5} + \beta_3 BPV_{t-1|t-22} + \epsilon_t \end{aligned}$$

  - SHARQ:

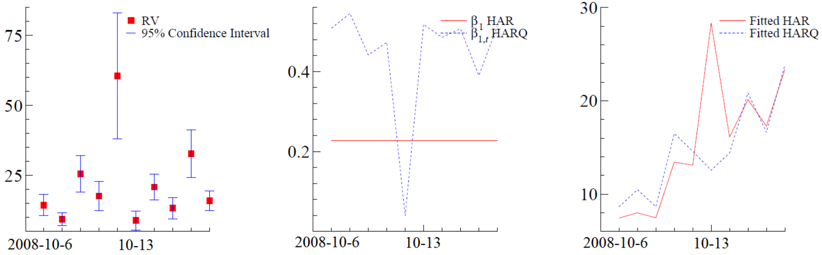
$$\begin{aligned} RV_t = \beta_0 &+ (\beta_1^+ + \beta_{1Q}^+\sqrt{RQ_{t-1}})RV_{t-1}^+ \\ &+ (\beta_1^- + \beta_{1Q}^-\sqrt{RQ_{t-1}})RV_{t-1}^- \\ &+ \beta_2 RV_{t-1|t-5} + \beta_3 RV_{t-1|t-22} + \epsilon_t. \end{aligned}$$

- HARQ-versions versus HAR-versions:

			HARQ	HARQ-J	CHARQ	SHARQ
			S&P500			
MSE	RW		0.8266	0.9243	0.8951	1.4412
	IW		0.8944	0.9335	1.0609	1.1027
QLIKE	RW		1.0168	0.9653	1.0235	1.4576
	IW		0.8809	0.9015	0.8825	1.2849
			Average and Median Ratio across stocks			
MSE	RW	Avg	0.9349	0.9397	0.9525	1.1308
		Med	0.9418	0.9513	0.9539	1.0840
	IW	Avg	0.9525	0.9666	0.9451	1.0870
		Med	0.9525	0.9662	0.9548	1.0554
QLIKE	RW	Avg	0.9902	0.9902	0.9879	1.1706
		Med	0.9916	0.9952	0.9900	1.1609
	IW	Avg	0.9487	0.9548	0.9306	1.1305
		Med	0.9550	0.9594	0.9277	1.1154

- We propose a class of realized-based volatility forecasting models that explicitly incorporate the heteroskedasticity of the measurement errors into the forecasts
    - Higher/lower weights on lagged *RV*s when estimated precisely/imprecisely
    - The HARQ model is particularly easy to estimate and implement and performs well empirically
- Same idea readily extends to other realized volatility based models and forecasting procedures
    - Currently working on multivariate vech-HARQ, HEAVY-Q and Realized GARCH-Q extensions
- Same idea may be applied to the forecasts of other economic time series subject to measurement errors
    - Currently working on surveys of professional forecasters

Thank You!


RV
95% Confidence Interval
75
50
25
2008-10-6
10-13
β_1 HAR
$\beta_{1,t}$ HARQ
0.4
0.2
2008-10-6
10-13
Fitted HAR
Fitted HARQ
30
20
10
2008-10-6
10-13