Properties of Optimal Forecasts

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Testing rationality and market efficiency

- Tests of market efficiency and investor rationality are usually done by testing properties of forecast errors

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- Tests of market efficiency and investor rationality are usually done by testing properties of forecast errors:
  - *Relating to the efficient markets hypothesis*: Cargill and Meyer (JF, 1980), De Bondt and Bange (JFQA, 1992), Mishkin (AER, 1981), *inter alia.*
  - *Relating to rationality of decision-makers*: Brown and Maital (EMA, 1981), Figlewski and Watchel (REStat, 1981), Keane and Runkle (AER, 1990), Lakonishok (JF, 1980), *inter alia.*
- All of these papers rely on properties of optimal forecasts derived assuming squared error loss:

The standard set-up

- The standard results on optimal forecasts were derived under the assumption that:

$$L\left(Y_{t+h}, \hat{Y}_{t+h,t}\right) = \left(Y_{t+h} - \hat{Y}_{t+h,t}\right)^2$$

$$\hat{Y}^*_{t+h,t} \equiv \arg\min_{\hat{y}} E_t\left[\left(Y_{t+h} - \hat{y}\right)^2\right]$$

- Which implies that:

$$\hat{Y}^*_{t+h,t} = E_t\left[Y_{t+h}\right]$$

Standard properties of optimal forecasts

- The properties of optimal forecasts in the standard set-up are:

1. Optimal forecasts are unbiased

2. Optimal h-step forecast errors are serially correlated only to lag (h-1)
   - So 1-step forecasts are have zero serial correlation

3. Optimal unconditional forecast error variance is an increasing function of the forecast horizon

Is MSE the right loss function?

- The assumption of MSE loss in economics has been questioned in the (mostly) recent literature:
  - Granger (1969), Granger and Newbold (1986), West, Edison and Cho (1996), Granger and Pesaran (2000), Pesaran and Skouas (2001).
- For example: financial analysts' forecasts have been found to be biased upwards
  - A result of analyst irrationality, or simply that the analyst is penalised more heavily for under-predictions than over-predictions?

What we do in this paper

- We extend the work of Christoffersen and Diebold (1997) and Granger (1969, 1999) to analytically consider the time series properties of optimal forecasts under asymmetric loss and nonlinear DGPs.

- We show that <u>all</u> the standard properties may be violated in quite reasonable situations
  - Thus the previous work on market efficiency and investor rationality may be disregarded if you do not believe in MSE loss

What we do in this paper (cont'd)

- We provide some general results on properties of optimal forecasts when the loss function is known, which may then be used in testing rationality

- We also provide some testable implications of forecast optimality that hold *without* knowledge of the forecaster's loss function

- Finally, we introduce a change of measure, from the objective to the "MSE-loss probability measure", under which the optimal forecast has the same properties as under MSE loss.

Notation and some assumptions

Y_{t+h}	the scalar random variable to be forecast
$\hat{Y}_{t+h,t}$	a forecast made at time t
$\hat{Y}^*_{t+h,t}$	the optimal forecast made at time t
$L = L\left(Y_{t+h}, \hat{Y}_{t+h,t}\right)$	the loss function
$e_{t+h,t} \equiv Y_{t+h} - \hat{Y}_{t+h,t}$	the forecast error
Ω_t	time t information set $\supseteq \sigma\left(Y_{t-j}; j \geq 0\right)$

$$\hat{Y}^*_{t+h,t} \equiv \arg\min_{\hat{y}} E\left[L\left(Y_{t+h}, \hat{y}\right) \middle| \Omega_t\right]$$

Properties in non-standard situations

1. Forecast error has zero conditional mean
   - *Granger (1969) and Christoffersen and Diebold (1997) showed that bias may be optimal under asym loss*
2. The optimal h - step forecast error exhibits zero serial correlation beyond the $(h - 1)^{th}$ lag.
   - *Right idea, but wrong object: standard forecast error is not (generally) the variable with zero serial correl.*
3. Unconditional forecast error variance is increasing in h.
   - *Variance is not (generally) right measure of forecast accuracy*

A counter-example

- We now present a realistic situation where all the standard properties of optimal forecasts and forecast errors break down.

- Our results are all analytical. We assume that the agent knows his loss function, *and* the DGP (including the parameters of the DGP)

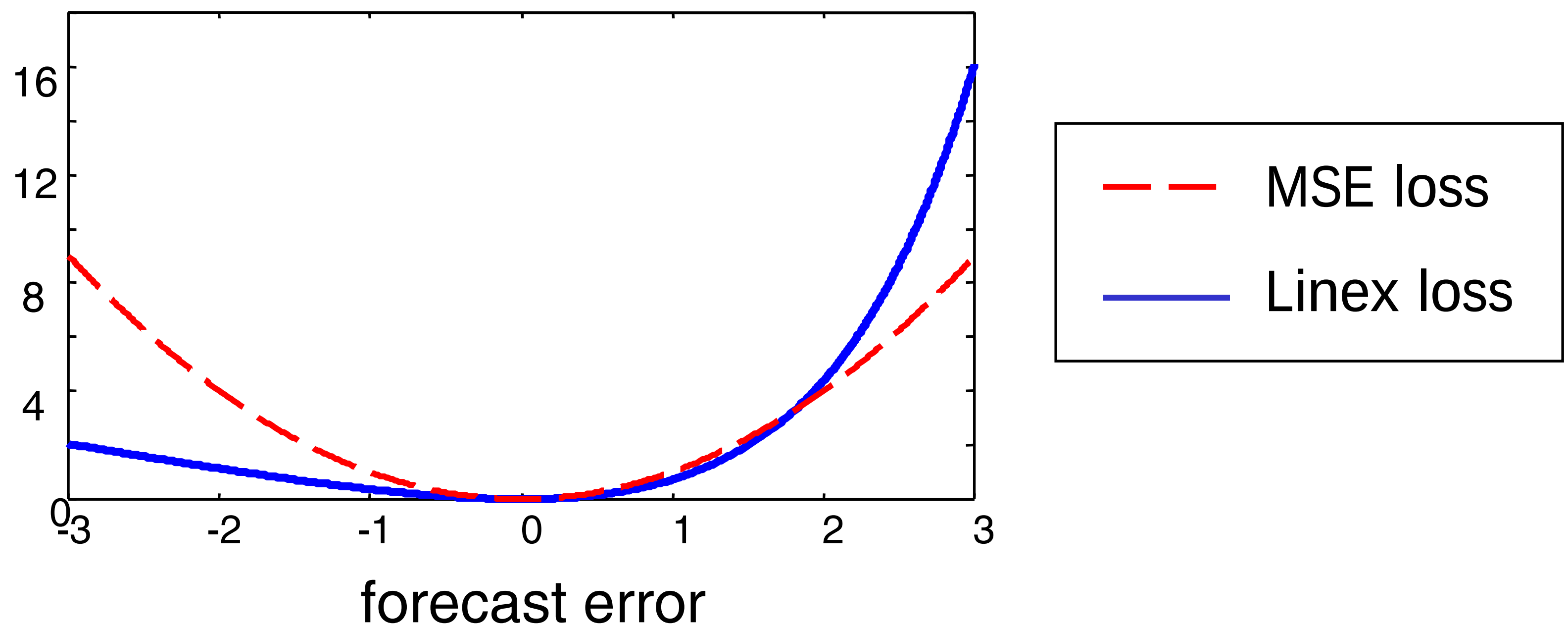
  $\rightarrow$ This agent is as optimal as can possibly be...

- We will define the loss function and DGP as follows:

Counter-example: loss function

- For tractability, we focus on the linear-exponential loss function of Varian (1975)

$$L\left(Y_{t+h},\hat{Y}_{t+h,t};a\right)=\exp\left\{a\left(Y_{t+h}-\hat{Y}_{t+h,t}\right)\right\}-a\left(Y_{t+h}-\hat{Y}_{t+h,t}\right)-1$$
$$=\exp\left\{ae_{t+h,t}\right\}-ae_{t+h,t}-1$$



Counter-example: DGP

- We consider a regime switching process, popular in both macroeconomics and finance

$$Y_{t+1} = \mu + \sigma_{st+1}\nu_{t+1}, \quad \nu_{t+1} \sim \text{iid } N(0,1)$$

$$S_{t+1} = \{1,2,\ldots,k\}$$

$$\Pr[S_{t+1} = j \mid S_t = i] = P_{[i,j]}$$

Counter-example: DGP

- For presentation purposes, I will focus on a particular case of the RS model:

$$Y_{t+1} = \mu + \sigma_{st+1} v_{t+1}, \quad v_{t+1} \sim iid\ N(0,1)$$

$$\mu = 0$$

$$\sigma = [0.5, 2]'$$

$$P = \begin{bmatrix} 0.95 & 0.05 \\ 0.10 & 0.90 \end{bmatrix}$$

$$\bar{\pi} = \left[\frac{2}{3}, \frac{1}{3} \right]'$$

1. First property: Bias

- Optimal *h* - step forecast in this special case is:

$$\hat{Y}^{*}_{t+h,t} = \mu + \frac{1}{a}\log\left(\hat{\pi}^{/}_{s_t|t} P^{h} \varphi\right)$$

$$\varphi = \exp\left\{0.5 a^{2} \sigma^{2}\right\}$$

1. First property: Bias

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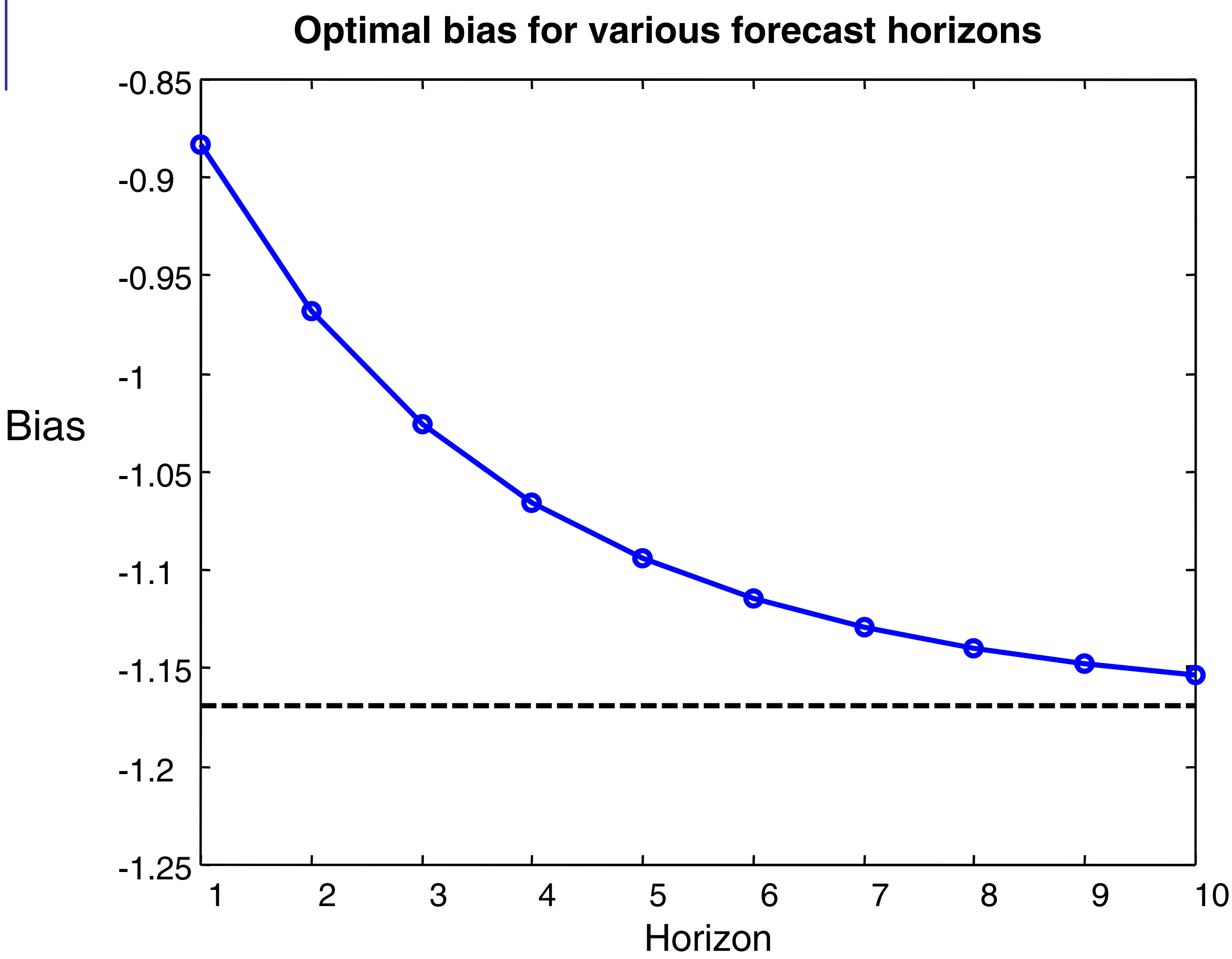
- which implies conditional and unconditional bias of:

$$E_t\left[e^{*}_{t+h,t}\right] = -\frac{1}{a}\log\left(\hat{\pi}^{/}_{s_t|t} P^h \varphi\right)$$

$$E\left[e^{*}_{t+h,t}\right] = -\frac{1}{a}\bar{\pi}'\log\left(P^h \varphi\right)$$

$$\rightarrow -\frac{1}{a}\log\left(\bar{\pi}'\varphi\right) \text{as } h \rightarrow \infty$$

1. Bias (cont'd)



Optimal bias for various forecast horizons
-0.85
-0.9
-0.95
-1
-1.05
-1.1
-1.15
-1.2
-1.25
Bias
1
2
3
4
5
6
7
8
9
10
Horizon

2. Second property: Serial correlation

- The j^th^-order serial correlation for the *h* – step forecast is given by:

$$Cov\left[e^{*}_{t+h,t}, e^{*}_{t+h-j,t-j}\right] = \bar{\pi}'\sigma^{2}1\{j=0\} + \frac{1}{a}\lambda'_{h}\left(\left(\bar{\pi}'\iota\right)\odot P^{j} - \bar{\pi}\bar{\pi}'\right)\lambda_{h}$$

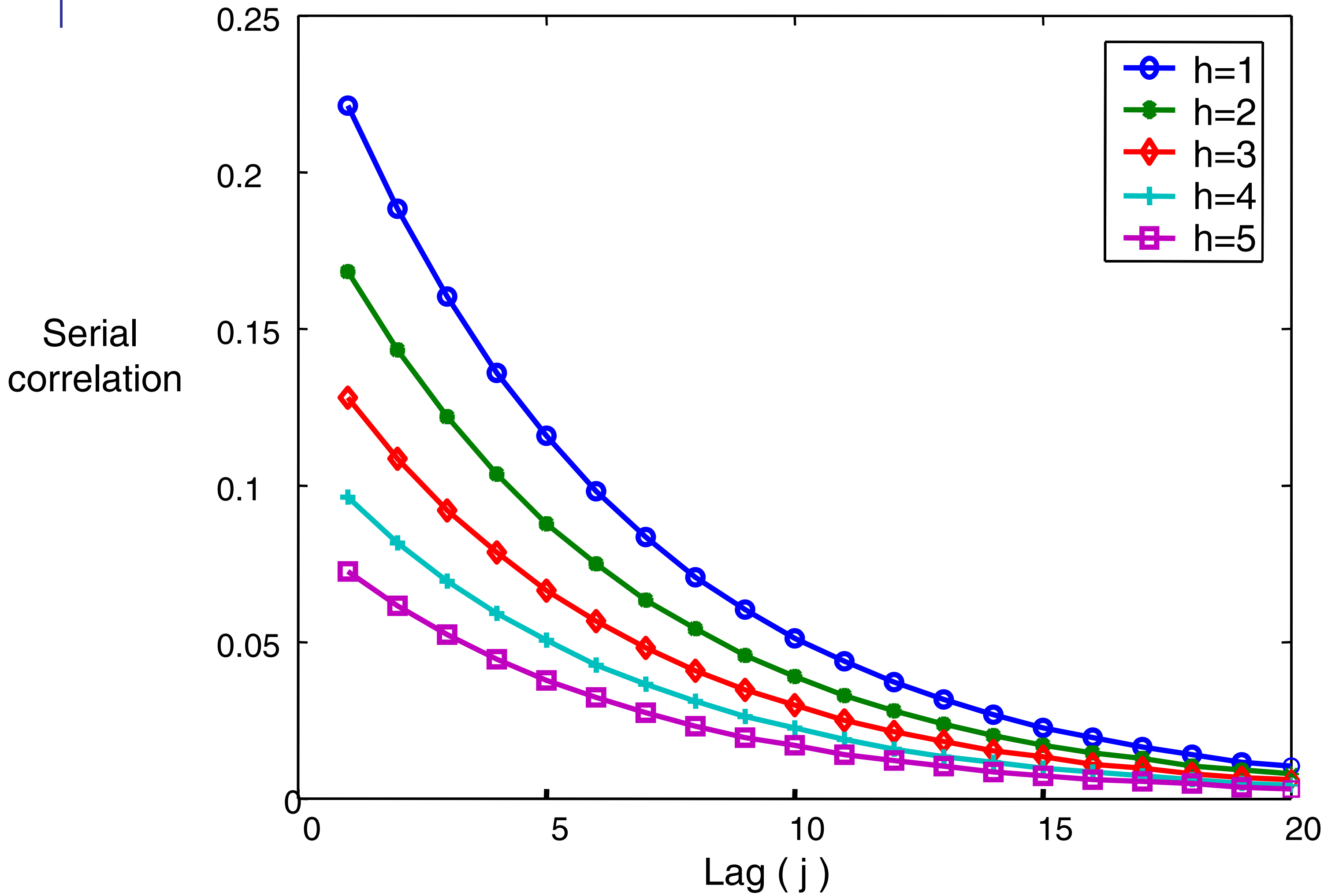
$$\to 0 \text{ as } j \to \infty$$

$$where\ \lambda_{h} = \log\left(P^{h}\phi\right)$$

- Notice that the only break-point is at j=0 $\Rightarrow$ serial correlation for $j > h\text{-}1$ may also be non-zero...

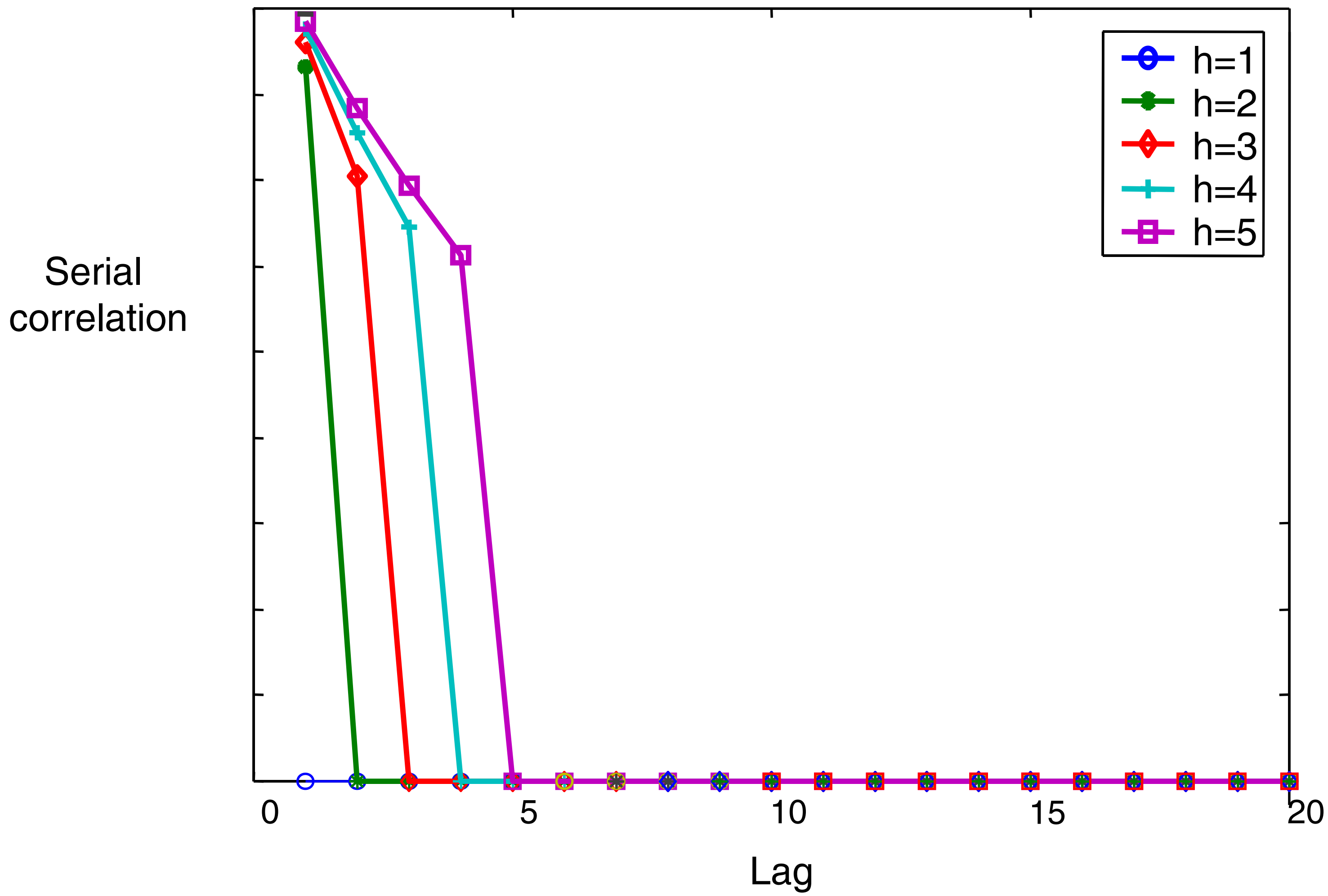
2. Serial correlation (cont'd)

Optimal forecast error autocorrelation function for various forecast horizons



2. Serial correlation (cont'd)

This is the shape that we would expect from standard results:



2. Serial correlation – intuition

- Some intuition for this result may be gleaned from a result of Christoffersen and Diebold, who show that:

$$\hat{Y}^{*}_{t+h,t} = E_t\left[Y_{t+h}\right] + \alpha_{t+h,t}$$

where $\alpha_{t+h,t}$ depends *only* on the time-varying moments of order higher than 1

- If $\alpha_{t+h,t}$ exihibits persistence, via its dependence on persistent second moments for example, then the forecast error may also exhibit persistence, ie serial correlation.

3. Third property: Forecast error variance

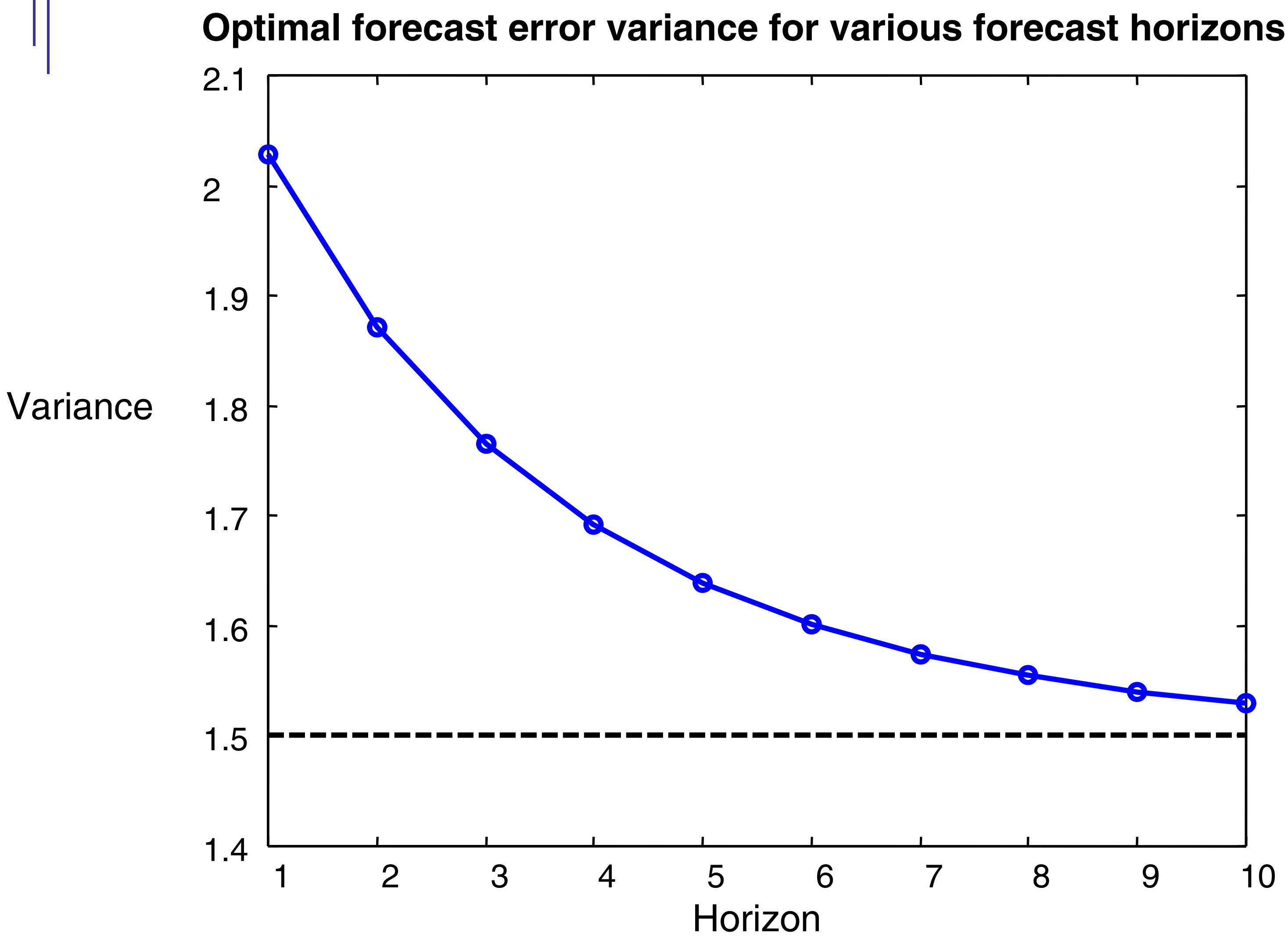
- The *conditional* forecast error variance can be increasing or decreasing in *h* under MSE loss – GARCH is a common example here

- We instead focus on *unconditional* forecast error variance as a function of *h*, which is non-decreasing for MSE loss. In the RS example it is:

$$V\left[e^{*}_{t+h,t}\right] = \bar{\pi}'\sigma^{2} + \frac{1}{a}\lambda'_{h}\left(\left(\bar{\pi}'\iota\right)\odot I - \bar{\pi}\bar{\pi}'\right)\lambda_{h}$$

$$\rightarrow \bar{\pi}'\sigma^{2} = V\left[Y_{t+h,t}\right] \text{ as } h \rightarrow \infty$$

3. Forecast error variance (cont'd)



3. Forecast error variance – intuition

- The main intuition here is that, in general, forecast error variance is *not* the right way to measure how difficult it is to forecast

- Given some loss function *L*, the right way to measure forecast accuracy is *expected loss*

- It happens that under MSE loss forecast error variance and expected loss coincide:

Expected forecast error loss and variance under MSE loss

- Under MSE loss:

$$L\left(Y_{t+h}, \hat{Y}^{*}_{t+h,t}\right) = \left(Y_{t+h} - \hat{Y}^{*}_{t+h,t}\right)^2 = e^{*2}_{t+h,t}$$

$$E\left[L\left(Y_{t+h}, \hat{Y}^{*}_{t+h,t}\right)\right] = E\left[e^{*2}_{t+h,t}\right] = V\left[e^{*}_{t+h,t}\right]$$

- And so it happens that here expected loss and error variance coincide. In general this is not the case.

A recap

- What we've shown up to this point:

1. Optimal forecast errors may have non-zero mean
2. Optimal forecast errors may be serially correlated
3. The forecast error variance may *decrease* with the forecast horizon

- But where are these "violations" coming from?

Causes of the violations

- Our counter-example involves *both:*
  - an asymmetric loss function, and
  - a non-linear DGP
- Earlier, we showed that the usual results hold under:
  - Squared error loss, and
  - any stationary DGP
- *What about the case of:*
  - *asymmetric loss and*
  - *a simple DGP, such as an ARMA?*

Asymmetric loss and DGP with mean-only dynamics

- Let $Y_{t+h} = E\left[Y_{t+h} \middle| \Omega_t\right] + \varepsilon_{t+h}\ ,\ \varepsilon_{t+h} | \Omega_t \sim D_h$

Asymmetric loss and DGP with mean-only dynamics

- Let $\mathrm{Y}_{\mathrm{t+h}} = \mathrm{E}\left[\mathrm{Y}_{\mathrm{t+h}} \middle| \Omega_{\mathrm{t}}\right] + \varepsilon_{\mathrm{t+h}}\ ,\ \varepsilon_{\mathrm{t+h}} \mid \Omega_{\mathrm{t}} \sim \mathrm{D}_{\mathrm{h}}$
- Let $L\left(Y_{t+h}, \hat{Y}_{t+h,t}\right) = L\left(Y_{t+h} - \hat{Y}_{t+h,t}\right) = L\left(e_{t+h,t}\right)$

Asymmetric loss and DGP with mean-only dynamics

- Let $\mathrm{Y}_{\mathrm{t+h}} = \mathrm{E}\left[\mathrm{Y}_{\mathrm{t+h}} \middle| \Omega_{\mathrm{t}}\right] + \varepsilon_{\mathrm{t+h}}\ ,\ \varepsilon_{\mathrm{t+h}} \mid \Omega_{\mathrm{t}} \sim \mathrm{D}_{\mathrm{h}}$
- Let $L\left(Y_{t+h}, \hat{Y}_{t+h,t}\right) = L\left(Y_{t+h} - \hat{Y}_{t+h,t}\right) = L\left(e_{t+h,t}\right)$

Then: $\hat{Y}^{*}_{t+h,t} = E_t\left[Y_{t+h}\right] + \alpha_h$

and

1. Optimal forcast is biased (bias is only a function of *h*, see Christoffersen and Diebold, 1997)

Asymmetric loss and DGP with mean-only dynamics

- Let $\mathrm{Y}_{\mathrm{t+h}} = \mathrm{E}\left[\mathrm{Y}_{\mathrm{t+h}} \middle| \Omega_{\mathrm{t}}\right] + \varepsilon_{\mathrm{t+h}} \,,\; \varepsilon_{\mathrm{t+h}} \,|\, \Omega_{\mathrm{t}} \sim \mathrm{D}_{\mathrm{h}}$
- Let $L\left(Y_{t+h}, \hat{Y}_{t+h,t}\right) = L\left(Y_{t+h} - \hat{Y}_{t+h,t}\right) = L\left(e_{t+h,t}\right)$

Then: $\hat{Y}^{*}_{t+h,t} = E_t\left[Y_{t+h}\right] + \alpha_h$

and

1. Optimal forcast is biased (bias is only a function of *h*, see Christoffersen and Diebold, 1997)
2. * The *h* - step forecast error has *MA*(*h-1*) ACF
3. * The forecast error variance is weakly increasing in *h*

h - step forecast error has *MA*(*h-1*) ACF

Proof:

$$Y_{t+h} = E_t[Y_{t+h}] + \varepsilon_{t+h}$$

$$\hat{Y}^*_{t+h,t} = E_t[Y_{t+h}] + \alpha_h$$

$$e^*_{t+h,t} = Y_{t+h} - \hat{Y}^*_{t+h,t}$$

$$= \varepsilon_{t+h} - \alpha_h$$

$$Cov[e^*_{t+h,t}, e^*_{t+h-j,t-j}] = Cov[\varepsilon_{t+h}, \varepsilon_{t+h-j}]$$

$$= 0 \; \forall \; \mathrm{j} \geq \mathrm{h}$$

$$\text{since } \varepsilon_{t+h} \mid \Omega_t \sim D_h$$

Interpretation

- This shows that the serial correlation properties are robust to the loss function under restrictions on the DGP
- This implies that if we can assume that there are only conditional mean dynamics, we can test for forecast optimality *without any knowledge of the forecaster's loss function.* This extends existing literature:

1. Assume MSE, allow arbitrary DGP
2. Elliott *et al.* (2002): assume loss function up to unknown parameter vector, assume linear forecast model

Error variance is weakly increasing in *h*

Proof:

$$Y_{t+h+j} = E_t[Y_{t+h+j}] + \eta_{t+h+j},\ \eta_{t+h+j} \mid \Omega_t \sim D_{h+j}$$

$$Y_{t+h+j} = E_{t+j}[Y_{t+h+j}] + \varepsilon_{t+h+j},\ \varepsilon_{t+h+j} \mid \Omega_{t+j} \sim D_h$$

$$e^*_{t+h+j,t} = \eta_{t+h+j} - \alpha_{h+j}$$

$$e^*_{t+h+j,t+j} = \varepsilon_{t+h+j} - \alpha_h$$

$$V_t[e^*_{t+h+j,t}] = \sigma^2_{h+j} = V[e^*_{t+h+j,t}]$$

$$V_t[e^*_{t+h+j,t+j}] = \sigma^2_h = V[e^*_{t+h+j,t+j}]$$

Want to show $\sigma^2_{h+j} \geq \sigma^2_h$

Error variance is weakly increasing in *h*

$$
\begin{aligned}
\sigma_{h+j}^2 &= V_t[e_{t+h+j,t}^*] \\
&= V_t[Y_{t+h+j} - E_t[Y_{t+h+j}]] \\
&= V_t[\varepsilon_{t+h+j} + E_{t+j}[Y_{t+h+j}] - E_t[Y_{t+h+j}]] \\
&= V_t[\varepsilon_{t+h+j}] + V_t[E_{t+j}[Y_{t+h+j}]] + 2Cov_t[\varepsilon_{t+h+j}, E_{t+j}[Y_{t+h+j}]] \\
&= V_t[\varepsilon_{t+h+j}] + V_t[E_{t+j}[Y_{t+h+j}]] \\
&\geq V_t[\varepsilon_{t+h+j}] \\
&= \sigma_h^2
\end{aligned}
$$

Some intuition

- What's behind the results violating standard properties?

  $\rightarrow$ A mis-match of the loss function and MSE
  $\rightarrow$ Dynamics in the process beyond the mean

- The standard results all follow from the use of the squared error as the loss function, and when a different loss is employed we find "violations"

- So what are the properties optimal forecasts in general situations?

The “generalised forecast error”

- Granger (1999) proposes looking at a generalised forecast error. We modify his definition slightly.

- The generalised forecast error comes out of the first-order condition for forecast optimality:

$$\hat{Y}^{*}_{t+h,t} \equiv \arg\min_{\hat{y}} E\left[L\left(Y_{t+h}, \hat{y}\right)\middle|\Omega_t\right]$$

$$FOC: \frac{\partial E\left[L\left(Y_{t+h}, \hat{Y}^{*}_{t+h,t}\right)\middle|\Omega_t\right]}{\partial \hat{y}} = 0$$

The generalised forecast error

- A natural alternative to the standard forecast error is thus:

$$\psi^*_{t+h,t} = \frac{\partial L\left(Y_{t+h}, \hat{Y}^*_{t+h,t}\right)}{\partial \hat{y}}$$

- Notice that under MSE the generalised and standard forecast errors are related by:

$$\psi^*_{t+h,t} = -2e^*_{t+h,t}$$

- The properties assigned to $e^*_{t+h,t}$ are actually properties of $\psi^*_{t+h,t}$ more generally

Properties of optimal forecast errors under general conditions

- By using the generalised forecast error and the arbitrary loss function *L* we can provide properties of optimal forecasts more generally:

1. $E_t\left[\psi^*_{t+h,t}\right] = E\left[\psi^*_{t+h,t}\right] = 0$.

2. The generalised forecast error from an optimal *h* – step forecast has the same ACF as some MA(*h* – 1) process.

3. Unconditional expected loss is non-decreasing in *h*.

1. Mean of generalised forecast error

- By the first-order condition for an optimal forecast we have:

$$0 = \frac{\partial E_t \left[L\left(Y_{t+h}, \hat{Y}^*_{t+h,t} \right) \right]}{\partial \hat{y}} = E_t \left[\psi^*_{t+h,t} \right]$$

$$so\ E\left[\psi^*_{t+h,t} \right] = 0 \text{ by the L.I.E.}$$

(assuming that we can interchange the differentiation and expectation operators.)

2. Serial correlation

- Instead of referring to an MA(h – 1) process, we show that the generalised forecast errors are uncorrelated for lags >h-1, ie, it has the *same ACF as some* MA(h-1) process.

$$E\left[\psi^*_{t+h,t} \middle| \Omega_t\right] = 0 \Rightarrow E\left[\psi^*_{t+h,t} \cdot \gamma\left(\psi^*_{t+h-j,t-j}\right)\right] = 0$$

$$\text{for all } j \geq h \text{ and any function } \gamma$$

$$\Rightarrow \left(\psi^*_{t+h,t}, \psi^*_{t+h-j,t-j}\right) \text{are uncorrelated for all } j \geq h$$

3. Expected loss

- The unconditional expected loss from an optimal forecast is non-decreasing in the forecast horizon.

By the optimality of $\hat{Y}^*_{t+h,t}$ we have, for all $j \geq 0$,

$$E\left[L\left(Y_{t+h}, \hat{Y}^*_{t+h,t}\right) \middle| \Omega_t\right] \leq E\left[L\left(Y_{t+h}, \hat{Y}^*_{t+h,t-j}\right) \middle| \Omega_t\right]$$

$$E\left[L\left(Y_{t+h}, \hat{Y}^*_{t+h,t}\right)\right] \leq E\left[L\left(Y_{t+h}, \hat{Y}^*_{t+h,t-j}\right)\right] \text{by the L.I.E.}$$

$$= E\left[L\left(Y_{t+h+j}, \hat{Y}^*_{t+h+j,t}\right)\right]$$

Properties under a different measure

- Here we propose retaining the object of interest, but changing its probability distribution

- This is akin to moving from the objective to the risk-neutral measure in asset pricing.
  - After a change of measure, assets may be priced as though agents are risk neutral

- Following our change of measure, the optimal forecast errors have the same properties as under MSE loss
  - So bias and serial correlation may be tested, for example

A change of measure - assumptions

- Suppose:

$$\frac{\partial L(Y_{t+h}, \hat{Y}_{t+h,t})}{\partial \hat{y}} \geq 0 \text{ if } Y_{t+h} - \hat{Y}_{t+h,t} < 0$$

$$\frac{\partial L(Y_{t+h}, \hat{Y}_{t+h,t})}{\partial \hat{y}} \leq 0 \text{ if } Y_{t+h} - \hat{Y}_{t+h,t} > 0$$

$$0 < \left| E_t \left[\frac{1}{e_{t+h}} \frac{\partial L(Y_{t+h}, \hat{Y}_{t+h,t})}{\partial \hat{y}} \right] \right| < \infty$$

A change of measure - formula

- Notice that:

$$f_{e_{t+h,t}}(e;\hat{Y}_{t+h,t}) = f_{t+h,t}(Y_{t+h} - \hat{Y}_{t+h,t}) \; \forall e, \hat{Y}_{t+h,t}$$

A change of measure - formula

- Notice that:

$$f_{e_{t+h,t}}(e;\hat{Y}_{t+h,t}) = f_{t+h,t}(Y_{t+h} - \hat{Y}_{t+h,t}) \; \forall e, \hat{Y}_{t+h,t}$$

Let the "MSE-loss probability measure" be defined as:

$$f^*_{e_{t+h,t}}(e;\hat{Y}_{t+h,t}) = \frac{\dfrac{1}{e} \cdot \left. \dfrac{\partial L(Y,\hat{Y}_{t+h,t})}{\partial \hat{y}} \right|_{Y=\hat{Y}_{t+h,t}+e} \cdot f_{e_{t+h,t}}(e;\hat{Y}_{t+h,t})}{E_t\left[\dfrac{1}{e_{t+h,t}} \dfrac{\partial L(Y_{t+h},\hat{Y}_{t+h,t})}{\partial \hat{y}}\right]}$$

Change of measure - validity

- Must show that the new measure is a valid probability measure.

- By assumption

$$\frac{1}{Y_{t+h} - \hat{Y}_{t+h,t}} \cdot \frac{\partial L(Y_{t+h}, \hat{Y}_{t+h,t})}{\partial \hat{y}} \leq 0 \ \forall \ Y_{t+h}, \hat{Y}_{t+h,t}$$

- So the denominator is negative, and numerator is weakly negative, thus entire expression is weakly positive

- By construction it integrates to 1, so it is a valid pdf.

Mean under MSE-loss measure

Proposition: Under the MSE-loss probability measure the optimal forecast error has conditional mean zero.

Proof:

$$E_t^*[e_{t+h,t}^*] = A^{-1} \cdot \int e \frac{1}{e} \cdot \frac{\partial L(Y, \hat{Y}_{t+h,t}^*)}{\partial \hat{y}} \Bigg|_{Y=\hat{Y}_{t+h,t}^*+e} f_{e_{t+h,t}}(e; \hat{Y}_{t+h,t}^*)$$

Mean under MSE-loss measure

Proposition: Under the MSE-loss probability measure the optimal forecast error has conditional mean zero.

Proof:

$$E_t^*[e_{t+h,t}^*] = A^{-1} \cdot \int e \frac{1}{e} \cdot \frac{\partial L(Y, \hat{Y}_{t+h,t}^*)}{\partial \hat{y}} \Bigg|_{Y=\hat{Y}_{t+h,t}^*+e} f_{e_{t+h,t}}(e; \hat{Y}_{t+h,t}^*)$$

$$= A^{-1} \cdot \int \frac{\partial L(Y, \hat{Y}_{t+h,t}^*)}{\partial \hat{y}} \Bigg|_{Y=\hat{Y}_{t+h,t}^*+e} f_{e_{t+h,t}}(e; \hat{Y}_{t+h,t}^*)$$

$$= 0$$

by the first-order condition for forecast optimality.

Serial correl under MSE-loss measure

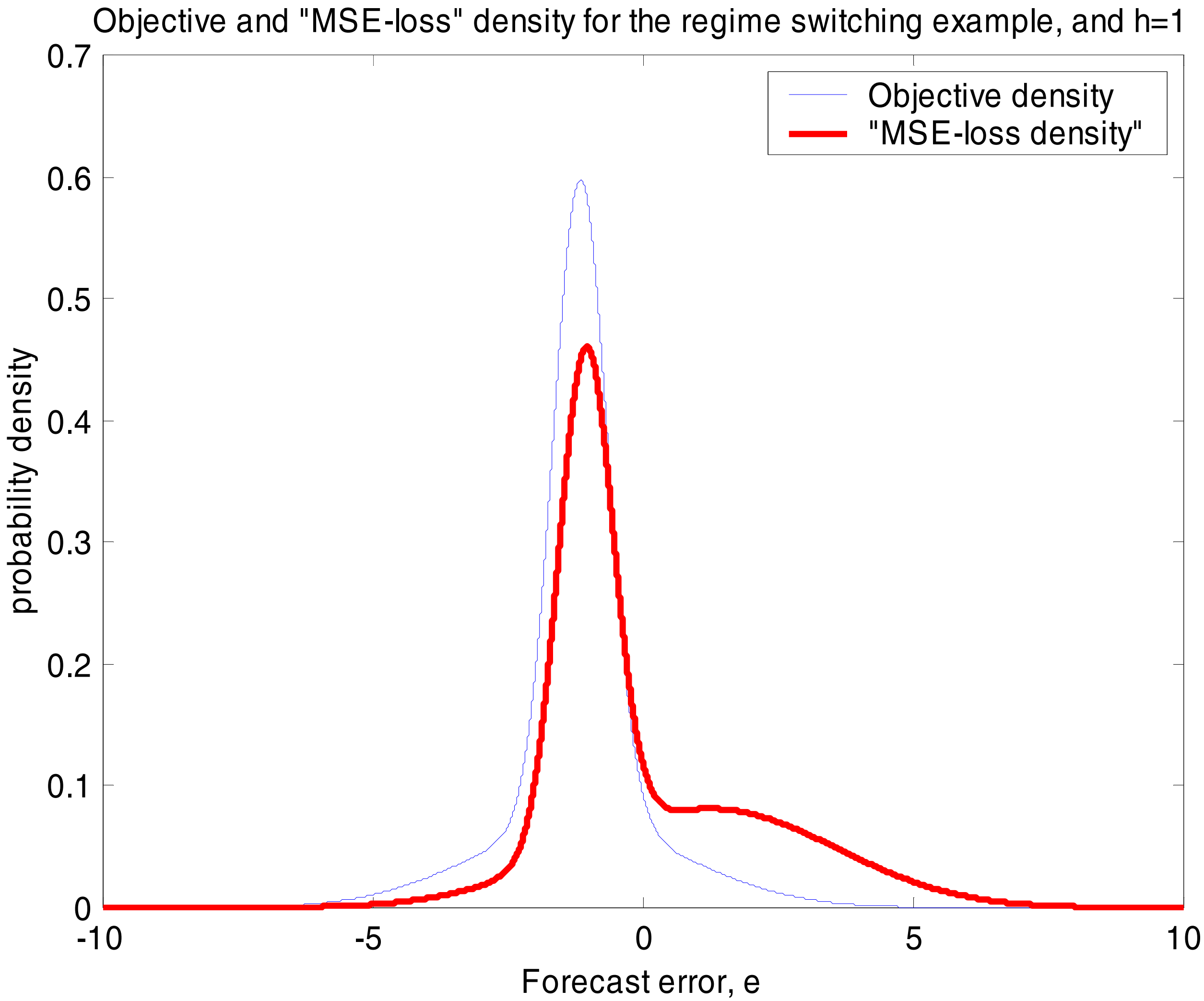
Proposition: The optimal *h*-step forecast error has zero serial correlation beyond lag *h* –1.

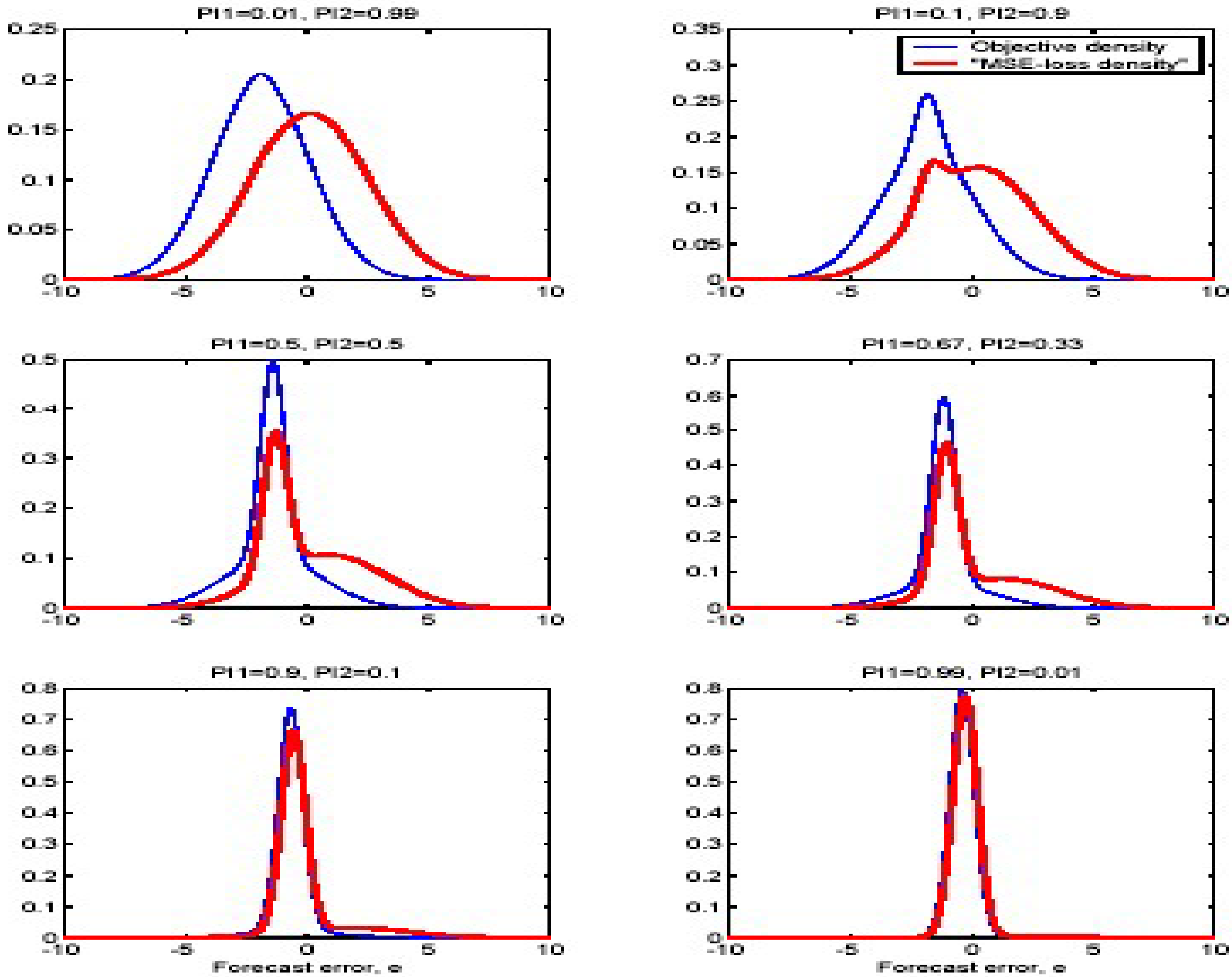
Proof:

$$\begin{aligned}\mathrm{Cov}^*[e^*_{t+h,t}, e^*_{t+h-j,t-j}] &= E^*[e^*_{t+h,t} \cdot e^*_{t+h-j,t-j}] \\ &= E^*[E^*_t[e^*_{t+h,t}] \cdot e^*_{t+h-j,t-j}] \quad \forall\, j \geq h \\ &= 0\end{aligned}$$

Objective and MSE-loss error densities

Pi = [2/3,1/3]'




PI1=0.01, PI2=0.99
PI1=0.1, PI2=0.9
Objective density
"MSE-loss density"
PI1=0.5, PI2=0.5
PI1=0.67, PI2=0.33
PI1=0.9, PI2=0.1
PI1=0.99, PI2=0.01
Forecast error, e
Forecast error, e

Summary of Results: Implications

- Tests of forecast optimality/forecaster rationality that are based on the standard forecast errors (generally) implicitly assume MSE loss

- If the forecast user/provider has a different loss function, the forecasts may be perfectly optimal and still violate standard properties

- Our results simply show that without some knowledge of the forecaster's loss function testing forecast optimality is an extremely difficult task

Summary: Testing optimality

- If the forecaster's loss function is known, the results in this paper may be used to construct tests of forecast optimality
  - Combine our results with the tests of Diebold-Mariano (1995) or West (1996)
- If the forecaster's loss function is known up to an unknown parameter, the work of Elliott, Komunjer and Timmermann (2002) may be used instead
- If the DGP is known to only have conditional mean dynamics we showed that forecast optimality may be tested with much robustness to the unknown loss function.