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Modeling Dependence in High Dimensions With Factor Copulas

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This article presents flexible new models for the dependence structure, or copula, of economic variables based on a latent factor structure. The proposed models are particularly attractive for relatively high-dimensional applications, involving 50 or more variables, and can be combined with semiparametric marginal distributions to obtain flexible multivariate distributions. Factor copulas generally lack a closed-form density, but we obtain analytical results for the implied tail dependence using extreme value theory, and we verify that simulation-based estimation using rank statistics is reliable even in high dimensions. We consider "scree" plots to aid the choice of the number of factors in the model. The model is applied to daily returns on all 100 constituents of the S&P 100 index, and we find significant evidence of tail dependence, heterogeneous dependence, and asymmetric dependence, with dependence being stronger in crashes than in booms. We also show that factor copula models provide superior estimates of some measures of systemic risk. Supplementary materials for this article are available online.

KEY WORDS: Copulas; Correlation; Dependence; Systemic risk; Tail dependence.

1. INTRODUCTION

One of the many surprises from the financial crisis of late 2007 to 2008 was the extent to which assets that had previously behaved mostly independently suddenly moved together. This was particularly prominent in the financial sector, where poor models of the dependence between certain asset returns (such as those on housing, and those related to mortgage defaults) are thought to be one of the causes of the collapse of the market for CDOs and related securities; see Coval, Jurek, and Stafford (2009) and Zimmer (2012) for example. Many models that were being used to capture the dependence between a large number of financial assets were revealed as being inadequate during the crisis. However, one of the difficulties in analyzing risks across many variables is the relative paucity of econometric models suitable for the task. Correlation-based models, while useful when risk can be summarized using the second moment, are often built on an assumption of multivariate Gaussianity, and face the risk of neglecting dependence between the variables in the tails, that is, neglecting the possibility that large crashes may be correlated across assets.

This article makes two primary contributions. First, we propose new models for the dependence structure, or copula, of economic variables based on a latent factor structure, the use of which makes them particularly attractive for relatively high-dimensional applications, involving 50 or more variables.[1] These copula models may be combined with existing parametric, semiparametric, or nonparametric models for univariate distributions to construct flexible yet tractable joint distributions for large collections of variables. The proposed copula models permit the researcher to determine the degree of flexibility based on the number of variables and the amount of data available. For example, by allowing for a fat-tailed common factor the model captures the possibility of correlated crashes, and by allowing the common factor to be asymmetrically distributed the model allows for the possibility that the dependence between the variables is stronger during downturns than during upturns. By allowing for multiple common factors, it is possible to capture heterogeneous pair-wise dependence within the overall multivariate copula. High-dimension economic applications often require some strong simplifying assumptions to keep the model tractable, and an important feature of the class of proposed models is that such assumptions can be made in an easily understandable manner, and can be tested and relaxed if needed.

Factor copulas do not generally have a closed-form density, but certain properties can nevertheless be obtained analytically. Using extreme value theory we obtain theoretical results on the tail dependence properties for general factor copulas, and for the specific parametric class of factor copulas that we use in our empirical work. Given the lack of closed-form density, maximum likelihood estimation for these copulas is difficult, and we employ the simulation-based estimator proposed in Oh and Patton (2013). In a supplemental appendix (available online) to this article we verify that this estimator, and its associated asymptotic distribution theory, has good finite-sample properties even in dimensions as high as 100, which is the relevant size given our empirical analysis. We also consider the use of "scree" plots, based on eigenvalues of the variables' rank correlation

[1]For recent work on high-dimensional covariance matrix estimation, see Engle, Shephard, and Sheppard (2008), Fan, Fan, and Lv (2008), Engle and Kelly (2012), Fan, Li, and Ke (2012), and Hautsch, Kyj, and Oomen (2012).

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matrix, to aid the choice of the number of factors in the factor copula model.

The second contribution of this article is a study of the dependence structure of all 100 constituent firms of the Standard and Poor's 100 index, using daily data over the period 2008–2010. This is one of the highest dimension applications of copula theory in the econometrics literature. We find significant evidence in favor of a fat-tailed common factor for these stocks (indicative of nonzero tail dependence), implying that the Normal (or Gaussian) copula is not suitable for these assets. Moreover, we find significant evidence that the common factor is asymmetrically distributed, with crashes being more highly correlated than booms. Our empirical results suggest that risk management decisions made using the Normal copula may be based on too benign a view of these assets, and derivative securities based on baskets of these assets, for example, CDOs, may be mispriced if based on a Normal copula. The fact that large negative shocks may originate from a fat-tailed common factor, and thus affect all stocks at once, makes the diversification benefits of investing in these stocks lower than under Normality. In an application to estimating systemic risk, we show that our factor copula model provides superior estimates of two measures of systemic risk.

Certain types of factor copulas have already appeared in the literature. The models we consider are extensions of Hull and White (2004), in that we retain a simple linear, additive factor structure, but allow for the variables in the structure to have flexibly specified distributions. Other variations on factor copulas are presented in Andersen and Sidenius (2004) and van der Voort (2005), who considered certain nonlinear factor structures, and in Rogge and Schönbucher (2003) and Laurent and Gregory (2005), who presented factor copulas for modeling times-to-default. See McNeil, Frey, and Embrechts (2005, chap. 9) for further discussion on similar applications. Krupskii and Joe (2013) also proposed a class of factor-vine copulas, where the factor structure is implied by the choice of copula linking each variable to the latent factor(s). With the exception of McNeil, Frey, and Embrechts (2005) and Krupskii and Joe (2013), the papers to date have not considered estimation of the unknown parameters of these copulas, instead examining calibration and derivatives pricing using these copulas. Our formal analysis of the estimation of copulas of dimension as high as 100 is new to the literature.

Some methods for modeling high-dimension copulas have previously been proposed in the literature, though few consider dimensions greater than 20.[2] The Normal copula, see Li (2000) among many others, is simple to implement but imposes the strong assumption of zero tail dependence, and symmetric dependence between booms and crashes. The Student's *t* copula and variants were discussed by Demarta and McNeil (2005). The "grouped *t*" copula was proposed by Daul et al. (2003), who applied this copula in analyses involving up to 100 variables. This copula allows for heterogeneous tail dependence between pairs of variables, but imposes that upper and lower tail dependence are equal, a finding we strongly reject for equity returns. Smith, Gan, and Kohn (2012) extracted the copula implied by a multivariate skew *t* distribution, Christoffersen et al. (2012) combined a skew *t* copula with a DCC model for conditional correlations in their study of 33 developed and emerging equity market indices, and Christoffersen et al. (2013) used the same model to study 233 equity returns and credit default swap spreads. Creal and Tsay (2015) proposed a stochastic copula model based on a factor structure, and used Bayesian estimation methods to apply it to an unbalanced panel of CDS spreads and equity returns on 100 firms. Archimedean copulas such as the Clayton or Gumbel allow for tail dependence and particular forms of asymmetry, but usually have only one or two parameter(s) to characterize the dependence between all variables, and are thus very restrictive in higher-dimension applications. "Vine" copulas are constructed by sequentially applying bivariate copulas to build up a higher-dimension copula; see Aas et al. (2009), Min and Czado (2010), and Almeida, Czado, and Manner (2012), for example. However, vine copulas are almost invariably based on an assumption that is hard to interpret and to test; see Acar, Genest, and Nešlehová (2012) for a critique. In our empirical application we compare our proposed factor models with several alternative existing models, and show that our model outperforms them all in terms of goodness of fit and in an application to measuring systemic risk.

The remainder of the article is structured as follows. Section 2 presents the class of factor copulas, derives their limiting tail properties, and considers some extensions and the use of "scree" plots to guide the choice of the number of factors. Section 3 describes the simulated method of moments (SMM) estimation method we use. Section 4 presents an empirical study of daily returns on individual constituents of the S&P 100 equity index over the period 2008–2010. An Appendix at the end of the article contains a discussion of the dependence measures used in estimation, and a supplemental appendix (available online) contains all proofs, and details of simulations used to study SMM estimation for applications with dimensions as large as ours.

2. FACTOR COPULAS

For simplicity of exposition we focus on unconditional distributions in this section, and discuss the extension to conditional distributions in the next section. Consider a vector of N variables, $\mathbf{Y}$, with some joint distribution $\mathbf{F}_y$, marginal distributions F_i, and copula $\mathbf{C}$:

$$[Y_1, \ldots, Y_N]' \equiv \mathbf{Y} \sim \mathbf{F}_y = \mathbf{C}(F_1, \ldots, F_N) \tag{1}$$

The copula completely describes the dependence between the variables $Y_1, \ldots, Y_N$. We will use existing models to estimate the marginal distributions F_i (which may be parametric, semiparametric, or nonparametric), and focus on constructing useful new models for the dependence between these variables, $\mathbf{C}$.[3] Decomposing the joint distribution in this way has two important advantages over considering the joint distribution $\mathbf{F}_y$ directly. First, it facilitates multi-stage estimation, which is particularly useful in high-dimension applications, where the sparseness of the data and the potential proliferation of parameters can cause problems. Second, it allows the researcher to draw on the large

[2]For general reviews of copulas in economics and finance, see Cherubini, Luciano, and Vecchiato (2004) and Patton (2012).

[3]Although we treat estimation of the marginal distributions as separate from copula estimation, the inference methods we consider do take estimation error from the marginal distributions into account.

literature on models for univariate distributions, leaving "only" the task of constructing a model for the copula, which is a simpler problem.

2.1 The Copula of a Latent Factor Structure

The class of copulas we consider are those that can be generated by the following factor structure, based on a set of $N + K$ latent variables:

$$\begin{aligned} &\text{Let} \quad X_i = \sum_{k=1}^{K} \beta_{ik} Z_k + \varepsilon_i, \quad i = 1, 2, \ldots, N \\ &\text{so} \quad [X_1, \ldots, X_N]' \equiv \mathbf{X} = \mathbf{B}\mathbf{Z} + \varepsilon \qquad (2) \\ &\text{where} \quad \varepsilon_i \sim \text{iid } F_\varepsilon(\gamma_\varepsilon), \\ &\qquad Z_k \sim \text{inid } F_{z_k}(\gamma_k), \; Z_k \perp\!\!\!\perp \varepsilon_i \forall i, k. \\ &\text{Then} \quad \mathbf{X} \sim \mathbf{F}_x = \mathbf{C}(G_1(\theta), \ldots, G_N(\theta); \theta), \end{aligned}$$

where $\theta \equiv [\text{vec}(\mathbf{B})', \gamma_\varepsilon', \gamma_1', \ldots, \gamma_K']'$. The copula of the latent variables $\mathbf{X}$, denoted $\mathbf{C}(\theta)$, is used as the model for the copula of the observable variables $\mathbf{Y}$. [4] An important point about the above construction is that the marginal distributions of X_i may be different from those of the original variables Y_i, so $F_i \neq G_i$ in general; we use the structure for the vector $\mathbf{X}$ *only* for its copula, and completely discard the resulting marginal distributions. This is motivated by our desire to use the dimension-reduction technique of imposing a factor structure only in the component of the joint distribution that is difficult to estimate in high dimensions, namely the copula. Marginal distributions, on the other hand, are usually able to be estimated flexibly, given the amount of time series data that is available in most financial applications. In our empirical application, for example, we employ semiparametric models for the marginal distributions, thus allowing for great flexibility, but impose a factor structure on the copula to avoid the "curse of dimensionality."

The copula implied by Equation (2) is generally not known in closed form. The leading case where it *is* known is when $\{F_\varepsilon, F_{z_1}, \ldots, F_{z_K}\}$ are all Gaussian distributions, in which case the variable $\mathbf{X}$ is multivariate Gaussian, implying a Gaussian copula. For other choices of $\{F_\varepsilon, F_{z_1}, \ldots, F_{z_K}\}$ the joint distribution of $\mathbf{X}$, and thus the copula of $\mathbf{X}$, is generally not known in closed form. However, it is simple to simulate from $\{F_\varepsilon, F_{z_1}, \ldots, F_{z_K}\}$ for many classes of distributions, and from simulated data we can extract properties of the copula, such as rank correlation, Kendall's tau, and quantile dependence. These simulated rank dependence measures can then be used in the SMM estimation method of Oh and Patton (2013), which is briefly described in Section 3.

[4]This method for constructing a copula model resembles the use of mixture models, for example, the Normal-inverse Gaussian or generalized hyperbolic distributions, where the distribution of interest is obtained by considering a function of a collection of latent variables; see Barndorff-Nielsen (1978, 1997), and McNeil, Frey, and Embrechts (2005). It can also be interpreted as a special case of the "conditional independence structure" of McNeil, Frey, and Embrechts (2005), which is used to describe a set of variables that are independent conditional on some smaller set of variables, $\mathbf{X}$ and $\mathbf{Z}$ in our notation. The variables $\mathbf{Z}$ are sometimes known as the "frailty" in the survival analysis and credit default literature; see Duffie et al. (2009) for example.

Key choices in specifying a factor copula include the following. First, the distributions to use for the common and idiosyncratic variables must be chosen. If simulation-based estimation methods are to be used, then these distributions should be such that random draws from these are easy to obtain. (This is true for most commonly used distributions.) Second, as discussed in more detail below, these distributions should be such that tail dependence and asymmetric dependence (here taken to mean that "booms" have a different dependence structure to "crashes") can be captured. Finally, the number of factors to consider (K) must be specified. Allowing for more than a single factor adds much flexibility to the model, at a cost of a substantial increase in the number of parameters. We discuss these choices empirically in Section 4.

2.2 Tail Dependence Properties of Factor Copulas

Although most factor copulas will not have a closed-form expression, using results from extreme value theory it is possible to obtain analytically results on the tail dependence implied by a given factor copula model. These results are relatively easy to obtain, given the simple linear structure generating the factor copula. Recall the definition of tail dependence for two variables X_i, X_j with marginal distributions G_i, G_j:

$$\begin{aligned} \tau_{ij}^L &\equiv \lim_{q \to 0} \frac{\Pr\left[X_i \leq G_i^{-1}(q), X_j \leq G_j^{-1}(q)\right]}{q} \\ \tau_{ij}^U &\equiv \lim_{q \to 1} \frac{\Pr\left[X_i > G_i^{-1}(q), X_j > G_j^{-1}(q)\right]}{1-q}. \end{aligned} \qquad (3)$$

That is, lower tail dependence measures the probability of both variables lying below their q quantile, for q limiting to zero, scaled by the probability of one of these variables lying below their q quantile. Upper tail dependence is defined analogously. In Proposition 1 we present results for a general single factor copula model:

Proposition 1 (Tail dependence for a factor copula). Consider the factor copula generated by Equation (2) with $K = 1$. Assume F_z and F_ε have regularly varying tails with a common tail index $\alpha > 0$, that is,

$$\begin{aligned} &\Pr[Z > s] \sim A_z^U s^{-\alpha} \;\text{ and }\; \Pr[\varepsilon_i > s] \sim A_\varepsilon^U s^{-\alpha}, \;\text{ as } s \to \infty \\ &\Pr[Z < -s] \sim A_z^L s^{-\alpha} \;\text{ and }\; \Pr[\varepsilon_i < -s] \sim A_\varepsilon^L s^{-\alpha} \;\text{ as } s \to \infty, \end{aligned} \qquad (4)$$

where A_z^L, A_z^U, A_ε^L, and A_ε^U are positive constants, and we write $x_s \sim y_s$ if $x_s/y_s \to 1$ as $s \to \infty$. Then (a) if $\beta_i, \beta_j > 0$ the lower and upper tail dependence coefficients are:

$$\begin{aligned} \tau_{ij}^L &= \frac{\min(\beta_i, \beta_j)^\alpha A_z^L}{\min(\beta_i, \beta_j)^\alpha A_z^L + A_\varepsilon^L}, \\ \tau_{ij}^U &= \frac{\min(\beta_i, \beta_j)^\alpha A_z^U}{\min(\beta_i, \beta_j)^\alpha A_z^U + A_\varepsilon^U}; \end{aligned} \qquad (5)$$

(b) if $\beta_i, \beta_j < 0$ the lower and upper tail dependence coefficients are:

$$\tau_{ij}^{L} = \frac{\min\left(|\beta_i|, |\beta_j|\right)^{\alpha} A_z^U}{\min\left(|\beta_i|, |\beta_j|\right)^{\alpha} A_z^U + A_\varepsilon^L},$$
$$\tau_{ij}^{U} = \frac{\min\left(|\beta_i|, |\beta_j|\right)^{\alpha} A_z^L}{\min\left(|\beta_i|, |\beta_j|\right)^{\alpha} A_z^L + A_\varepsilon^U}; \tag{6}$$

(c) if $\beta_i\beta_j = 0$ or (d) if $\beta_i\beta_j < 0$, the lower and upper tail dependence coefficients are zero.

All proofs are presented in the supplemental appendix (available online). This proposition shows that when the coefficients on the common factor have the same sign, and the common factor and idiosyncratic variables have the same tail index, the factor copula generates upper and lower tail dependence. If either Z or ε is asymmetrically distributed, then the upper and lower tail dependence coefficients can differ, which provides this model with the ability to capture differences in the probabilities of joint crashes and joint booms. If Z has a thinner upper (for example) tail than lower tail, while ε is symmetric with the same tail index as Z's lower tail, then upper tail dependence will be zero while the lower tail dependence will generally be positive. When either of the coefficients on the common factor are zero, or if they have differing signs, then the upper and lower tail dependence coefficients are both zero.

The above proposition considers the case that the common factor and idiosyncratic variables have the same tail index; when these indices differ we obtain a boundary result: if the tail index of Z is strictly greater than that of ε and $\beta_i\beta_j > 0$ then tail dependence is one, while if the tail index of Z is strictly less than that of ε then tail dependence is zero.

In our empirical analysis in Section 4, we will focus on the Skew *t* distribution of Hansen (1994) as a model for the common factor and the standardized *t* distribution for the idiosyncratic shocks. Proposition 2 below presents the analytical tail dependence coefficients for a factor copula based on these distributions.

Proposition 2 (Tail dependence for a Skew *t*-*t* factor copula). Consider the factor copula generated by Equation (2) with $K = 1$. If $F_z = \text{Skew } t\,(\nu, \lambda)$ and $F_\varepsilon = t\,(\nu)$, then the tail indices of Z and ε_i equal ν, and the constants A_z^L, A_z^U, A_ε^L, and A_ε^U from Proposition 1 are given by

$$A_z^L = \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1-\lambda)^2}\right)^{-(\nu+1)/2},$$
$$A_z^U = \frac{bc}{\nu}\left(\frac{b^2}{(\nu-2)(1+\lambda)^2}\right)^{-(\nu+1)/2},$$
$$A_\varepsilon^L = A_\varepsilon^U = \frac{c}{\nu}\left(\frac{1}{\nu-2}\right)^{-(\nu+1)/2}, \tag{7}$$

where $a = 4\lambda c(\nu-2)/(\nu-1)$, $b = \sqrt{1+3\lambda^2-a^2}$, $c = \Gamma\left(\frac{\nu+1}{2}\right) / \left(\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi(\nu-2)}\right)$. Given Proposition 1 and the expressions for A_z^L, A_z^U, A_ε^L, and A_ε^U above, we then obtain the tail dependence coefficients for this copula.

We next generalize Proposition 1 to consider a multi-factor copula model, which will prove useful in our empirical application in Section 4.

Proposition 3 (Tail dependence for a multi-factor copula). Consider the factor copula generated by Equation (2). Assume $F_\varepsilon, F_{z_1}, \ldots, F_{z_K}$ have regularly varying tails with a common tail index $\alpha > 0$, and upper and lower tail coefficients $A_\varepsilon^U, A_1^U, \ldots, A_K^U$ and $A_\varepsilon^L, A_1^L, \ldots, A_K^L$. Then if $\beta_{ik} \geq 0\ \forall\, i, k$, the lower and upper tail dependence coefficients are

$$\tau_{ij}^{L} = \frac{\sum_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^L \beta_{ik}^{\alpha} \delta_{L,ijk}^{\alpha}}{A_\varepsilon^L + \sum_{k=1}^{K} A_k^L \beta_{ik}^{\alpha}}, \tag{8}$$
$$\tau_{ij}^{U} = \frac{\sum_{k=1}^{K} \mathbf{1}\left\{\beta_{ik}\beta_{jk} > 0\right\} A_k^U \beta_{ik}^{\alpha} \delta_{U,ijk}^{\alpha}}{A_\varepsilon^U + \sum_{k=1}^{K} A_k^U \beta_{ik}^{\alpha}},$$

where

$$\delta_{Q,ijk}^{-1} \equiv \begin{cases} \max\left\{1, \gamma_{Q,ij}\beta_{ik}/\beta_{jk}\right\}, & \text{if } \beta_{ik}\beta_{jk} > 0 \\ 1, & \text{if } \beta_{ik}\beta_{jk} = 0 \end{cases},$$
$$\text{for } Q \in \{L, U\}, \tag{9}$$

$$\gamma_{Q,ij} \equiv \left(\frac{A_\varepsilon^Q + \sum_{k=1}^{K} A_k^Q \beta_{jk}^{\alpha}}{A_\varepsilon^Q + \sum_{k=1}^{K} A_k^Q \beta_{ik}^{\alpha}}\right)^{1/\alpha}, \text{ for } Q \in \{L, U\}. \tag{10}$$

The extensions to consider the case that some have opposite signs to the others can be accommodated using the same methods as in the proof of Proposition 1. In the one-factor copula model the variables $\delta_{L,ijk}$ and $\delta_{U,ijk}$ can be obtained directly and are determined by $\min\{\beta_i, \beta_j\}$; in the multi-factor copula model these variables can be determined using Equation (9), but do not generally have a simple expression.

2.3 Illustration of Some Factor Copulas

To illustrate the flexibility of the class of factor copulas, Figure 1 presents 1000 random draws from bivariate distributions constructed using four different factor copulas. In all cases the marginal distributions, F_i, are set to $N(0, 1)$, and the variances of the latent variables in the factor copula are set to $\sigma_z^2 = \sigma_\varepsilon^2 = 1$, so that the common factor accounts for one-half of the variance of each X_i. The first copula is generated from a factor structure with $F_z = F_\varepsilon = N(0, 1)$, implying that the copula is Normal. The second sets $F_z = F_\varepsilon = t(4)$, generating a symmetric copula with positive tail dependence. The third copula sets $F_\varepsilon = N(0, 1)$ and $F_z = \text{Skew } t\ (\infty, -0.25)$, corresponding to a skewed Normal distribution. This copula exhibits asymmetric dependence, with crashes being more correlated than booms, but zero tail dependence. The fourth copula sets $F_\varepsilon = t(4)$ and $F_z = \text{Skew } t(4, -0.25)$, which generates asymmetric dependence and positive tail dependence. Figure 1 shows that when the distributions in the factor structure are Normal or skewed Normal, tail events tend to be uncorrelated across the two variables. When the degrees of freedom is set to 4, on the other hand, we observe several draws in the joint upper and lower tails. When the skewness parameter is negative, as in the lower two panels of Figure 1, we observe stronger clustering of

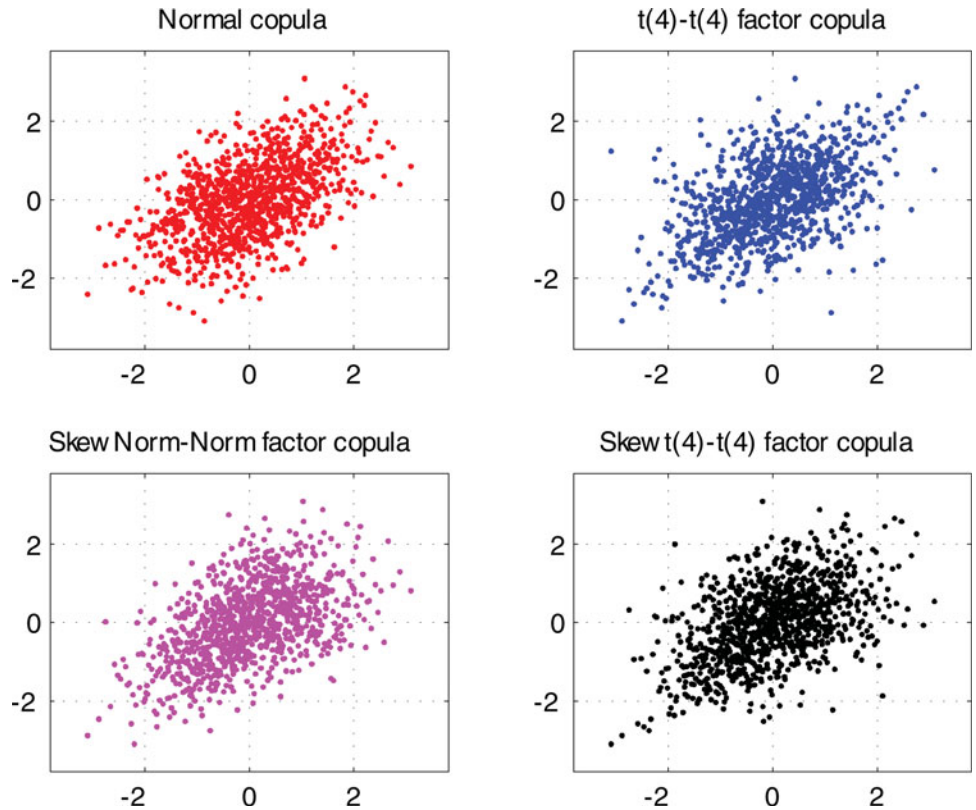

Figure 1. Scatterplots from four bivariate distributions, all with $N(0,1)$ margins and linear correlation of 0.5, constructed using four different factor copulas.

observations in the joint negative quadrant compared with the joint positive quadrant.

Figure 2 illustrates the differences between copulas using a multivariate approach related to our study of systemic risk below. Conditional on observing j out of 100 stocks crashing, we present the expected number, or proportion, of the remaining $(100 - j)$ stocks that will crash, a measure based on Hartmann, Straetmans, and de Vries (2006) and Geluk, de Haan, and de Vries (2007). Define

$$N_q^* \equiv \sum_{i=1}^{N} \mathbf{1}\{U_i \leq q\},$$
$$\kappa^q(j) = E\left[N_q^* | N_q^* \geq j\right] - j, \tag{11}$$

and

$$\pi^q(j) \equiv \frac{\kappa^q(j)}{N-j}.$$

For this illustration we define a "crash" as a realization in the lower 1/66 tail, corresponding to a once-in-a-quarter event for daily asset returns. We consider four copulas: the familiar Normal, Student's $t(4)$, and Clayton copula, as well as the Skew $t(4)$-$t(4)$ factor copula, all with parameters chosen so that linear correlation of 1/2 is implied. The upper panel shows that as we condition on more variables crashing, the expected number of other variables that will crash, $\kappa^q(j)$, initially increases, and peaks at around $j = 30$. At that point, the Skew $t(4)$-$t(4)$ factor copula predicts that around another 38 variables will crash, while under the Normal copula we expect only around 12 more variables to crash. As we condition on even more variables having crashed, the plot converges inevitably to zero (since conditioning on having observed more crashes, there are fewer variables left to crash). The lower panel of Figure 2 shows that the expected *proportion* of remaining stocks that will crash, $\pi^q(j)$, generally increases all the way to $j = 99$.[5] This figure illustrates some of the features of dependence that are unique to high-dimension applications, and further motivates our proposal for a class of flexible, parsimonious models for such applications.

2.4 Guidance on Choosing the Number of Factors

In this section we consider a graphical tool to obtain guidance on the choice of the number of factors to include in a factor copula model, namely the famous "scree" plot of Cattell (1966). Given that factor copula models are parametric, formal tests for the correct number of factors should exploit that parametric structure, and in our empirical analysis below we use model specification tests described in the next section for this purpose. However, it is still of interest to have some prior guidance on just how many common factors might reasonably be needed to describe the dependence.

A "scree" plot shows the eigenvalues of a covariance or correlation matrix from largest to smallest, and it is commonly found that the number of factors is equal to the number of "large"

[5]For the Normal copula this is not the case, however this is likely due to simulation error: even with the 10 million simulations used to obtain this figure, joint 1/66 tail crashes are so rare under a Normal copula that there is a fair degree of simulation error in this plot for $j \geq 80$.

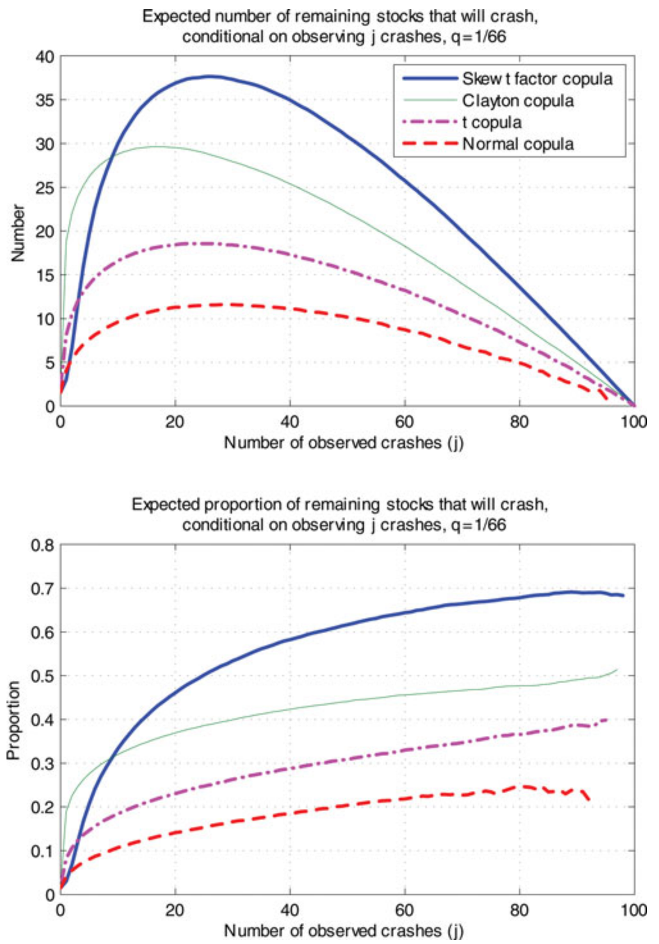

Figure 2. Conditional on observing j out of 100 stocks crashing, this figure presents the expected number (upper panel) and proportion (lower panel) of the remaining $(100 - j)$ stocks that will crash. "Crash" events are defined as returns in the lower 1/66 tail.

eigenvalues. Employing scree plots for factor copulas is made difficult by the fact that we want to apply them to the rank correlation matrix of the data, not the covariance matrix, and results for the latter do not generally carry over to the former. Furthermore, the sampling variability of rank correlations differs from those of correlations, as the latter are (ratios of) moments while the former are rank statistics.

In the proposition below we provide conditions under which scree plots can aid in the identification of the number of factors in a factor copula. We discuss the assumptions below.

Proposition 4. Assume (1) $\mathbf{Y}_t \sim$iid $\mathbf{F}_y$, $\mathbf{F}_y$ and $\mathbf{F}_x$ from Equations (1) and (2) are continuous, and every bivariate marginal copula C_{ij} of $\mathbf{C}$ has continuous partial derivatives with respect to u_i and u_j; (2) $\hat{\mathbf{R}}_T^L = \hat{\mathbf{R}}_T + o_p(1)$, where $\hat{\mathbf{R}}_T^L$ and $\hat{\mathbf{R}}_T$ are the sample linear and rank correlation matrices of $\{\mathbf{X}\}_{t=1}^T$; and (3) the eigenvalues of $\mathbf{BB}'$ are "large," in the sense that they imply $g_K(\mathbf{R}) > 1$. Let

$$\hat{K}_T = \max\left\{k : g_k(\hat{\mathbf{R}}_T^y) > 1\right\}, \tag{12}$$

where $\hat{\mathbf{R}}_T^y$ is the sample rank correlation matrix of $\{\mathbf{Y}_t\}_{t=1}^T$, and $g_k(\mathbf{A})$ returns the kth-largest eigenvalue of the matrix $\mathbf{A}$.

(i) Under assumptions (1)–(2), $\Pr[\hat{K}_T \leq K] \to 1$ as $T \to \infty$.

(ii) Under assumptions (1)–(3), $\Pr[\hat{K}_T = K] \to 1$ as $T \to \infty$.

The first assumption given in the statement of Proposition 4 simply requires that the distributions and copulas are continuous, and the iid part of this assumption can be relaxed by invoking assumption 2 of Oh and Patton (2013) and then analyzing estimated standardized residuals rather than the original data. The second assumption is stronger, requiring rank correlations and linear correlations to be "close." A sufficient condition for this is that the marginal distributions of $\mathbf{X}$ are Uniform, and in other cases it may or may not be a reasonable approximation. In the supplemental appendix (available online), we present evidence that this assumption holds very well for a variety of factor copula models based on t or skew t distributions. If other distributions are considered, in particular those that are far from "bell shaped," it is possible that this assumption will not be plausible. If the copula is *elliptical*, then Klüppelberg and Kuhn (2009) suggest using Kendall's tau rather than Spearman's rank correlation, as the former is a known monotonic function of linear correlation for such copulas (see Fang, Fang, and Kotz 2002), and assumption (2) is not needed. Elliptical copulas cannot accommodate asymmetric dependence, which we find to be important in our empirical application, consistent with several existing papers, and so we do not attempt to exploit that result.

With assumptions (1)–(2) we find that $\hat{K}_T$ provides, asymptotically, a lower bound on the true number of factors; it will miss factors that are such that $g_k(\mathbf{R}) \leq 1$ for $k \in [1, K]$. If N diverges with T then this cannot happen and assumption (3) will hold automatically (see Chamberlain and Rothschild 1983; Bai and Ng 2002), while in our setting of finite N this assumption may not hold. In such cases using a threshold of one provides a lower bound on the true number of factors. In the supplemental appendix (available online) we undertake a simulation study of $\hat{K}_T$ based on realistic parameter values, and we find that it correctly estimates the number of factors in 90%–99% of simulations.

In Figure 3 we present four examples of scree plots for a single simulation from a factor copula described in detail in the supplemental appendix (available online). In all cases we set $N = 100$ and $T = 1000$, and we vary the number of factors, K. We see similar shapes to these plots in other applications, and we clearly see how this sort of figure might provide guidance on the choice of the number of factors: the first K eigenvalues are large, and the remaining $N - K$ eigenvalues gradually tail off. In two of these cases ($K = 2$ and 4) the bound of one clearly "works" in the sense that it correctly separates the first K from the remaining eigenvalues. In this simulation of the $K = 1$ case, the second eigenvalue is just above one, due to sampling variability in the eigenvalue, and in this case $\hat{K}_T$ would overestimate the true number of factors. In the $K = 8$ case, we see that the bound of one almost "cuts off" the eighth eigenvalue, which would lead to the underestimation of the true number of factors.

2.5 Nonlinear Factor Copula Models

The class of factor copula models proposed in Equation (2) can be generalized to more flexible factor structures, by considering "link" functions that are not linear and additive. Consider

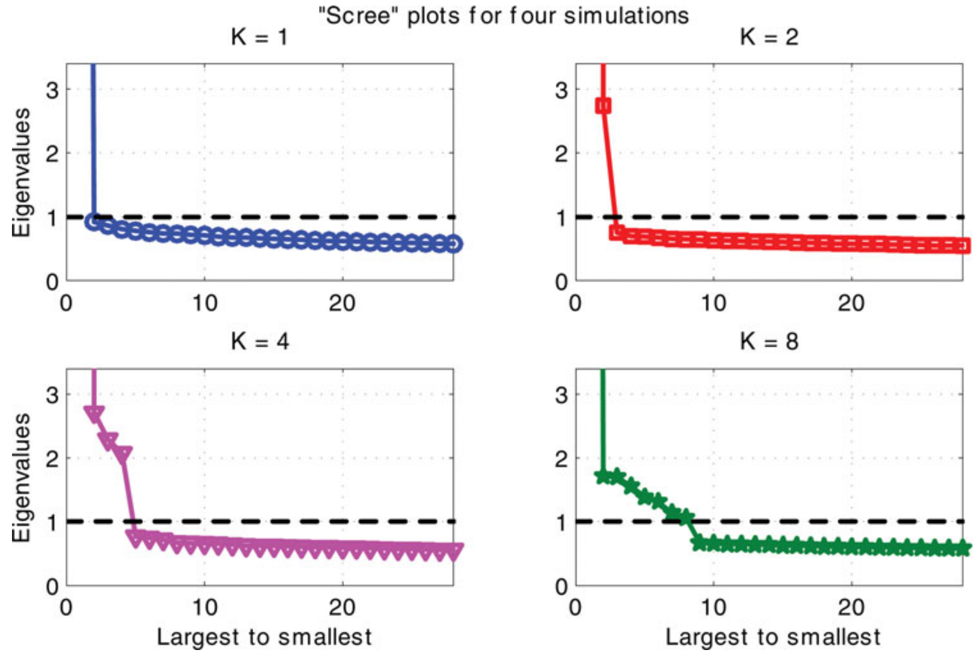

Figure 3. Each panel of this figure shows the ordered eigenvalues of the sample rank correlation matrix from a 100-dimensional factor copula with K common factors. In all cases the first eigenvalue is much larger than 3 and is cropped from the figure, and the horizontal axis is truncated at 28 for clarity.

the following general one-factor structure:

$$\begin{aligned} X_i &= h\left(Z, \varepsilon_i\right), \ i = 1, 2, \ldots, N \\ Z &\sim F_z, \ \varepsilon_i \sim \text{iid } F_\varepsilon, \ Z \perp\!\!\!\perp \varepsilon_i \forall i \\ [X_1, \ldots, X_N]' \equiv \mathbf{X} &\sim \mathbf{F}_x = \mathbf{C}\left(G_1, \ldots, G_N\right) \end{aligned} \tag{13}$$

for some function $h : \mathbb{R}^2 \to \mathbb{R}$. This general structure allows us to nest a variety of well-known copulas in the literature. Examples of copula models that fit in this framework are summarized below:

Copula	$h\left(Z, \varepsilon\right)$	F_Z	F_ε
Normal	$Z + \varepsilon$	$N(0, \sigma_z^2)$	$N(0, \sigma_\varepsilon^2)$
Student's t	$Z^{1/2}\varepsilon$	$\text{Ig}(\nu/2, \nu/2)$	$N(0, \sigma_\varepsilon^2)$
Skew t	$\lambda Z + Z^{1/2}\varepsilon$	$\text{Ig}(\nu/2, \nu/2)$	$N(0, \sigma_\varepsilon^2)$
Gen hyperbolic	$\gamma Z + Z^{1/2}\varepsilon$	$GIG(\lambda, \chi, \psi)$	$N(0, \sigma_\varepsilon^2)$
Clayton	$(1 + \varepsilon/Z)^{-\alpha}$	$\Gamma\left(\alpha, 1\right)$	$\text{Exp}\left(1\right)$
Gumbel	$-\left(\log Z/\varepsilon\right)^{\alpha}$	$\text{Stable}\left(1/\alpha, 1, 1, 0\right)$	$\text{Exp}\left(1\right)$

where Ig represents the inverse gamma distribution, GIG is the generalized inverse gaussian distribution, and Γ is the gamma distribution. The skew t and Generalized hyperbolic copulas listed here are from McNeil, Frey, and Embrechts (2005, chap. 5), the representation of a Clayton copula in this form is from Cook and Johnson (1981), and the representation of the Gumbel copula is from Marshall and Olkin (1988).

The above copulas all have closed-form densities via judicious combinations of the "link" function h and the distributions F_z and F_ε. By removing this requirement and employing simulation-based estimation methods to overcome the lack of closed-form likelihood, one can obtain a much wider variety of models for the dependence structure. In this article we will focus on linear, additive factor copulas, and generate flexible models by flexibly specifying the distribution of the common factor(s).

3. SIMULATION-BASED ESTIMATION OF FACTOR COPULAS

Factor copula models do not generally have a closed-form likelihood, making maximum likelihood estimation difficult. Oh and Patton (2013) proposed an estimation method similar to the simulated method of moments (SMM), which is readily applied in such cases. We adopt that estimation method here, and briefly describe it below. An extensive simulation study of this estimation method for applications involving up to 100 variables is presented in the supplemental appendix (available online).

The class of data-generating processes (DGPs) covered by Oh and Patton (2013) is the same as that in Chen and Fan (2006) and Rémillard (2010). This class allows each variable to have time-varying conditional mean and variance, each governed by parametric models, with an unknown marginal distribution. The marginal distributions are estimated using empirical distribution function, making the complete marginal distribution models dynamic and semiparametric. The conditional copula of the data is assumed to belong to a parametric family and is assumed constant. The combination of time-varying conditional means and variance and a constant conditional copula makes this model similar in spirit to the "CCC" model of Bollerslev (1990). The DGP is then

$$\mathbf{Y}_t = \boldsymbol{\mu}_t\left(\boldsymbol{\phi}\right) + \boldsymbol{\sigma}_t\left(\boldsymbol{\phi}\right)\boldsymbol{\eta}_t, \tag{14}$$

where

$$\boldsymbol{\mu}_t(\boldsymbol{\phi}) \equiv [\mu_{1t}(\boldsymbol{\phi}), \ldots, \mu_{Nt}(\boldsymbol{\phi})]'$$
$$\boldsymbol{\sigma}_t(\boldsymbol{\phi}) \equiv \text{diag}\{\sigma_{1t}(\boldsymbol{\phi}), \ldots, \sigma_{Nt}(\boldsymbol{\phi})\}$$
$$\boldsymbol{\eta}_t \equiv [\eta_{1t}, \ldots, \eta_{Nt}]' \sim \text{iid } \mathbf{F}_{\boldsymbol{\eta}} = \mathbf{C}(F_1, \ldots, F_N; \boldsymbol{\theta}),$$

where $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are $\mathcal{F}_{t-1}$-measurable and independent of $\boldsymbol{\eta}_t$. $\mathcal{F}_{t-1}$ is the sigma-field generated by $\{\mathbf{Y}_{t-1}, \mathbf{Y}_{t-2}, \ldots\}$. The $r \times 1$ vector of parameters governing the dynamics of the variables, $\boldsymbol{\phi}$, is assumed to be $\sqrt{T}$ -consistently estimable in a stage prior to copula estimation. If $\boldsymbol{\phi}_0$ is known, or if $\boldsymbol{\mu}_t$ and $\boldsymbol{\sigma}_t$ are constant, then the model becomes one for iid data. The copula is parameterized by a $p \times 1$ vector of parameters, $\boldsymbol{\theta}$, which is estimated using the following approach.

The estimation method of Oh and Patton (2013) is closely related to SMM estimation, though it is not strictly SMM, as the "moments" that are used in estimation are functions of rank statistics. They propose estimating $\boldsymbol{\theta}$ based on the standardized residual $\{\hat{\boldsymbol{\eta}}_t \equiv \boldsymbol{\sigma}_t^{-1}(\hat{\boldsymbol{\phi}})[\mathbf{Y}_t - \boldsymbol{\mu}_t(\hat{\boldsymbol{\phi}})]\}_{t=1}^T$ and simulations from some parametric joint distribution, $\mathbf{F}_x(\boldsymbol{\theta})$, with implied copula $\mathbf{C}(\boldsymbol{\theta})$. Let $\tilde{\mathbf{m}}_S(\boldsymbol{\theta})$ be an $m \times 1$ vector of dependence measures computed using S simulations from $\mathbf{F}_x(\boldsymbol{\theta})$, $\{\mathbf{X}_s\}_{s=1}^S$, and let $\hat{\mathbf{m}}_T$ be the corresponding vector of dependence measures computed using the standardized residuals $\{\hat{\boldsymbol{\eta}}_t\}_{t=1}^T$. We discuss the empirical choice of which dependence measures to match in the Appendix.

The SMM estimator is then defined as

$$\hat{\boldsymbol{\theta}}_{T,S} \equiv \underset{\boldsymbol{\theta}\in\Theta}{\arg\min}\, Q_{T,S}(\boldsymbol{\theta}), \tag{15}$$

where

$$Q_{T,S}(\boldsymbol{\theta}) \equiv \mathbf{g}'_{T,S}(\boldsymbol{\theta})\hat{\mathbf{W}}_T\mathbf{g}_{T,S}(\boldsymbol{\theta}),$$
$$\mathbf{g}_{T,S}(\boldsymbol{\theta}) \equiv \hat{\mathbf{m}}_T - \tilde{\mathbf{m}}_S(\boldsymbol{\theta}),$$

and $\hat{\mathbf{W}}_T$ is some positive definite weight matrix, which may depend on the data. Under regularity conditions, Oh and Patton (2013) showed that if $S/T \to \infty$ as $T \to \infty$, the SMM estimator is consistent and asymptotically normal:

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{T,S} - \boldsymbol{\theta}_0\right) \xrightarrow{d} N(0, \boldsymbol{\Omega}_0) \text{ as } T, S \to \infty, \tag{16}$$

where

$$\boldsymbol{\Omega}_0 = \left(\mathbf{G}_0'\mathbf{W}_0\mathbf{G}_0\right)^{-1}\mathbf{G}_0'\mathbf{W}_0\boldsymbol{\Sigma}_0\mathbf{W}_0\mathbf{G}_0\left(\mathbf{G}_0'\mathbf{W}_0\mathbf{G}_0\right)^{-1}$$

$\boldsymbol{\Sigma}_0 \equiv \text{avar}[\hat{\mathbf{m}}_T]$, $\mathbf{G}_0 \equiv \nabla_\theta \mathbf{g}_0(\boldsymbol{\theta}_0)$, $\mathbf{g}_0(\boldsymbol{\theta}) = \text{p-}\lim_{T,S\to\infty} \mathbf{g}_{T,S}(\boldsymbol{\theta})$, and $\mathbf{W}_0 = \text{p-}\lim_{T\to\infty} \hat{\mathbf{W}}_T$. The asymptotic variance of the estimator has the same form as in standard GMM applications, however the components $\boldsymbol{\Sigma}_0$ and $\mathbf{G}_0$ require different estimation methods than in standard applications. Oh and Patton (2013) also presented the distribution of a test of the over-identifying restrictions (the "J" test), which we will use for specification testing in our empirical application.

Our empirical application below involves 100 variables, and it is well known that properties of estimators can deteriorate as the dimension grows; Oh and Patton (2013) verified that their asymptotic theory provides a good approximation to finite-sample behavior for applications involving only up to ten variables. In the supplemental appendix (available online) we undertake an extensive simulation study of this estimator in applications involving up to 100 variables. In brief, these simulations show that the SMM estimator and its associated distribution theory continue to have satisfactory properties even in high-dimension applications: finite-sample bias is small, confidence intervals have good coverage rates, and the J-test has reasonable finite-sample size. This provides reassurance for using this estimator in our empirical application below.

4. HIGH-DIMENSION COPULA MODELS FOR S&P 100 RETURNS

Having proposed a new class of models for copulas in high dimensions and discussed their estimation, we now apply these models to a difficult empirical problem. We study of the dependence between all 100 stocks that were constituents of the S&P 100 index in December 2010. Our sample period is April 2008 to December 2010, a total of $T = 696$ trade days. The starting point for our sample period was determined by the date of the latest addition to the S&P 100 index (Philip Morris Inc.), which has had no additions or deletions since April 2008. The stocks in our study are listed in Table 1, along with their three-digit Standard Industrial Classification (SIC) codes, which we will use in part of our analysis below.

Table 2 presents some summary statistics of the data used in this analysis. The top panel presents sample moments of the daily returns for each stock. The means and standard deviations are comparable to those observed in other studies. The skewness and kurtosis coefficients reveal a substantial degree of heterogeneity in the shape of the distribution of these asset returns, motivating our use of a nonparametric estimate (the empirical distribution function, EDF) in our analysis.

In the second panel of Table 2 we present information on the parameters of the AR(1)-GJR-GARCH(1,1) models, augmented with lagged market return information, that are used to filter each of the individual return series:

$$\begin{aligned} r_{it} &= \phi_{0i} + \phi_{1i}r_{i,t-1} + \phi_{mi}r_{m,t-1} + \varepsilon_{it} \\ \sigma_{it}^2 &= \omega_i + \beta_i\sigma_{i,t-1}^2 + \alpha_i\varepsilon_{i,t-1}^2 + \gamma_i\varepsilon_{i,t-1}^2\mathbf{1}\left\{\varepsilon_{i,t-1} \le 0\right\} \\ &\quad + \alpha_{mi}\varepsilon_{m,t-1}^2 + \gamma_{mi}\varepsilon_{m,t-1}^2\mathbf{1}\left\{\varepsilon_{m,t-1} \le 0\right\} \end{aligned} \tag{17}$$

We estimate the parameters of the mean and variance models using quasi-maximum likelihood, and we estimate the distribution of the standardized residuals using the EDF, which allows us to nonparametrically capture skewness and excess kurtosis in the residuals, if present, and importantly it allows these characteristics to differ across the 100 variables.

Our estimates of the parameters of these models are consistent with those reported in numerous other studies, with a small negative AR(1) coefficient found for most though not all stocks, and with the lagged market return entering significantly in 37 out of the 100 stocks. The estimated GJR-GARCH parameters are strongly indicative of persistence in volatility, and the asymmetry parameter, γ, in this model is positive for all but three of the 100 stocks in our sample, supporting the widespread finding of a "leverage effect" in the conditional volatility of equity

[6]We considered GARCH (Bollerslev 1986), EGARCH (Nelson 1991), and GJR-GARCH (Glosten, Jagannathan, and Runkle 1993) models for the conditional variance of these returns, and for almost all stocks the GJR-GARCH model was preferred according to the BIC.

Table 1. Stocks used in the empirical analysis

Ticker	Name	SIC	Ticker	Name	SIC	Ticker	Name	SIC
AA	Alcoa	333	EXC	Exelon	493	NKE	Nike	302
AAPL	Apple	357	F	Ford	371	NOV	National Oilwell	353
ABT	Abbott Lab.	283	FCX	Freeport	104	NSC	Norfolk Sth	671
AEP	American Elec	491	FDX	Fedex	451	NWSA	News Corp	271
ALL	Allstate Corp	633	GD	GeneralDynam	373	NYX	NYSE Euronxt	623
AMGN	Amgen Inc.	283	GE	General Elec	351	ORCL	Oracle	737
AMZN	Amazon.com	737	GILD	GileadScience	283	OXY	OccidentalPetrol	131
AVP	Avon	284	GOOG	Google Inc	737	PEP	Pepsi	208
AXP	American Ex	671	GS	GoldmanSachs	621	PFE	Pfizer	283
BA	Boeing	372	HAL	Halliburton	138	PG	Procter&Gamble	284
BAC	Bank of Am	602	HD	Home Depot	525	QCOM	Qualcomm Inc	366
BAX	Baxter	384	HNZ	Heinz	203	RF	Regions Fin	602
BHI	Baker Hughes	138	HON	Honeywell	372	RTN	Raytheon	381
BK	Bank of NY	602	HPQ	HP	357	S	Sprint	481
BMY	Bristol-Myers	283	IBM	IBM	357	SLB	Schlumberger	138
BRK	Berkshire Hath	633	INTC	Intel	367	SLE	Sara Lee Corp.	203
C	Citi Group	602	JNJ	Johnson&J.	283	SO	Southern Co.	491
CAT	Caterpillar	353	JPM	JP Morgan	672	T	AT&T	481
CL	Colgate	284	KFT	Kraft	209	TGT	Target	533
CMCSA	Comcast	484	KO	Coca Cola	208	TWX	Time Warner	737
COF	Capital One	614	LMT	Lock'dMartn	376	TXN	Texas Inst	367
COP	Conocophillips	291	LOW	Lowe's	521	UNH	UnitedHealth	632
COST	Costco	533	MA	Master card	615	UPS	United Parcel	451
CPB	Campbell	203	MCD	MaDonald	581	USB	US Bancorp	602
CSCO	Cisco	367	MDT	Medtronic	384	UTX	United Tech	372
CVS	CVS	591	MET	Metlife Inc.	671	VZ	Verizon	481
CVX	Chevron	291	MMM	3M	384	WAG	Walgreen	591
DD	DuPont	289	MO	Altria Group	211	WFC	Wells Fargo	602
DELL	Dell	357	PM	Philip Morris	211	WMB	Williams	492
DIS	Walt Disney	799	MON	Monsanto	287	WMT	WalMart	533
DOW	Dow Chem	282	MRK	Merck	283	WY	Weyerhauser	241
DVN	Devon Energy	131	MS	MorganStanley	671	XOM	Exxon	291
EMC	EMC	357	MSFT	Microsoft	737	XRX	Xerox	357
ETR	ENTERGY	491						

	Description	Num		Description	Num
SIC 1	Mining, construct.	6	SIC 5	Trade	8
SIC 2	Manuf: food, furn.	26	SIC 6	Finance, Ins	18
SIC 3	Manuf: elec, mach	25	SIC 7	Services	6
SIC 4	Transprt, comm's	11	ALL		100

NOTE: This table presents the ticker symbols, names, and three-digit SIC codes of the 100 stocks used in the empirical analysis of this article. The lower panel reports the number of stocks in each one-digit SIC group.

returns. The lagged market residual is also found to be important for volatility in many cases, with the null that $\alpha_{mi} = \gamma_{mi} = 0$ being rejected at the 5% level for 32 stocks.

In the lower panel of Table 2 we present summary statistics for four measures of dependence between pairs of standardized residuals: linear correlation, rank correlation, average upper and lower 1% tail dependence (equal to $(\tau_{0.99} + \tau_{0.01})/2$), and the difference in upper and lower 10% tail dependence (equal to $\tau_{0.90} - \tau_{0.10}$). The two correlation statistics measure the sign and strength of dependence, the third and fourth statistics measure the strength and symmetry of dependence in the tails. The two correlation measures are similar, and are 0.42 and 0.44 on average. Across all 4950 pairs of assets the rank correlation varies from 0.37 to 0.50 from the 25th and 75th percentiles of the cross-sectional distribution, indicating the presence of mild heterogeneity in the correlation coefficients. The 1% tail dependence measure is 0.06 on average, and varies from 0.00 to 0.07 across the interquartile range. The difference in the 10% tail dependence measures is negative on average, and indeed is negative for over 75% of the pairs of stocks, strongly indicating asymmetric dependence between these stocks.

In Figure 4 we present the "scree" plot of eigenvalues of the rank correlation matrix of the standardized residuals, motivated by the discussion in Section 2.4. This plot shows that the first three eigenvalues are very large, all greater than four, indicating the presence of multiple common factors in the copula. The next five eigenvalues are all appreciably above one, while the ninth and tenth eigenvalues are just above one. Thus, the estimator proposed in Proposition 4 would suggest that 10 common factors are required, although taking estimation error into account we might suspect that only eight are needed. We investigate these suggestions more formally below.

Table 2. Summary statistics

	Cross-sectional distribution					
	Mean	5%	25%	Median	75%	95%
Mean	0.0004	−0.0003	0.0001	0.0003	0.0006	0.0013
Std dev	0.0287	0.0153	0.0203	0.0250	0.0341	0.0532
Skewness	0.3458	−0.4496	−0.0206	0.3382	0.6841	1.2389
Kurtosis	11.3839	5.9073	7.5957	9.1653	11.4489	19.5939
ϕ_0	0.0004	−0.0004	0.0001	0.0004	0.0006	0.0013
ϕ_1	−0.0345	−0.2045	−0.0932	−0.0238	0.0364	0.0923
ϕ_m	−0.0572	−0.2476	−0.1468	−0.0719	0.0063	0.1392
$\omega \times 1000$	0.0126	0.0024	0.0050	0.0084	0.0176	0.0409
β	0.8836	0.7983	0.8639	0.8948	0.9180	0.9436
α	0.0240	0.0000	0.0000	0.0096	0.0354	0.0884
γ	0.0593	0.0000	0.0017	0.0396	0.0928	0.1628
α_m	0.0157	0.0000	0.0000	0.0000	0.0015	0.0646
γ_m	0.1350	0.0000	0.0571	0.0975	0.1577	0.3787
ρ	0.4155	0.2643	0.3424	0.4070	0.4749	0.5993
ρ_s	0.4376	0.2907	0.3690	0.4292	0.4975	0.6143
$(\tau_{0.99} + \tau_{0.01})/2$	0.0572	0.0000	0.0000	0.0718	0.0718	0.1437
$(\tau_{0.90} - \tau_{0.10})$	−0.0922	−0.2011	−0.1293	−0.0862	−0.0431	0.0144

NOTE: This table presents some summary statistics of the daily equity returns data used in the empirical analysis. The top panel presents simple unconditional moments of the daily return series. The second panel presents summaries of the estimated AR(1)–GJR-GARCH(1,1) models estimated on these returns. The lower panel presents linear correlation, rank correlation, average 1% upper and lower tail dependence, and the difference between the 10% tail dependence measures, computed using the standardized residuals from the estimated AR–GJR-GARCH model. The columns present the mean and quantiles from the cross-sectional distribution of the measures listed in the rows. The top two panels present summaries across the $N = 100$ marginal distributions, while the lower panel presents a summary across the $N(N-1)/2 = 4950$ distinct pairs of stocks.

4.1 Results From Equidependence Copula Specifications

We now present our first empirical results on the dependence structure of these 100 stock returns: the estimated parameters of eight different models for the copula. In this section we consider an "equidependence" model, similar to the equicorrelation model of Engle and Kelly (2012), where we assume a single common factor and impose that all assets have the same coefficient on the common factor. This is clearly a restrictive model, and we test whether it is rejected by the data below.

We consider four existing copulas: the Clayton copula, the Normal copula, the Student's t copula, and the Skew t copula, with equicorrelation imposed on the latter three models (the Clayton copula implies equicorrelation by construction), and four factor copulas, described by the distributions assumed for the common factor and the idiosyncratic shock: t -Normal, Skew t-Normal, t-t, Skew t-t. All models are estimated using the SMM-type method described in Section 3. The value of the SMM objective function at the estimated parameters, Q_{SMM}, is presented for each model, along with the p-value from the J-test of the over-identifying restrictions. Standard errors are based on 1000 bootstraps to estimate $\Sigma_{T,S}$, and step size $\varepsilon_T = 0.1$ to compute $\hat{G}$. The rank dependence measures that are used in the SMM estimation of this model are presented in the Appendix.

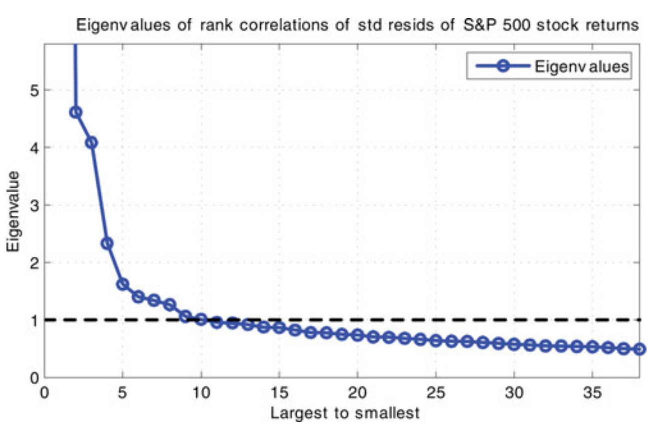

Figure 4. Plot of the ordered eigenvalues of the sample rank correlation matrix of the estimated standardized residuals. The largest eigenvalue is much larger than 5 and is truncated, and the horizontal axis is truncated at 38 for clarity.

Table 3 reveals that the coefficient on the common factor, β, is estimated by all models to be around 0.95, implying an average correlation coefficient of around 0.47. The estimated inverse degrees of freedom parameter in these models is around 1/25, and the standard errors on ν^{-1} reveal that this parameter is significant[7] at the 10% level for the three models that allow for asymmetric dependence, but not significant for the three models that impose symmetric dependence. The asymmetry parameter, λ, is significantly negative in all models in which it is estimated, with t-statistics ranging from 2.1 to 4.4. This implies that the dependence structure between these stock returns is significantly asymmetric, with large crashes being more likely than large booms. Other papers have considered equicorrelation models for the dependence between large collections of stocks, see

[7]Note that the case of zero tail dependence corresponds to $\nu_z^{-1} = 0$, which is on the boundary of the parameter space, implying that a standard t-test is strictly not applicable. In such cases the squared t-statistic no longer has an asymptotic χ_1^2 distribution under the null, rather it is distributed as an equal-weighted mixture of a χ_1^2 and χ_0^2; see Gouriéroux and Monfort (1996, chap. 21). The 90% and 95% critical values for this distribution are 1.64 and 2.71, which correspond to t-statistics of 1.28 and 1.65.

Table 3. Estimation results for daily returns on S&P 100 stocks

	β		ν^{-1}		λ			
	Est	Std Err	Est	Std Err	Est	Std Err	Q_{SMM}	p-Value
Clayton[†]	0.6017	0.0345	—	—	—	—	0.0449	0.0000
Normal	0.9534	0.0311	—	—	—	—	0.0090	0.0000
Student's t	0.9268	0.0296	0.0272	0.0292	—	—	0.0119	0.0000
Skew t	0.8196	0.0557	0.0532	0.0133	−8.3015	4.0202	0.0010	0.0020
Factor $t - N$	0.9475	0.0293	0.0233	0.0325	—	—	0.0101	0.0000
Factor skew $t - N$	0.9463	0.0299	0.0432	0.0339	−0.2452	0.0567	0.0008	0.0002
Factor $t - t$	0.9503	0.0311	0.0142	0.0517	—	—	0.0098	0.0000
Factor skew $t - t$	0.9375	0.0314	0.0797	0.0486	−0.2254	0.0515	0.0007	0.0005

NOTE: This table presents estimation results for various copula models applied to 100 daily stock returns over the period April 2008 to December 2010. Estimates and asymptotic standard errors for the copula model parameters are presented, as well as the value of the SMM objective function at the estimated parameters and the p-value of the overidentifying restriction test. Note that the parameter λ lies in $(-1, 1)$ for the factor copula models, but in $(-\infty, \infty)$ for the Skew t copula; in all cases the copula is symmetric when $\lambda = 0$.
[†]Note that the parameter of the Clayton copula is not β but we report it in this column for simplicity.

Engle and Kelly (2012) for example, but empirically showing the importance of allowing the implied common factor to be fat tailed and asymmetric is novel.

Figure 5 exploits the high-dimensional nature of our analysis, and plots the expected proportion of "crashes" in the remaining $(100 - j)$ stocks, conditional on observing a crash in j stocks. We consider a "crash" defined as a once-in-a-month (1/22, around 4.6%) event and as a once-in-a-quarter (1/66, around 1.5%) event. We obtain pointwise (in j) 90% bootstrap confidence intervals for these estimates based on the theory in Rémillard (2010); see Patton (2012) for discussion. For once-in-a-month crashes, the observed proportions track the Skew t-t factor copula well for j up to around 25 crashes, and again for j of around 70. For j in between 30 and 65 the Normal copula appears to fit quite well. For once-in-a-quarter crashes, displayed in the lower panel of Figure 5, the empirical plot tracks that for the Normal copula well for j up to around 30, but for $j = 35$ the empirical plot jumps and follows the Skew t-t factor copula. Thus, it appears that the Normal copula may be adequate for modeling moderate tail events, but a copula with greater tail dependence (such as the Skew t-t factor copula) is needed for more extreme tail events. It is worth noting, however, that we have few observations in our sample for these extreme tail events, and thus the confidence intervals are quite wide, making it difficult to make precise statements about relative fit.

The last two columns of Table 3 report the value of the objective function (Q_{SMM}) and the p-value from a test of the over-identifying restrictions. The Q_{SMM} values reveal that the three models that allow for asymmetry (Skew t copula, and the two Skew t factor copulas) out perform all the other models, and reinforce the above conclusion that allowing for a skewed common factor is important for this collection of assets. The p-values, however, are near zero for all models, indicating that none of them pass this specification test. Two likely sources of these rejections are the assumption of equidependence, which was shown in the summary statistics in Table 2 to be questionable for this large set of stock returns, and the assumption of a single common factor, which is not consistent with the "scree" plot in Figure 4. We relax both of these assumptions in the next section.

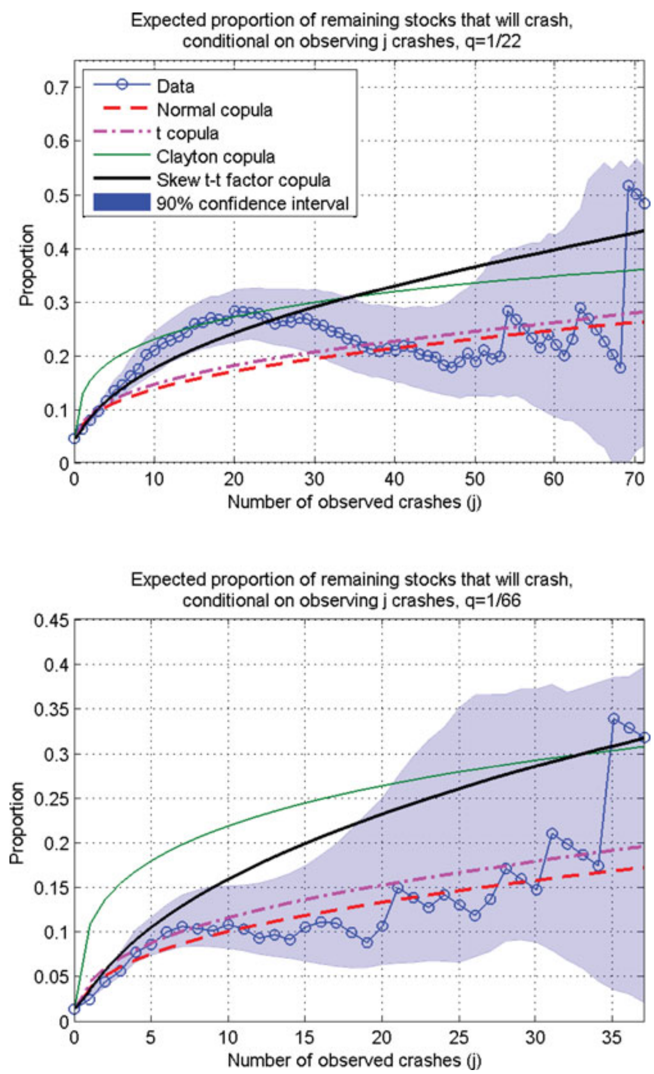

Figure 5. Conditional on observing j out of 100 stocks crashing, this figure presents the proportion of the remaining (100 j) stocks that will crash. "Crash" events are defined as returns in the lower 1/22 (upper panel) and 1/66 (lower panel) tail. Note that the horizontal axes in these two panels are different, due to the limited information in the joint tails.

Table 4. Estimation results for daily returns on S&P 100 stocks for multi-factor copula models

	Normal		Student's t		Skew t		Factor $t-t$		Factor skew $t-t$	
	Est	Std Err	Est	Std Err	Est	Std Err	Est	Std Err	Est	Std Err
ν^{-1}	—	—	0.0728	0.0269	0.0488	0.0069	0.0663	0.0472	0.0992	0.0455
λ	—	—	—	—	−9.6597	1.0860	—	—	−0.2223	0.0550
β_1	1.3027	0.0809	1.2773	0.0754	1.1031	0.1140	1.2936	0.0800	1.2457	0.0839
β_2	0.8916	0.0376	0.8305	0.0386	0.7343	0.0733	0.8528	0.0362	0.8847	0.0392
β_3	0.9731	0.0363	0.9839	0.0380	0.9125	0.0638	1.0223	0.0389	1.0320	0.0371
β_4	0.9426	0.0386	0.8751	0.0367	0.7939	0.0715	0.9064	0.0375	0.9063	0.0415
β_5	1.0159	0.0555	0.9176	0.0523	0.8171	0.0816	0.9715	0.0527	0.9419	0.0551
β_6	1.1018	0.0441	1.0573	0.0435	0.9535	0.0733	1.0850	0.0441	1.0655	0.0457
β_7	1.0954	0.0574	1.0912	0.0564	1.0546	0.0839	1.1057	0.0535	1.1208	0.0601
γ_1	1.0339	0.0548	0.9636	0.0603	1.0292	0.0592	1.0566	0.0595	1.0892	0.0582
γ_2	0.4318	0.0144	0.3196	0.0388	0.3474	0.0411	0.3325	0.0215	0.2201	0.0457
γ_3	0.4126	0.0195	0.3323	0.0394	0.2422	0.0458	0.2147	0.0727	0.1701	0.0996
γ_4	0.4077	0.0235	0.3726	0.0328	0.3316	0.0289	0.3470	0.0273	0.2740	0.0495
γ_5	0.4465	0.0403	0.5851	0.0300	0.5160	0.0324	0.5214	0.0312	0.5459	0.0448
γ_6	0.6122	0.0282	0.5852	0.0351	0.5581	0.0286	0.5252	0.0274	0.5686	0.0407
γ_7	0.5656	0.0380	0.5684	0.0464	0.2353	0.0889	0.3676	0.0434	0.3934	0.0548
Q_{SMM}	0.1587		0.1567		0.0266		0.1391		0.0189	
J p-value	0.0000		0.0000		0.0437		0.0000		0.0722	
$\gamma_i = 0\ \forall i$	0.0000		0.0000		0.0000		0.0000		0.0000	
$\beta_i = 0\ \forall i$	0.0000		0.0000		0.0000		0.0000		0.0000	
$\beta_i = \beta_j, \gamma_i = \gamma_j \forall i, j$	0.0000		0.0000		0.0000		0.0000		0.0000	

NOTE: This table presents estimation results for various multi-factor copula models applied to filtered daily returns on collections of 100 stocks over the period April 2008 to December 2010. Estimates and asymptotic standard errors for the model parameters are presented. Note that the parameter λ lies in $(-1, 1)$ for the factor copula models, but in $(-\infty, \infty)$ for the Skew t copula; in all cases the copula is symmetric when $\lambda = 0$. The bottom three rows present p-values from tests of constraints on the coefficients on the factors.

4.2 Results From Multi-Factor Copula Specifications

In response to the rejection of the copula models based on equidependence, we now consider a generalization to allow for heterogeneous dependence. We propose a multi-factor model that allows for a common, market-wide, factor, and a set of industry factors. We use the first digit of SIC to form seven groups of stocks; see Table 1. The model we consider is the copula generated by the following structure:

$$\begin{aligned} X_i &= \beta_{S(i)} Z_0 + \gamma_{S(i)} Z_{S(i)} + \varepsilon_i,\ i = 1, 2, \ldots, 100 \\ Z_0 &\sim \text{skew } t\,(\nu, \lambda) \\ Z_S &\sim \text{iid } t\,(\nu),\ S = 1, 2, \ldots, 7;\ Z_S \perp\!\!\!\perp Z_0 \forall S \\ \varepsilon_i &\sim \text{iid } t\,(\nu),\ i = 1, 2, \ldots, 100; \varepsilon_i \perp\!\!\!\perp Z_j \forall i, j, \end{aligned} \tag{18}$$

where $S(i)$ is the SIC group for stock i. There are eight latent common factors in total in this model, but any given variable is only affected by two factors, simplifying its structure and reducing the number of free parameters. Note here we impose that the industry factors and the idiosyncratic shocks are symmetric, and only allow asymmetry in the market-wide factor, Z_0. It is feasible to consider allowing the industry factors to have differing levels of asymmetry, but we rule this out in the interests of parsimony. We impose that all stocks in the same SIC group have the same factor loadings, but allow stocks in different groups to have different factor loadings. This generates a "block equidependence" model that greatly increases the flexibility of the model, but without generating too many additional parameters to estimate. In total, this copula model has a total of 16 parameters, providing more flexibility than the three-parameter equidependence model considered in the previous section, but still more parsimonious (and tractable) than a completely unstructured approach to this 100-dimensional problem.[8]

The results of this model are presented in Table 4. The Clayton copula is not presented here as it imposes equidependence by construction, and so is not comparable to the other models. The estimated inverse degrees of freedom parameter, ν^{-1}, is around 1/14, which is larger and more significant than for the equidependence model, indicating stronger evidence of tail dependence. The asymmetry parameters are also larger (in absolute value) and more significantly negative in this more flexible model than in the equidependence model. It appears that when we add variables that control for intra-industry dependence (i.e., industry-specific factors), we find the market-wide common factor is more fat tailed and left skewed than when we impose a single factor structure.

Focusing on our preferred *Skew t-t* factor copula model, the coefficients on the market factor, β_i, range from 0.88 (for SIC group 2, Manufacturing: Food, apparel, etc.) to 1.25 (SIC group 1, Mining and construction), indicating the varying degrees of interindustry dependence. The coefficients on the industry factors, γ_i, measure the degree of additional intra-industry dependence, beyond that coming from the market-wide factor. These range from 0.17 to 1.09 for SIC groups 3 and 1, respectively. Even for the smaller estimates, these are significantly different

[8]We also considered a one-factor model that allowed for different factor loadings, generalizing the equidependence model of the previous section but simpler than this multi-factor copula model. That model provided a significantly better fit than the equidependence model, but was also rejected using the J-test of over-identifying restrictions, and so is not presented here to conserve space.

Table 5. Rank correlation and tail dependence implied by a multi-factor copula model

	SIC 1	SIC 2	SIC 3	SIC 4	SIC 5	SIC 6	SIC 7
				Rank correlation			
SIC 1	0.72						
SIC 2	0.41	0.44					
SIC 3	0.44	0.45	0.51				
SIC 4	0.41	0.42	0.45	0.46			
SIC 5	0.39	0.40	0.44	0.41	0.53		
SIC 6	0.42	0.43	0.47	0.43	0.42	0.58	
SIC 7	0.45	0.46	0.50	0.46	0.44	0.47	0.57
				Lower\Upper tail dependence			
SIC 1	0.99\0.74	0.02	0.07	0.02	0.03	0.09	0.13
SIC 2	0.70	0.70\0.02	0.02	0.02	0.02	0.02	0.02
SIC 3	0.92	0.70	0.92\0.07	0.02	0.03	0.07	0.07
SIC 4	0.75	0.70	0.75	0.75\0.02	0.02	0.02	0.02
SIC 5	0.81	0.70	0.81	0.75	0.81\0.03	0.03	0.03
SIC 6	0.94	0.70	0.92	0.75	0.81	0.94\0.09	0.09
SIC 7	0.96	0.70	0.92	0.75	0.81	0.94	0.96\0.14

NOTE: This table presents the dependence measures implied by the estimated skew $t-t$ factor copula model reported in Table 4. This model implies a block equidependence structure based on the industry to which a stock belongs, and the results are presented with intra-industry dependence in the diagonal elements, and cross-industry dependence in the off-diagonal elements. The top panel presents rank correlation coefficients based on 50,000 simulations from the estimated model. The bottom panel presents the theoretical upper tail dependence coefficients (upper triangle) and lower tail dependence coefficients (lower triangle) based on Propositions 2 and 3.

from zero, indicating the presence of industry factors beyond a common market factor. The intra- and interindustry rank correlations and tail dependence coefficients implied by this model[9] are presented in Table 5, and reveal the degree of heterogeneity and asymmetry that this copula captures: rank correlations range from 0.39 (for pairs of stocks in SIC groups 1 and 5) to 0.72 (for stocks within SIC group 1). The upper and lower tail dependence coefficients further reinforce the importance of asymmetry in the dependence structure, with lower tail dependence measures being substantially larger than upper tail measures: lower tail dependence averages 0.82 and ranges from 0.70 to 0.99, while upper tail dependence averages 0.07 and ranges from 0.02 to 0.74.

With this more flexible model we can test restrictions on the factor coefficients, to see whether the additional flexibility is required to fit the data. The p-values from these tests are in the bottom rows of Table 4. First, we can test whether all of the industry factor coefficients are zero, which reduces this model to a one-factor model with flexible weights. The p-values from these tests are zero to four decimal places for all models, providing strong evidence in favor of including industry factors. We can also test whether the market factor is needed given the inclusion of industry factors by testing whether all betas are equal to zero, and as expected this restriction is strongly rejected by the data. We further can test whether the coefficients on the market and industry factors are common across all industries, reducing this model to an equidependence model, and this too is strongly rejected. Finally, we use the J-test of over-identifying restrictions to check the specification of these models. Using this test, we see that the models that impose symmetry are strongly rejected. The Skew t copula has a p-value of 0.04, indicating a marginal rejection, and the Skew t-t factor copula performs best, passing this test at the 5% level, with a p-value of 0.07. It is worth noting that even this multi-factor specification is still restrictive, both in the number of assumed factors, and in that it imposes equidependence within industry groups. If the computational challenge of allowing for $\mathcal{O}(N)$ unknown parameters could be overcome, one would expect the goodness of fit to improve. We leave such an extension for future work.

Thus, it appears that a multi-factor model with heterogeneous weights on the factors, and that allows for positive tail dependence and stronger dependence in crashes than booms, is needed to fit the dependence structure of these 100 stock returns.

4.3 Measuring Systemic Risk: Marginal Expected Shortfall

The recent financial crisis has highlighted the need for the management and measurement of systemic risk; see Acharya et al. (2010) for discussion. Brownlees and Engle (2011) proposed a measure of systemic risk they call "marginal expected shortfall," or MES. It is defined as the expected return on stock i given that the market return is below some (low) threshold:

$$\mathrm{MES}_{it} = -E_{t-1}\left[r_{it} \mid r_{mt} < C\right] \tag{19}$$

An appealing feature of this measure of systemic risk is that it can be computed with only a bivariate model for the conditional distribution of (r_{it}, r_{mt}), and Brownlees and Engle (2011) proposed a semiparametric model based on a bivariate DCC-GARCH model to estimate it. A corresponding drawback of this measure is that by using a market index to identify periods of crisis, it may overlook periods with crashes in individual firms. With a model for the entire set of constituent stocks, such as the high-dimension copula models considered in this article, combined with standard AR-GARCH type models for the marginal distributions, we can estimate the MES measure proposed in Brownlees and Engle (2011), as well as alternative measures that use crashes in individual stocks as flags for peri-

[9]Rank correlations from this model are not available in closed form, and we use 50,000 simulations to estimate these. Upper and lower tail dependence coefficients are based on Propositions 2 and 3.

Table 6. Performance of methods for predicting systemic risk

	MSE		RelMSE	
Cut-off	2%	4%	2%	4%
	Marginal expected shortfall (MES)			
Brownlees-Engle	**0.9961**	**1.2023**	0.7169	0.3521
Historical	1.1479	1.6230	1.0308	0.4897
CAPM	1.1532	1.5547	0.9107	0.4623
Normal copula	1.0096	1.2521	0.6712	0.3420
t copula	1.0118	1.2580	0.6660	0.3325
Skew *t* copula	1.0051	1.2553	0.6030	0.3040
Skew $t-t$ factor copula	1.0012	1.2445	**0.5885**	**0.2954**
	k-Expected shortfall (*k*ES)			
Historical	1.1632	1.6258	1.4467	0.7653
Normal copula	1.0885	1.4855	1.3220	0.5994
t copula	1.0956	1.4921	1.4496	0.6372
Skew *t* copula	1.0898	1.4923	1.3370	0.5706
Skew $t-t$ factor copula	**1.0822**	**1.4850**	**1.1922**	**0.5204**

NOTE: This table presents the MSE (left panel) and relative MSE (right panel) for various methods of estimating measures of systemic risk. The top panel presents results for marginal expected shortfall (MES), defined in Equation (19), and the lower panel presents results for k-expected shortfall (kES), defined in Equation (20), with k set to 30. Two thresholds are considered, $C=-2\%$ and $C=-4\%$. There are 70 and 21 "event" days for MES under these two thresholds, and 116 and 36 "event" days for kES. The best-performing model for each threshold and performance metric is highlighted in bold.

ods of turmoil.[10] For example, one might consider the expected return on stock i conditional on k stocks in the market having returns below some threshold, a "kES" :

$$k\text{ES}_{it} = -E_{t-1}\left[r_{it}\left|\left(\sum_{j=1}^{N}\mathbf{1}\left\{r_{jt}<C\right\}\right)>k\right.\right] \qquad (20)$$

Brownlees and Engle (2011) proposed a simple method for ranking estimates of MES:

$$\text{MSE}_i = \frac{1}{T}\sum_{t=1}^{T}(r_{it}-\text{MES}_{it})^2\,\mathbf{1}\{r_{mt}<C\}$$

$$\text{RelMSE}_i = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_{it}-\text{MES}_{it}}{\text{MES}_{it}}\right)^2\mathbf{1}\{r_{mt}<C\} \qquad (21)$$

Corresponding metrics immediately follow for estimates of "kES."

In Table 6 we present the MSE and RelMSE for estimates of MES and kES, for threshold choices of 2% and 4%. We implement the model proposed by Brownlees and Engle (2011), as well as their implementations of a model based on the CAPM, and one based purely on rolling historical information. Along with these, we present results for four copulas: the Normal, Student's t, Skew t, and Skew t-t factor copula, all with the multi-factor structure from Section 4.2. In the upper panel of Table 6 we see that the Brownlees-Engle model performs the best for both thresholds under the MSE performance metric, with the Skew $t-t$ factor copula as the second-best performing model. Under the relative MSE metric, the factor copula is best performing model, for both thresholds, followed by the Skew t copula. Like Brownlees and Engle (2011), we find that the worst-performing methods under both metrics are the Historical and CAPM methods.

The lower panel of Table 6 presents the performance of various methods for estimating kES, with k set to 30.[11] This measure requires an estimate of the conditional distribution for the entire set of 100 stocks, and thus the CAPM and Brownlees-Engle methods cannot be applied. We evaluate the remaining five methods, and find that the Skew $t-t$ factor copula performs the best for both thresholds, under both metrics. Thus, our proposed factor copula model for high-dimensional dependence allows us to gain some insights into the structure of the dependence between this large collection of assets, and also provides improved estimates of measures of systemic risk.

5. CONCLUSION

While there are numerous bivariate copula specifications for applied researchers to use, there are very few copula models for high-dimensional applications. This article proposes new models for the copula of economic variables based on a latent factor structure, which is particularly attractive for high-dimensional applications. This class of models allows the researcher to increase or decrease the flexibility of the model according to the amount of data available and the dimension of the problem, and, importantly, to do so in a manner that is easily interpreted and understood. The factor copulas presented in this article do not generally have a closed-form likelihood, but we use extreme value theory to obtain new analytical results on their implied tail dependence, and we verify that simulation-based methods can reliably be used for estimation and specification testing in applications involving up to 100 variables.

We employ our proposed factor copulas to study daily returns on all 100 constituents of the S&P 100 index over the period 2008–2010, and find significant evidence of a skewed, fat-tailed common factor, which generates asymmetric dependence and tail dependence. Using a multi-factor copula, we find evidence of the importance of industry factors, which generates heterogeneous dependence. We also consider an application to the estimation of systemic risk, and we show that the proposed factor copula model provides superior estimates of two measures of systemic risk. An interesting avenue for future research is to compare the various recently proposed methods for modeling high-dimensional dependence, such as Aas et al. (2009), Christoffersen, et al. (2013), Krupskii and Joe (2013), Creal and Tsay (2015), and this article, in terms of both statistical fit and various economic measures of fit.

APPENDIX: CHOICE OF DEPENDENCE MEASURES FOR ESTIMATION

To implement the SMM estimator of these copula models we must first choose which dependence measures to use in the SMM estimation.

[10]Note that here we study 100 large cap firms, which are in some ways relatively homogeneous. In the systemic risk literature, a common question is whether crises in one subgroup of firms (e.g., large firms, or financial firms) spill over to another group (small firms, or nonfinancial firms). We do not pursue such a question here.

[11]We choose this value of k so that the number of identified "crisis" days is broadly comparable to the number of such days for MES. Results for alternative values of k are similar.

We draw on "pure" measures of dependence, in the sense that they are solely affected by changes in the copula, and not by changes in the marginal distributions. For examples of such measures, see Joe (1997, chap. 2) or Nelsen (2006, chap. 5). Our preliminary studies of estimation accuracy and identification lead us to use pair-wise rank correlation, and quantile dependence with $q = [0.05, 0.10, 0.90, 0.95]$, giving us five dependence measures for each pair of variables.

Let δ_{ij} denote one of the dependence measures (i.e., rank correlation or quantile dependence at different levels of q) between variables i and j, and define the "pair-wise dependence matrix":

$$D = \begin{bmatrix} 1 & \delta_{12} & \cdots & \delta_{1N} \\ \delta_{12} & 1 & \cdots & \delta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1N} & \delta_{2N} & \cdots & 1 \end{bmatrix}. \tag{A.1}$$

Where applicable, we exploit the (block) equidependence feature of the models in defining the "moments" to match. For the equidependence model in Section 4.1, the model implies equidependence, and we use as "moments" the average of these five dependence measures across all pairs, reducing the number of moments to match from $5N(N-1)/2$ to just 5:

$$\bar{\delta} \equiv \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\delta}_{ij} \tag{A.2}$$

For the multi-factor copula model in Section 4.2, we exploit the fact that (a) all variables in the same group exhibit equidependence, and (b) any pair of variables (i, j) in groups (r, s) has the same dependence as any other pair (i', j') in the same two groups (r, s). This allows us to average all intra- and intergroup dependence measures. Consider the following general design, where we have N variables, M groups, and k_m variables per group, where $\Sigma_{m=1}^{M} k_m = N$. Then decompose the $(N \times N)$ matrix D into sub-matrices according to the groups:

$$\underset{(N\times N)}{D} = \begin{bmatrix} D_{11} & D_{12}' & \cdots & D_{1M}' \\ D_{12} & D_{22} & \cdots & D_{2M}' \\ \vdots & \vdots & \ddots & \vdots \\ D_{1M} & D_{2M} & \cdots & D_{MM} \end{bmatrix}, \text{ where } D_{ij} \text{ is } \left(k_i \times k_j\right). \tag{A.3}$$

Then create a matrix of average values from each of these matrices, taking into account the fact that the diagonal blocks are symmetric:

$$\underset{(M\times M)}{D^*} = \begin{bmatrix} \delta_{11}^* & \delta_{12}^* & \cdots & \delta_{1m}^* \\ \delta_{12}^* & \delta_{22}^* & \cdots & \delta_{2m}^* \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{1m}^* & \delta_{2m}^* & \cdots & \delta_{mm}^* \end{bmatrix}, \tag{A.4}$$

where $\delta_{ss}^* \equiv \frac{2}{k_s(k_s-1)} \sum\sum \hat{\delta}_{ij}$, avg of *all upper triangle* values in D_{ss}, $\delta_{rs}^* = \frac{1}{k_r k_s} \sum\sum \hat{\delta}_{ij}$, avg of *all* elements in matrix $D_{rs}, r \neq s$.

Finally, create the vector of average measures $[\bar{\delta}_1^*, \ldots, \bar{\delta}_M^*]$, where

$$\bar{\delta}_i^* \equiv \frac{1}{M} \sum_{j=1}^{M} \delta_{ij}^*. \tag{A.5}$$

This gives as a total of M moments for each dependence measure, so $5M$ in total.

SUPPLEMENTAL MATERIALS

The supplemental file includes an appendix that contains all proofs and details of simulations used to study SMM estimation.

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