Supplemental Appendix for:

Time-Varying Systemic Risk: Evidence from a Dynamic Copula Model of CDS Spreads

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A.1: Empirical implementation details

A.1.1: Obtaining a factor copula likelihood

The factor copula introduced in Oh and Patton (2015) does not have a likelihood in closed form, but it is relatively simple to obtain the likelihood using numerical integration. Consider the factor structure in equations (5) and (6) of the main paper. Our objective is to obtain the copula density of $\mathbf{X}_t$:

$$\mathbf{c}_t\left(u_1,...,u_N\right)=\frac{\mathbf{g}_t\left(G_{1t}^{-1}\left(u_1\right),...,G_{Nt}^{-1}\left(u_N\right)\right)}{g_{1t}\left(G_{1t}^{-1}\left(u_1\right)\right)\cdot\cdots\cdot g_{Nt}\left(G_{Nt}^{-1}\left(u_N\right)\right)} \tag{1}$$

where $\mathbf{g}_t\left(x_1,...,x_N\right)$ is the joint density of $\mathbf{X}_t$, $g_{it}\left(x_i\right)$ is the marginal density of X_{it}, and $\mathbf{c}_t\left(u_1,...,u_N\right)$ is the copula density. To construct the copula density, we need each of the functions that appear on the right-hand side above: $g_{it}\left(x_i\right)$, $G_{it}\left(x_i\right)$, $\mathbf{g}_t\left(x_1,...,x_N\right)$ and $G_{it}^{-1}\left(u_i\right)$.

The independence of Z and ε_i implies that:

$$\begin{aligned}
f_{X_i|Z,t}\left(x_i|z\right) &= f_{\varepsilon_i}\left(x_i-\lambda_{it}z\right)\\
F_{X_i|Z,t}\left(x_i|z\right) &= F_{\varepsilon_i}\left(x_i-\lambda_{it}z\right)\\
\mathbf{g}_{\mathbf{X}|Z,t}\left(x_{1,...,}x_N|z\right) &= \prod_{i=1}^{N} f_{\varepsilon_i}\left(x_i-\lambda_{it}z\right)
\end{aligned} \tag{2}$$

With these conditional distributions, one dimensional integration gives the marginals:

$$g_{it}\left(x_i\right)=\int_{-\infty}^{\infty}f_{X_i,Z,t}\left(x_i,z\right)dz=\int_{-\infty}^{\infty}f_{X_i|Z,t}\left(x_i|z\right)dF_{Zt}\left(z\right)=\int_{-\infty}^{\infty}f_{\varepsilon_i}\left(x_i-\lambda_{it}z\right)dF_{Zt}\left(z\right) \tag{3}$$

and similarly

$$
\begin{aligned}
G_{it}(x_i) &= \int_{-\infty}^{\infty} F_{\varepsilon_i}(x_i - \lambda_{it} z)\, dF_{Zt}(z) \\
\mathbf{g}_t(x_1, ..., x_N) &= \int_{-\infty}^{\infty} \prod_{i=1}^{N} f_{\varepsilon_i}(x_i - \lambda_{it} z)\, dF_{Zt}(z)
\end{aligned} \tag{4}
$$

We use a change of variables, $u \equiv F_{Zt}(z)$, to convert these to bounded integrals:

$$
\begin{aligned}
g_{it}(x_i) &= \int_0^1 f_{\varepsilon_i}\left(x_i - \lambda_{it} F_{Zt}^{-1}(u)\right) du \\
G_{it}(x_i) &= \int_0^1 F_{\varepsilon_i}\left(x_i - \lambda_{it} F_{Zt}^{-1}(u)\right) du \\
\mathbf{g}_t(x_1, ..., x_N) &= \int_0^1 \prod_{i=1}^{N} f_{\varepsilon_i}\left(x_i - \lambda_{it} F_{Zt}^{-1}(u)\right) du
\end{aligned} \tag{5}
$$

Thus the factor copula density requires the computation of just one-dimensional integrals. (For a factor copula with J common factors the integral would be J-dimensional.) We use Gauss-Legendre quadrature for the integration, using Q "nodes," (see Judd (1998) for details) and we choose Q on the basis of a small simulation study described below.

Finally, we need a method to invert $G_{it}(x_i)$, and note from above that this is a function of both x and the factor loading λ_{it}, with $G_{it} = G_{js}$ if $\lambda_{it} = \lambda_{js}$. We estimate the inverse of G_{it} by creating a grid of 100 points for x in the interval $[x_{\min}, x_{\max}]$ and 50 points for λ in the interval $[\lambda_{\min}, \lambda_{\max}]$, and then evaluating G at each of those points. We then use two-dimensional linear interpolation to obtain $G^{-1}(u; \lambda)$ given u and λ. This two-dimensional approximation substantially reduces the computational burden, especially when λ is time-varying, as we can evaluate the function G prior to estimation, rather than re-estimating it for each likelihood evaluation.

We conducted a small Monte Carlo simulation to evaluate the accuracy of the numerical approximations for G and G^{-1}. We use quadrature nodes $Q \in \{10, 50, 150\}$ and $[x_{\min}, x_{\max}] = [-30, 30]$, $[\lambda_{\min}, \lambda_{\max}] = [0, 6]$ for the numerical inversion. For this simulation, we considered the factor copula implied by the following structure:

$$
\begin{aligned}
X_i &= \lambda_0 Z + \varepsilon_i, \;\; i = 1, 2 \\
\text{where} \quad Z &\sim \mathit{Skew}\ t\ (\nu_0, \psi_0), \;\; \varepsilon_i \sim iid\ t(\nu_0), \;\; Z \perp\!\!\!\perp \varepsilon_i \ \forall\, i
\end{aligned} \tag{6}
$$

where $\lambda_0 = 1$, $\nu_0^{-1} = 0.25$ and $\psi_0 = -0.5$. At each replication, we simulate $\mathbf{X} = [X_1, X_2]$ 1000 times, and apply the empirical distribution function to transform $\mathbf{X}$ to $\mathbf{U} = [U_1, U_2]$. With this $[U_1, U_2]$ we estimate $[\lambda, \nu^{-1}, \psi]$ by our numerically approximated maximum likelihood method.

Table A3 contains estimation results for 100 replications. We find that estimation with only 10 nodes introduces a relatively large bias, in particular for ν^{-1}, consistent with this low number of nodes providing a poor approximation of the tails of this density. Estimation with 50 nodes gives accurate results, and is comparable to those with 150 nodes in that bias and standard deviation are small.

Table A4 studies the Normal factor copula (which can be obtained from the previous equation when $\nu_0 \to \infty$ and $\psi_0 = 0$) for which we have a closed-form likelihood. We compare estimating λ based on numerical integration with 10, 50 and 150 nodes against using the closed-form likelihood (corresponding to ∞ nodes). We again see that using only 10 nodes leads to a decrease in accuracy, while 50 and 150 nodes give approximately equally-accurate results, that are just slightly worse than using the analytical likelihood. We thus use 50 nodes throughout the paper.

A.1.2: "Variance targeting" assumptions

We investigate the plausibility of Assumptions 1(c) and 1(d) via simulations. We use 50,000 observations to estimate the true (unknown) mappings. Note that while the copula is time-varying, we only need to study the mapping for a given day (set of shape parameters), and so we do not consider dynamics in this simulation study. Moreover, the mappings are pair-wise, and so we need only consider the (i, j) bivariate case. We consider three different factor copulas: (i) $\nu_z = \nu_\varepsilon = 5, \psi = 0.1$ (ii) $\nu_z = \nu_\varepsilon = 4, \psi = 0.25$ (iii) $\nu_z = \nu_\varepsilon = \infty, \psi = 0$ (corresponding to the Normal copula). We fix $\lambda_i = \lambda_j = \lambda$, and let λ vary so that the model-implied rank correlation ranges from 0.1 to 0.7, which covers the range of pair-wise rank correlations we observe in our data.

The results are summarized in Figure A1. The left panel of this figure reveals that the mapping from rank correlation to linear correlation changes only slightly with the shape parameters $(\nu_z, \nu_\varepsilon, \psi)$, and in all cases the function is strictly increasing, supporting Assumption 1(d). In fact, we observe that for all cases the function is close to being the identity function, and we invoke this approximation in our estimation to increase computational speed. The right panel of the figure plots the mapping from rank correlation to log factor loadings, and shows that the true mapping is reasonably approximated by a straight line, particularly for values of rank correlation near the sample average rank correlation in our application, which is around 0.4, supporting Assumption 1(c). Violations of either of these assumptions, if large relative to sampling variability and other sources of estimation error, would manifest in poor performance of the estimation of the "heterogeneous dependence" model. As discussed in Section 3, the simulation results in Tables 1 and A1 provide evidence that this estimation has good finite-sample properties.

References

[1] Judd, K.L, 1998, *Numerical Methods in Economics*, MIT Press, Cambridge, USA.

[2] Oh, D.H. and A.J. Patton, 2015, Modelling Dependence in High Dimensions with Factor Copulas, *Journal of Business and Economic Statistics*, forthcoming.

[3] Nelsen, R.B., 2006, *An Introduction to Copulas*, Second Edition, Springer, U.S.A.

A.2: Additional tables and figures

Table A1: Simulation results for the "heterogeneous dependence" model

	True	Bias	Std	Median	90%	10%	Diff (90%-10%)
ω_1	-0.030	0.004	0.017	-0.022	-0.005	-0.052	0.047
ω_2	-0.029	0.004	0.018	-0.022	-0.005	-0.048	0.043
ω_3	-0.029	0.004	0.016	-0.021	-0.005	-0.043	0.038
ω_4	-0.028	0.003	0.017	-0.023	-0.005	-0.047	0.042
ω_5	-0.028	0.004	0.016	-0.020	-0.005	-0.046	0.041
ω_6	-0.027	0.004	0.016	-0.020	-0.003	-0.044	0.040
ω_7	-0.026	0.003	0.016	-0.022	-0.004	-0.042	0.038
ω_8	-0.026	0.003	0.016	-0.020	-0.005	-0.043	0.038
ω_9	-0.025	0.003	0.015	-0.019	-0.005	-0.041	0.036
ω_{10}	-0.025	0.002	0.016	-0.019	-0.005	-0.041	0.036
ω_{11}	-0.024	0.002	0.015	-0.018	-0.004	-0.038	0.033
ω_{12}	-0.023	0.003	0.013	-0.018	-0.004	-0.037	0.032
ω_{13}	-0.023	0.003	0.014	-0.018	-0.004	-0.038	0.033
ω_{14}	-0.022	0.003	0.012	-0.018	-0.004	-0.035	0.031
ω_{15}	-0.022	0.002	0.013	-0.019	-0.004	-0.043	0.039
ω_{16}	-0.021	0.002	0.013	-0.016	-0.003	-0.034	0.031
ω_{17}	-0.020	0.003	0.011	-0.015	-0.003	-0.032	0.029
ω_{18}	-0.020	0.002	0.013	-0.015	-0.003	-0.033	0.030
ω_{19}	-0.019	0.002	0.012	-0.016	-0.003	-0.031	0.028
ω_{20}	-0.019	0.002	0.011	-0.015	-0.003	-0.033	0.030
ω_{21}	-0.018	0.003	0.010	-0.013	-0.003	-0.028	0.025
ω_{22}	-0.017	0.002	0.010	-0.013	-0.003	-0.028	0.025
ω_{23}	-0.017	0.003	0.009	-0.013	-0.003	-0.025	0.022
ω_{24}	-0.016	0.002	0.010	-0.013	-0.003	-0.024	0.021
ω_{25}	-0.016	0.000	0.010	-0.014	-0.003	-0.030	0.027

Table A1: Simulation results for the "heterogeneous dependence" model (continued)

	True	Bias	Std	Median	90%	10%	Diff (90%-10%)
ω_{26}	-0.015	0.001	0.010	-0.012	-0.003	-0.028	0.025
ω_{27}	-0.014	0.000	0.011	-0.011	-0.003	-0.028	0.025
ω_{28}	-0.014	0.001	0.009	-0.011	-0.002	-0.023	0.021
ω_{29}	-0.013	0.000	0.009	-0.011	-0.002	-0.025	0.023
ω_{30}	-0.012	0.001	0.008	-0.010	-0.002	-0.022	0.020
ω_{31}	-0.012	0.000	0.008	-0.010	-0.002	-0.022	0.020
ω_{32}	-0.011	0.001	0.008	-0.008	-0.002	-0.019	0.017
ω_{33}	-0.011	0.001	0.007	-0.009	-0.002	-0.017	0.015
ω_{34}	-0.010	-0.001	0.008	-0.009	-0.002	-0.021	0.019
ω_{35}	-0.009	0.000	0.008	-0.008	-0.002	-0.020	0.018
ω_{36}	-0.009	0.000	0.007	-0.008	-0.002	-0.018	0.016
ω_{37}	-0.008	0.001	0.005	-0.006	-0.001	-0.014	0.013
ω_{38}	-0.008	0.001	0.006	-0.005	-0.001	-0.016	0.015
ω_{39}	-0.007	0.001	0.005	-0.005	-0.001	-0.013	0.012
ω_{40}	-0.006	-0.001	0.005	-0.006	-0.002	-0.015	0.014
ω_{41}	-0.006	-0.003	0.007	-0.007	-0.002	-0.019	0.017
ω_{42}	-0.005	0.000	0.004	-0.005	-0.001	-0.010	0.009
ω_{43}	-0.005	0.001	0.004	-0.003	0.000	-0.009	0.008
ω_{44}	-0.004	0.000	0.004	-0.003	0.000	-0.010	0.010
ω_{45}	-0.003	-0.001	0.005	-0.003	0.000	-0.010	0.010
ω_{46}	-0.003	0.001	0.003	-0.002	0.002	-0.007	0.008
ω_{47}	-0.002	-0.001	0.003	-0.002	0.000	-0.006	0.006
ω_{48}	-0.002	-0.001	0.003	-0.001	0.001	-0.006	0.008
ω_{49}	-0.001	-0.001	0.004	-0.001	0.002	-0.006	0.008
ω_{50}	0.000	-0.002	0.004	-0.002	0.001	-0.007	0.008
ω_{51}	0.000	-0.002	0.003	-0.001	0.001	-0.006	0.007
ω_{52}	0.001	-0.001	0.004	0.000	0.004	-0.003	0.007
ω_{53}	0.002	0.000	0.004	0.000	0.006	-0.003	0.009
ω_{54}	0.002	-0.003	0.004	-0.001	0.002	-0.005	0.007
ω_{55}	0.003	-0.001	0.004	0.001	0.007	-0.003	0.010
ω_{56}	0.003	-0.002	0.003	0.001	0.007	-0.002	0.009
ω_{57}	0.004	0.000	0.004	0.003	0.010	0.000	0.010
ω_{58}	0.005	-0.002	0.004	0.001	0.008	-0.001	0.009
ω_{59}	0.005	-0.002	0.003	0.003	0.008	0.000	0.008
ω_{60}	0.006	-0.002	0.005	0.003	0.010	0.000	0.010

Table A1: Simulation results for the "heterogeneous dependence" model (continued)

	True	Bias	Std	Median	90%	10%	Diff (90%-10%)
ω_{61}	0.006	-0.002	0.005	0.003	0.010	0.000	0.010
ω_{62}	0.007	-0.001	0.005	0.004	0.013	0.001	0.012
ω_{63}	0.008	-0.003	0.005	0.003	0.012	0.000	0.011
ω_{64}	0.008	-0.002	0.005	0.004	0.013	0.001	0.012
ω_{65}	0.009	-0.002	0.006	0.005	0.013	0.000	0.013
ω_{66}	0.009	-0.003	0.005	0.006	0.014	0.001	0.013
ω_{67}	0.010	-0.004	0.006	0.005	0.013	0.000	0.013
ω_{68}	0.011	-0.004	0.006	0.005	0.013	0.001	0.013
ω_{69}	0.011	-0.002	0.007	0.008	0.022	0.002	0.020
ω_{70}	0.012	-0.004	0.007	0.007	0.017	0.001	0.016
ω_{71}	0.012	-0.003	0.007	0.009	0.019	0.001	0.017
ω_{72}	0.013	-0.005	0.007	0.007	0.016	0.001	0.015
ω_{73}	0.014	-0.004	0.008	0.008	0.020	0.001	0.019
ω_{74}	0.014	-0.004	0.009	0.008	0.023	0.002	0.021
ω_{75}	0.015	-0.004	0.008	0.009	0.019	0.002	0.017
ω_{76}	0.016	-0.005	0.009	0.008	0.025	0.002	0.023
ω_{77}	0.016	-0.003	0.009	0.011	0.026	0.002	0.024
ω_{78}	0.017	-0.004	0.009	0.010	0.024	0.002	0.022
ω_{79}	0.017	-0.004	0.010	0.011	0.032	0.002	0.030
ω_{80}	0.018	-0.004	0.009	0.012	0.026	0.002	0.024
ω_{81}	0.019	-0.006	0.009	0.011	0.024	0.002	0.022
ω_{82}	0.019	-0.005	0.010	0.012	0.026	0.003	0.024
ω_{83}	0.020	-0.005	0.010	0.012	0.029	0.002	0.027
ω_{84}	0.020	-0.004	0.012	0.013	0.033	0.004	0.030
ω_{85}	0.021	-0.006	0.011	0.014	0.032	0.002	0.030
ω_{86}	0.022	-0.006	0.011	0.014	0.029	0.003	0.026
ω_{87}	0.022	-0.006	0.013	0.015	0.032	0.003	0.029
ω_{88}	0.023	-0.006	0.011	0.014	0.032	0.004	0.028
ω_{89}	0.023	-0.006	0.012	0.016	0.033	0.003	0.030
ω_{90}	0.024	-0.006	0.012	0.016	0.036	0.003	0.033
ω_{91}	0.025	-0.005	0.014	0.017	0.036	0.004	0.033
ω_{92}	0.025	-0.005	0.014	0.018	0.039	0.003	0.036
ω_{93}	0.026	-0.007	0.012	0.018	0.038	0.003	0.035
ω_{94}	0.026	-0.006	0.015	0.018	0.040	0.004	0.036
ω_{95}	0.027	-0.006	0.014	0.018	0.040	0.004	0.036

Table A1: Simulation results for the "heterogeneous dependence" model (continued)

	True	Bias	Std	Median	90%	10%	Diff (90%-10%)
ω_{96}	0.028	-0.007	0.015	0.019	0.042	0.004	0.038
ω_{97}	0.028	-0.006	0.015	0.019	0.042	0.004	0.038
ω_{98}	0.029	-0.006	0.015	0.020	0.045	0.004	0.041
ω_{99}	0.029	-0.008	0.013	0.020	0.038	0.004	0.034
ω_{100}	0.030	-0.007	0.016	0.021	0.040	0.004	0.036
α	0.050	-0.006	0.015	0.045	0.062	0.023	0.039
β	0.980	0.002	0.012	0.983	0.997	0.966	0.031
ν^{-1}	0.200	-0.002	0.009	0.199	0.209	0.186	0.023
ψ_z	0.100	0.008	0.032	0.111	0.152	0.064	0.088

Notes: This table presents results from the simulation study described in Section 3 of the paper, complementing those in Table 1 of the paper.

Table A2: Static copula model estimation results

	Equidep				Block equidep				Heterog			
	Normal		Factor		Normal		Factor		Normal		Factor	
	Pre	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*	*Pre*	*Post*
$\lambda_{1 \to G}$	0.326	0.817	0.336	0.802	–	–	–	–	–	–	–	–
	(0.126)	(0.099)	(0.118)	(0.098)								
ν_z^{-1}	–	–	0.108	0.009	–	–	0.097	0.011	–	–	0.111	0.008
			(0.027)	(0.025)			(0.028)	(0.019)			(0.034)	(0.019)
ν_ϵ^{-1}	–	–	0.214	0.205	–	–	0.198	0.213	–	–	0.191	0.187
			(0.015)	(0.010)			(0.014)	(0.009)			(0.013)	(0.009)
ψ_z	–	–	0.055	0.080	–	–	0.085	0.092	–	–	0.105	0.114
			(0.062)	(0.056)			(0.066)	(0.060)			(0.063)	(0.061)
$\log \mathcal{L}$	36,185		39,508		36,477		39,757		37,652		40,628	
AIC	-72,366		-79,000		-72,934		-79,482		-74,904		-80,844	
BIC	-72,364		-78,990		-72,922		-79,463		-74,661		-80,594	

Notes: This table presents parameter estimates for two versions of the factor copula (Normal and Skew t-t), each with one of three degrees of heterogeneity of dependence (equidependence, block equidependence, and heterogeneous dependence). All models are imposed to be constant through time, and so the GAS parameters are fixed at zero. Standard errors based on the stationary bootstrap of Politis and Romano (1994) are presented below the estimated parameters. All models are allowed to have a structural break on April 8, 2009 (see Section 4.4 of the paper), and we denote parameters from the first and second sub-samples as "Pre" and "Post." The log-likelihood at the estimated parameters and the Akaike and Bayesian Information criteria are presented in the bottom three rows. The factor loadings (λ_i) for the block equidependence and heterogeneous dependence models are not reported to conserve space.

Table A3: Simulation results for MLE with different numbers of quadrature nodes, for the skew t-t factor copula

	10 nodes			50 nodes			150 nodes		
	λ	ν^{-1}	ψ	λ	ν^{-1}	ψ	λ	ν^{-1}	ψ
True	1.000	0.250	-0.500	1.000	0.250	-0.500	1.000	0.250	-0.500
Bias	0.039	-0.062	-0.072	0.023	-0.018	-0.039	0.026	-0.002	-0.026
Std	0.126	0.038	0.203	0.144	0.049	0.168	0.135	0.045	0.144
Median	1.021	0.193	-0.532	1.014	0.236	-0.510	1.027	0.251	-0.500
90% quant	1.204	0.239	-0.341	1.208	0.293	-0.339	1.187	0.305	-0.349
10% quant	0.878	0.136	-0.990	0.846	0.163	-0.751	0.856	0.188	-0.704
diff (90%-10%)	0.326	0.103	0.649	0.362	0.130	0.412	0.331	0.118	0.355

Notes: This table presents the results from the simulation study described in Appendix A.1.

Table A4: Simulation results for MLE with different numbers of quadrature nodes, for the Normal factor copula

	10 nodes	50 nodes	150 nodes	∞ nodes
	λ	λ	λ	λ
True	1.000	1.000	1.000	1.000
Bias	0.0065	0.0051	0.0051	0.0049
Std	0.0476	0.0473	0.0473	0.0472
Median	1.0056	1.0040	1.0042	1.0038
90% quant	1.0704	1.0694	1.0698	1.0677
10% quant	0.9453	0.9430	0.9436	0.9430
diff (90%-10%)	0.1252	0.1264	0.1262	0.1248

Notes: This table presents the results from the simulation study described in Appendix A.1.

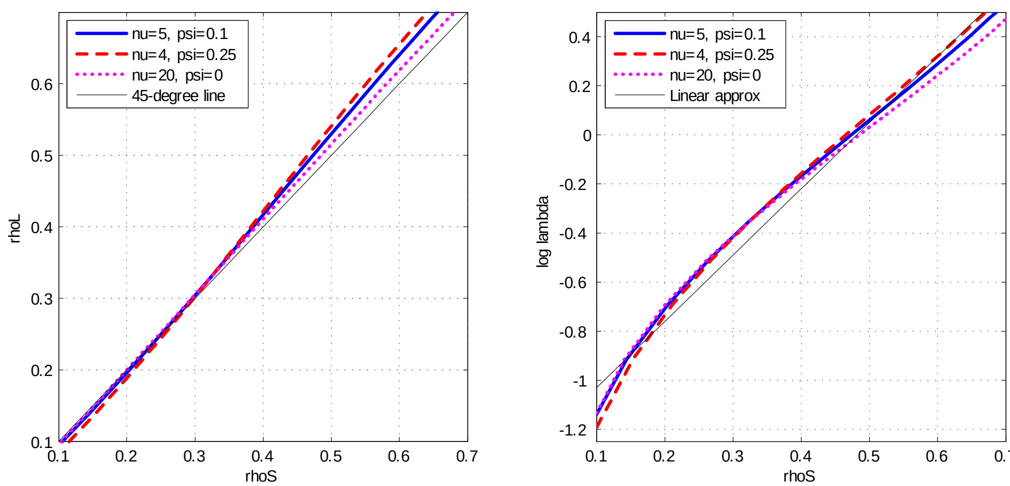

Figure A1: *The left panel plots the mapping from rank correlation to linear correlation for various choices of shape parameters in the factor copula. The right panel compares the true mappings from rank correlation to log-lambda with a linear approximation.*

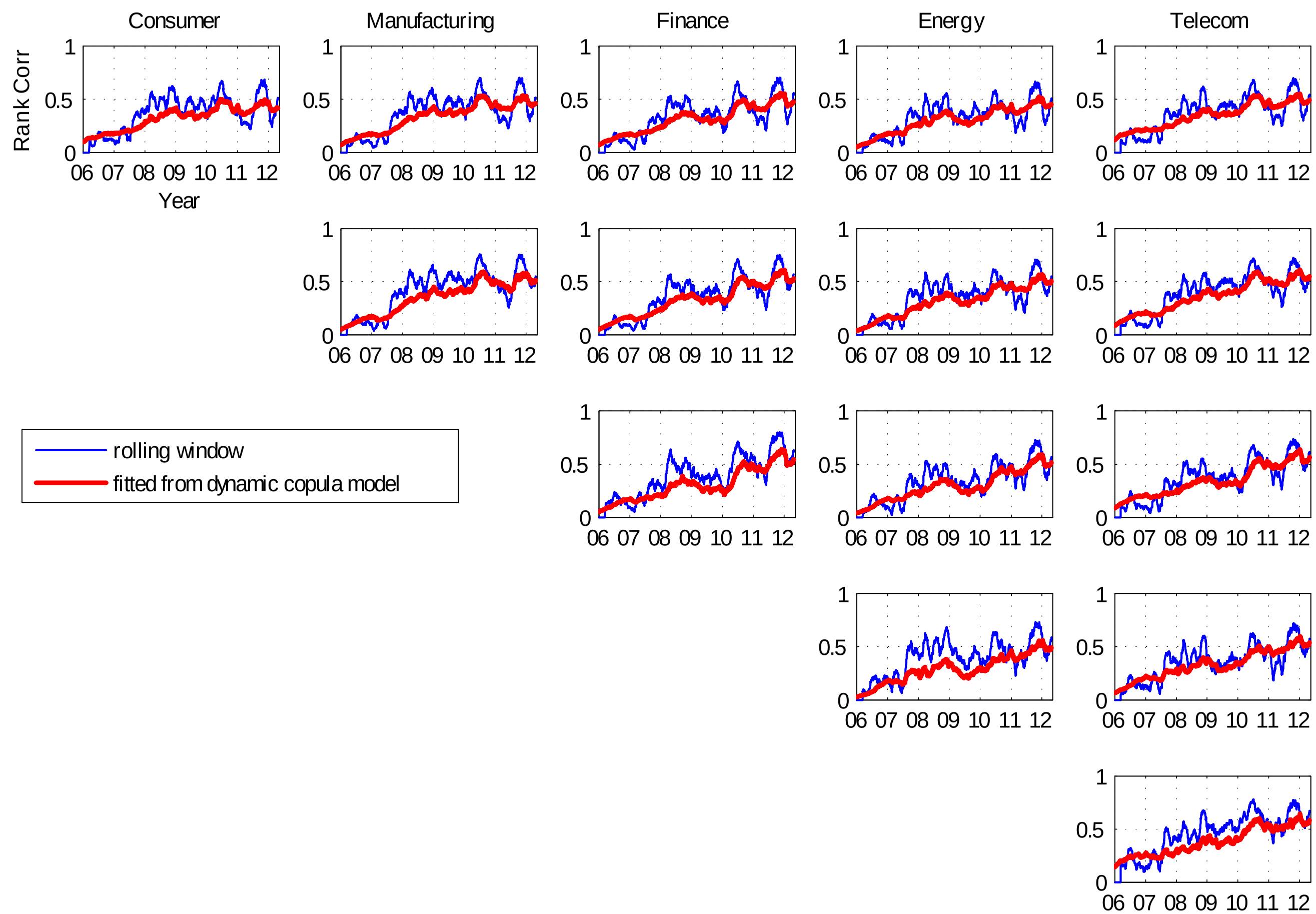

Figure A2: *This figure plots the time-varying rank correlations implied by the heterogeneous dependence factor copula, as well as 60-day rolling window rank correlations. Each element of this figure is an average of all pair-wise correlations for firms in the same pair of industries. The diagonal elements present rank correlations between firms in the same industry; the off-diagonal elements for firms in different industries.*