Realized SemiCovariances

Looking for Signs of Direction Inside the Covariance Matrix

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Motivation

- How do variances and covariances react to **positive vs. negative returns**? Do market participants process positive vs. negative returns differently?
- Is there any information in the **signs** of high frequency, intra-daily, returns?
    - It seems hard to believe, but we will show that indeed there is.
- Is it possible to extend the idea of **semi-variances** to covariances in a sensible way?
    - Are there any gains from doing so?

The main idea: Realized SemiCovariances

- Let $\mathbf{r}_{k,t} = [r_{1,k,t}, ..., r_{N,k,t}]'$ be the vector of returns on N assets over the k^{th} high frequency period on day t. Standard realized covariance is obtained as:

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- Define the vectors of positive and negative returns as:

$$\mathbf{r}_{k,t}^{+} \equiv \mathbf{r}_{k,t} \odot \mathbf{1}\{\mathbf{r}_{k,t} \geq 0\}, \qquad \mathbf{r}_{k,t}^{-} \equiv \mathbf{r}_{k,t} \odot \mathbf{1}\{\mathbf{r}_{k,t} < 0\}$$

where the indicator function is applied element by element.

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- Define four **"realized semicovariance"** matrices:

$$\mathbf{P}_t^{(m)} = \sum_{k=1}^{m} \mathbf{r}_{k,t}^{+}\mathbf{r}_{k,t}^{+\prime},$$

$$\mathbf{N}_t^{(m)} = \sum_{k=1}^{m} \mathbf{r}_{k,t}^{-}\mathbf{r}_{k,t}^{-\prime}$$

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$$\mathbf{M}_t^{-(m)} = \sum_{k=1}^{m} \mathbf{r}_{k,t}^{-}\mathbf{r}_{k,t}^{+\prime}, \qquad \mathbf{N}_t^{(m)} = \sum_{k=1}^{m} \mathbf{r}_{k,t}^{-}\mathbf{r}_{k,t}^{-\prime}$$

- Note that

$$\mathbf{RCOV}_t^{(m)} = \mathbf{P}_t^{(m)} + \mathbf{M}_t^{+(m)} + \mathbf{M}_t^{-(m)} + \mathbf{N}_t^{(m)} \quad \text{for all } m$$

The bivariate case: Positive and negative semicovariances

- Consider the positive semicovariance matrix:

$$\mathbf{P}_t^{(m)} = \begin{bmatrix} \sum_{k=1}^{m} r_{1,k,t}^{+2} & \sum_{k=1}^{m} r_{1,k,t}^{+} r_{2,k,t}^{+} \\ \bullet & \sum_{k=1}^{m} r_{2,k,t}^{+2} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{V}_{1,t}^{+} & \mathcal{P}_t \\ \bullet & \mathcal{V}_{2,t}^{+} \end{bmatrix}$$

- The negative semicovariance matrix is analogous:

$$\mathbf{N}_t^{(m)} = \begin{bmatrix} \sum_{k=1}^{m} r_{1,k,t}^{-2} & \sum_{k=1}^{m} r_{1,k,t}^{-} r_{2,k,t}^{-} \\ \bullet & \sum_{k=1}^{m} r_{2,k,t}^{-2} \end{bmatrix} \equiv \begin{bmatrix} \mathcal{V}_{1,t}^{-} & \mathcal{N}_t \\ \bullet & \mathcal{V}_{2,t}^{-} \end{bmatrix}$$

- The diagonal entries are the *realized semivariances* studied by Barndorff-Nielsen, Kinnebrock and Shephard (2010, book) and Patton and Sheppard (2015, *REStat*).

The bivariate case: Mixed semicovariances

- Next consider a "mixed" semicovariance matrix:

$$\mathbf{M}_t^{+(m)} = \begin{bmatrix} \sum_{k=1}^{m} r_{1,k,t}^{+} r_{1,k,t}^{-} & \sum_{k=1}^{m} r_{1,k,t}^{+} r_{2,k,t}^{-} \\ & \sum_{k=1}^{m} r_{2,k,t}^{+} r_{2,k,t}^{-} \end{bmatrix} \equiv \begin{bmatrix} 0 & \mathcal{M}_t^{+} \\ & 0 \end{bmatrix}$$

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$$= \left(\mathbf{M}_t^{-(m)}\right)^{\intercal}$$

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\end{aligned}
$$

  - Clearly, it is the **off-diagonal** entries of the realized semicovariance matrices that are novel.

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- Note that if the order of the assets is **arbitrary**, then it is natural to **combine** the mixed matrices, $\mathbf{M}_t^{+(m)}$ and $\mathbf{M}_t^{-(m)}$ into the (*symmetric*) matrix:

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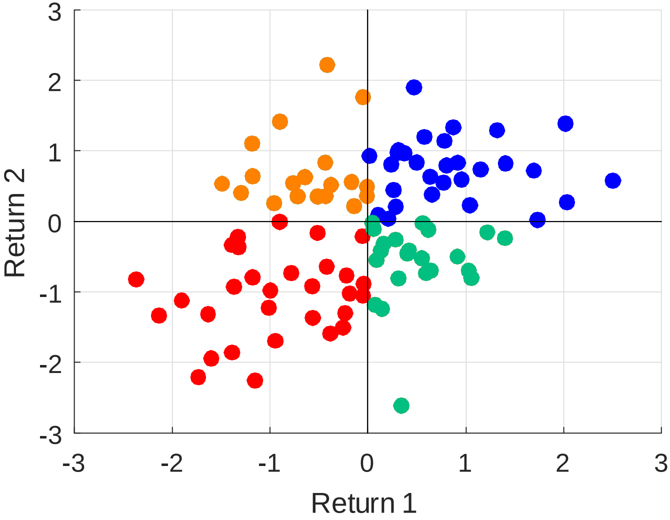
- In this case the decomposition has just three elements:

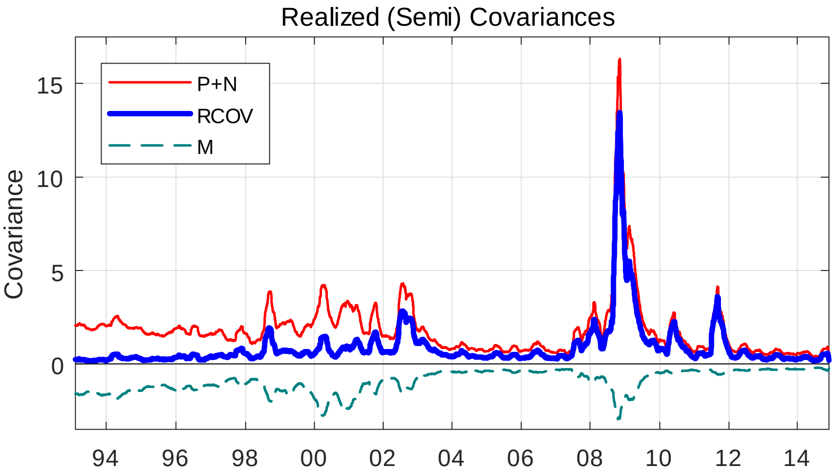
$$\mathbf{RCOV}_t^{(m)} = \mathbf{P}_t^{(m)} + \mathbf{N}_t^{(m)} + \mathbf{M}_t^{(m)} \quad \text{for all } m$$

Example



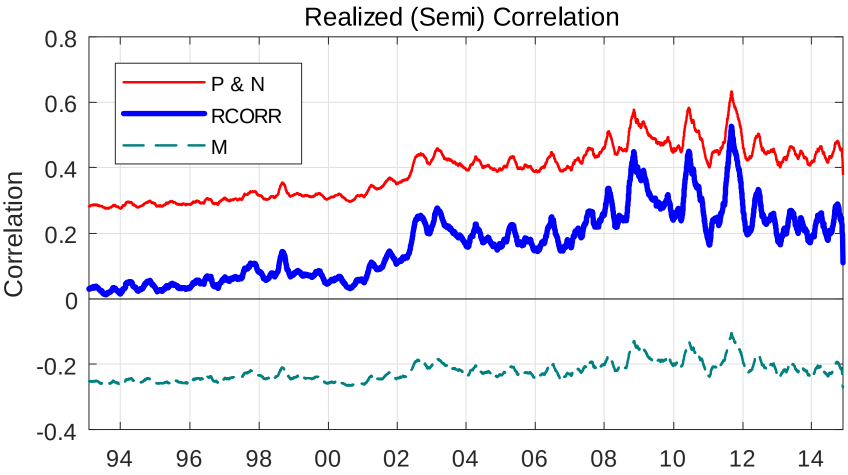
Realized Semicovariances across 500 pairs of stocks

Jan 1993 – Dec 2014.



Realized Semicorrelations across 500 pairs of stocks

Jan 1993 – Dec 2014.



Summary of main findings

1. **Asymptotic properties** available under fairly general conditions
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2. We find striking differences in the **empirical properties** of these measures:
   - *Negative* semicovariance is much more useful for forecasting positive, negative, or total covariances.
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2. We find striking differences in the **empirical properties** of these measures:
   - *Negative* semicovariance is much more useful for forecasting positive, negative, or total covariances.
   - *Mixed* semicovariances are markedly more persistent than concordant semicovariances.
3. **Portfolio variance forecasts** can be significantly improved by using our proposed decomposition.
   - We consider portfolios ranging from 2 to 100 assets; gains kick in early and plateau at around 30.
   - Decomposing using semicovariances significantly better than using semivariances; both are better than ignoring sign information completely.

Related work

- **Covariance matrix estimation:** Kendall (1953, *JRSS*), Elton and Gruber (1973, *JF*), Bauwens, Laurent and Rombouts (2006, *JAE*).
- **Semivariances:** Markowitz (1959, book), Mao (1970, *JF*), Hogan and Warren (1972, 1974 *JFQA*), Fishburn (1977, *AER*).
- **Realized semivariance:** Barndorff-Nielsen, Kinnebrock and Shephard (2010, book), Patton and Sheppard (2015, *REStat*), Segal, Shaliastovich, Yaron (2015, *JFE*).
- **Asymmetric correlations:** Longin and Solnik (2001, *JF*), Ang and Chen (2002, *JFE*), Patton (2004, *JFEC*), Hong, Tu and Zhou (2007, *RFS*).
- **Jumps and Co-jumps:** Das and Uppal (2004, *JF*), Bollerslev, Law and Tauchen (2008, *JoE*), Jacod and Todorov (2009, *AoS*), Li, Todorov and Tauchen (2017, *ECMA*).

Outline

1. Introduction
2. Theoretical properties of realized semicovariances
   - Definitions
   - Limit theory
   - Simulation results
3. Empirical properties of realized semicovariances
   - Daily measures for 749 US equities
   - Differences in dynamic dependencies
   - Application to portfolio volatility forecasting
4. Summary and conclusion

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4. Summary and conclusion

Semicovariances for a standard Normal

- To build some intuition for the main theoretical results, consider these measures in population for a bivariate standard Normal:

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim N\left(0, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$$

- Clearly, here the (total) covariance is

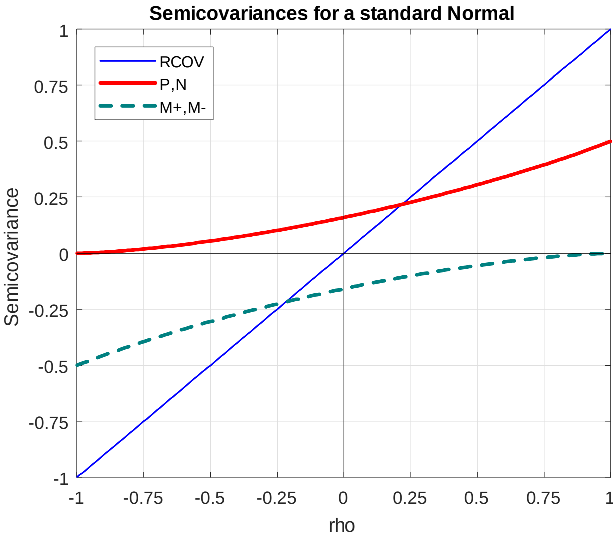
$$Cov\left[Z_1, Z_2\right] = \mathbb{E}\left[Z_1 Z_2\right] = \rho$$

- Now consider semicovariances:

$$\begin{aligned} \mathbb{E}\left[Z_1^+ Z_2^+\right] &= \mathbb{E}\left[Z_1^- Z_2^-\right] = \psi\left(\rho\right) \\ \mathbb{E}\left[Z_1^+ Z_2^-\right] &= \mathbb{E}\left[Z_1^- Z_2^+\right] = -\psi\left(-\rho\right) \\ \text{where} \quad \psi\left(\rho\right) &\equiv \frac{\sqrt{1-\rho^2} + \rho\left(\pi - \text{arccos}\rho\right)}{2\pi} \end{aligned}$$

Semicovariances for a standard Normal

At rho = 0, P = M = 1/(2*pi) and M+ = M- = -1/(2*pi)



Asymptotic analysis

- Let $X_t \equiv [X_{1t}, ..., X_{dt}]'$ denote the d-dim log-price process. As in Jacod and Protter (2012) we assume X_t is an Itô semimartingale of the form

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t$$

  - b is a $\mathbb{R}^d$-valued drift process
  - W is a q-dim standard Brownian motion, with $q \geq d$
  - J is a pure jump process
  - σ is a $d \times q$ stochastic volatility matrix

- Define $c_s \equiv \sigma_s \sigma_s'$ as the spot covariance matrix of X_s, with $v_{js} \equiv \sqrt{c_{jj,s}}$ and $\rho_{jks} \equiv c_{jk,s} / (v_{js} v_{ks})$.
- **Assumption 1:** (i) b and c are càdlàg and adapted, (ii) J has finite variation, (iii) X is sampled on a regular time grid with sampling interval $\Delta \to 0$ over a fixed span $T > 0$.

Consistency

- **Theorem 1:** Under Assumption 1, the (j,k) elements of each realized semicovariance matrix satisfy:

$$\begin{bmatrix} P_{jk,T} \\ N_{jk,T} \\ M^{+}_{jk,T} \\ M^{-}_{jk,T} \end{bmatrix} \xrightarrow{p} \int_0^T v_{js} v_{ks} \begin{bmatrix} \psi\left(\rho_{jks}\right) \\ \psi\left(\rho_{jks}\right) \\ \left(-\psi\left(-\rho_{jks}\right)\right) \\ \left(-\psi\left(-\rho_{jks}\right)\right) \end{bmatrix} ds + \sum_{s \leq T} \begin{bmatrix} \Delta X^{+}_{js} \Delta X^{+}_{ks} \\ \Delta X^{-}_{js} \Delta X^{-}_{ks} \\ \Delta X^{+}_{js} \Delta X^{-}_{ks} \\ \Delta X^{-}_{js} \Delta X^{+}_{ks} \end{bmatrix}$$

  - This holds for $j = k$ and so covers semivariances as well.

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  - This holds for $j = k$ and so covers semivariances as well.

- Note that the first-order asymptotic behavior of $P_{jk,T} - N_{jk,T}$ is completely determined by the "directional co-jumps"

$$P_{jk,T} - N_{jk,T} \xrightarrow{p} \sum_{s \le T} \left(\Delta X^{+}_{js} \Delta X^{+}_{ks} - \Delta X^{-}_{js} \Delta X^{-}_{ks} \right)$$

Limit variation of the estimators

- **Assumption 2:** J is of finite activity.
- **Assumption 3:** The process σ has the form

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \tilde{M}_t + \tilde{L}_t + \sum_{s \leq t} \Delta\sigma_s \mathbf{1}\{\|\Delta\sigma_s\| > 1\}$$

such that:

- $\tilde{\sigma}$ is a $d \times d \times d$ càdlàg adapted process of full rank
- $\tilde{M}_t$ is a local martingale orthog to W with $\left\|\Delta\tilde{M}\right\| \leq 1$ and its predictable quadratic covariation process has the form $\int_0^t \tilde{a}_s ds$ for some locally bounded process $\tilde{a}$
- $\tilde{L}_t$ is a long-memory component: locally α-Hölder continuous for some $\alpha \in (1/2, 1)$
- the compensator of the pure jump process $\sum_{s \leq t} \Delta\sigma_s \mathbf{1}\{\|\Delta\sigma_s\| > 1\}$ has the form $\int_0^t a_s ds$ for some locally bounded process a.

Second-order asymptotic analysis

- **Theorem 2:** Under assumptions 2 and 3,

$$\Delta^{-1/2}\begin{bmatrix} vech\left(\mathbf{P}_T-\mathbf{P}\right) \\ vech\left(\mathbf{N}_T-\mathbf{N}\right) \\ vec^*\left(\mathbf{M}_T-\mathbf{M}\right)\end{bmatrix} \xrightarrow{\mathcal{L}\text{-}s} \begin{bmatrix} B_P^{(1)} \\ B_N^{(1)} \\ B_M^{(1)}\end{bmatrix} + \begin{bmatrix} B_P^{(2)} \\ B_N^{(2)} \\ B_M^{(2)}\end{bmatrix} + \begin{bmatrix} \zeta_P \\ \zeta_N \\ \zeta_M\end{bmatrix} + \begin{bmatrix} \tilde{\zeta}_P \\ \tilde{\zeta}_N \\ \tilde{\zeta}_M\end{bmatrix} + \begin{bmatrix} \xi_P \\ \xi_N \\ \xi_M\end{bmatrix}$$

  - $B^{(1)}$: bias term related to price drift
  - $B^{(2)}$: bias term related to leverage effects
  - ζ: sampling error term spanned by diffusive price risk
  - $\tilde{\zeta}$: sampling error term orthogonal to diffusive price risk
  - ξ: sampling error term related to jump price risk

- *Cannot* use this theorem for confidence intervals or the like.

Truncated realized semicovariances I

- It may be of interest to separate the contribution of jumps to semicovariances.
- We do this using a truncation method (Mancini, 2009 *SJS*), which exploits a sequence $u_T \asymp \Delta^{\varpi}$ for some $\varpi \in (0, 1/2)$.
- Then

$$\mathbf{P}_t^{C(m)} \equiv \sum\nolimits_{k=1}^{m} \mathbf{r}_{k,t}^{+} \mathbf{r}_{k,t}^{+\prime} \mathbf{1}\{|\mathbf{r}_{kt}| \leq u_T\} \xrightarrow{p} \int_{t-1}^{t} v_{js} v_{ks} \psi\left(\rho_{jks}\right) ds$$

$$\mathbf{P}_t^{J(m)} \equiv \mathbf{P}_t^{(m)} - \mathbf{P}_t^{C(m)} \xrightarrow{p} \sum_{s \in (t-1,t]} \Delta X_{js}^{+} \Delta X_{ks}^{+}$$

- Truncated versions of negative and mixed semicovariances are defined analogously.

Truncated realized semicovariances II

- **Theorem 4:** Under Assumptions 2 and 3,

$$\Delta^{-1/2} \begin{bmatrix} vech\left(\mathbf{P}_T^C - \mathbf{P}^C\right) \\ vech\left(\mathbf{N}_T^C - \mathbf{N}^C\right) \\ vec^*\left(\mathbf{M}_T^C - \mathbf{M}^C\right) \end{bmatrix} \xrightarrow{\mathcal{L}\text{-}s} \begin{bmatrix} B_P^{(1)} \\ B_N^{(1)} \\ B_M^{(1)} \end{bmatrix} + \begin{bmatrix} B_P^{(2)} \\ B_N^{(2)} \\ B_M^{(2)} \end{bmatrix} + \begin{bmatrix} \zeta_P \\ \zeta_N \\ \zeta_M \end{bmatrix} + \begin{bmatrix} \tilde{\zeta}_P \\ \tilde{\zeta}_N \\ \tilde{\zeta}_M \end{bmatrix}$$

and

$$\Delta^{-1/2} \begin{bmatrix} vech\left(\mathbf{P}_T^J - \mathbf{P}^J\right) \\ vech\left(\mathbf{N}_T^J - \mathbf{N}^J\right) \\ vec^*\left(\mathbf{M}_T^J - \mathbf{M}^J\right) \end{bmatrix} \xrightarrow{\mathcal{L}\text{-}s} \begin{bmatrix} \xi_P \\ \xi_N \\ \xi_M \end{bmatrix}$$

Truncated realized semicovariances III

- **Assumption 4:** $\tilde{\sigma} = 0$ and the volatility process σ more generally is independent of the Brownian motion W (ie, no leverage effects).
- **Theorem 5:** Under Assumptions 2–4,

$$\left(\hat{Q} + \tilde{Q}\right)^{-1/2} \left(\Delta^{-1/2} \begin{bmatrix} vech\left(\mathbf{P}_T^C - \mathbf{P}^C\right) \\ vech\left(\mathbf{N}_T^C - \mathbf{N}^C\right) \\ vec^*\left(\mathbf{M}_T^C - \mathbf{M}^C\right) \end{bmatrix} - \begin{bmatrix} B_P^{(1)} + B_P^{(2)} \\ B_N^{(1)} + B_N^{(2)} \\ B_M^{(1)} + B_M^{(2)} \end{bmatrix} \right)$$

$$\xrightarrow{\mathcal{L}\text{-}s} N\left(0, I\right)$$

where

$$Q^* \equiv \int_0^T \Gamma_s ds \quad \text{and} \quad Q^{**} \equiv \int_0^T \tilde{\Gamma}_s ds$$

and Γ_s and $\tilde{\Gamma}_s$ are known, continuous (but messy) functions of the spot covariance matrix, which can be consistently estimated using Li, Todorov and Tauchen (2017, *JoE*).

- The bias terms cannot be consistently estimated (at least using only infill asymptotics).

Link to the Kyle model (tentative) I

- The bias terms that appear in the limit distribution can be interpreted using the continuous-time version of the Kyle (1985, *ECMA*) model due to Back (1992, *RFS*).
  - Simplify presentation and focus on single asset case $\Rightarrow$ semivariances only
- Assume there are $m \geq 1$ periods per day.
  - At start of k^{th} period, the asset value is drawn $\log(V_k) \sim N\left(\bar{V}_k, \sigma^2_{Vk}\right)$.
  - **Informed trader** observes V_k, and trades continuously through the period to maximize her profit.
  - **Uninformed traders** have price inelastic net demand, with order flow given by $\sigma_{Lt} dW_t$.
  - **Market maker** observes aggregated order flow Y_t and sets $P_t = E\left[V_k | (Y_s)_{s \leq t}\right]$, $t \in [k-1, k]$

Link to the Kyle model (tentative) II

- In equilibrium of this model:

$$\begin{aligned} d \log P_s &= b_s ds + \sigma_{Vk} dW_s, \ \text{ for } s \in [k-1, k] \\ \text{where } \ b_s &= \frac{\log V_k - \log P_s}{k - s} \end{aligned}$$

i.e. a continuous Itô process with stochastic drift and constant volatility.

- Note that the drift b_s is proportional to the **amount of mispricing** $\Leftrightarrow$ the "profit margin" of the informed trader.

- In this set-up, the second bias term $B^{(2)} = 0$ and

$$B_P^{(1)} = -B_N^{(1)} = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{T} \sigma_{Vk} \int_{k-1}^{k} b_s ds$$

- Thus $B_P^{(1)}$ can be interpreted as the weighted average mis-pricing across the day, with larger weights when asymmetric information is greater.

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2. Theoretical properties of realized semicovariances
   - Definitions
   - Limit theory
   - **Simulation results**
3. Empirical properties of realized semicovariances
   - Daily measures for 749 US equities
   - Differences in dynamic dependencies
   - Application to portfolio volatility forecasting
4. Summary and conclusion

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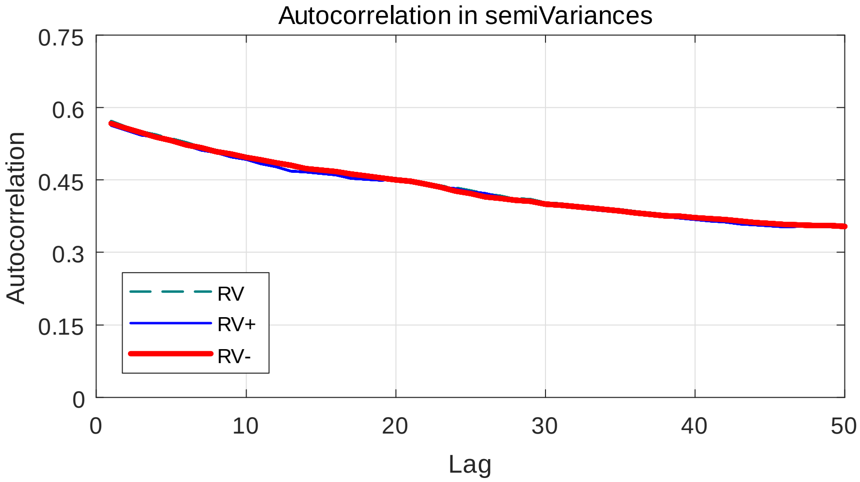
Data

- We use data from TAQ, January 1993 to December 2014.
  - $T = 5541$ days (maximum).
- We consider all stocks that were ever a constituent of the S&P 500 index and have at least 2000 daily observations.
  - $N = 749$ unique stocks.
- We use 15-minute sampling.
  - $m = 26$ observations per day.

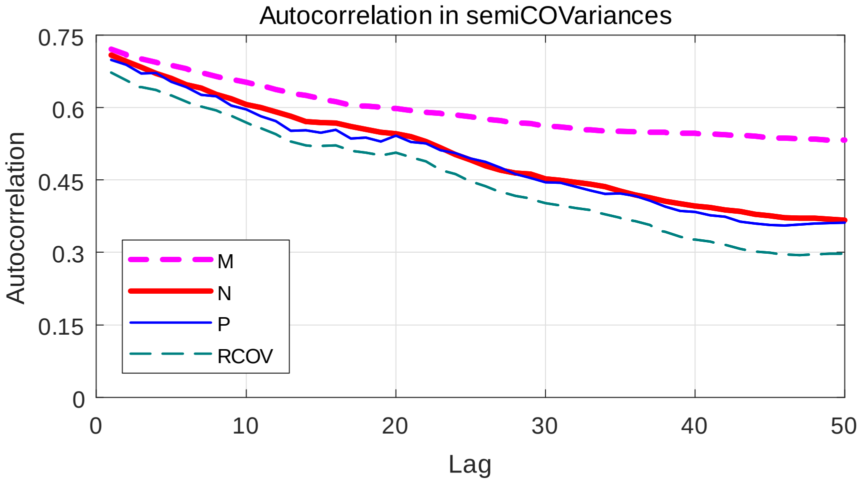
Time series dependence in semivariances

ACF-IV of Hansen and Lunde (2014, ET). Essentially no differences.



Time series dependence in semicovariances

ACF-IV of Hansen and Lunde (2014, ET). M is most persistent, RCOV is least.



A HAR model for realized semicovariances

- We consider predictions using the HAR model of Corsi (2009, *JFEC*).
- For each of 500 randomly-chosen pairs of assets (i,j), we estimate:

$$\begin{bmatrix} \mathcal{P}_{ij,t} \\ \mathcal{N}_{ij,t} \\ \mathcal{M}_{ij,t} \end{bmatrix} = \begin{bmatrix} \phi_{\mathcal{P}ij} \\ \phi_{\mathcal{N}ij} \\ \phi_{\mathcal{M}ij} \end{bmatrix} + \Phi_{ij,D} \begin{bmatrix} \mathcal{P}_{ij,t-1} \\ \mathcal{N}_{ij,t-1} \\ \mathcal{M}_{ij,t-1} \end{bmatrix} + \Phi_{ij,W} \begin{bmatrix} \mathcal{P}_{ij,t-2:t-5} \\ \mathcal{N}_{ij,t-2:t-5} \\ \mathcal{M}_{ij,t-2:t-5} \end{bmatrix}$$

$$+\Phi_{ij,M} \begin{bmatrix} \mathcal{P}_{ij,t-6:t-22} \\ \mathcal{N}_{ij,t-6:t-22} \\ \mathcal{M}_{ij,t-6:t-22} \end{bmatrix} + \begin{bmatrix} \epsilon_t^{\mathcal{P}ij} \\ \epsilon_t^{\mathcal{N}ij} \\ \epsilon_t^{\mathcal{M}ij} \end{bmatrix}$$

where $\mathcal{P}_{ij,t-2:t-5} \equiv \frac{1}{4} \sum_{j=2}^{5} \mathcal{P}_{ij,t-j}$.

- Recall that $RCOV_{ij,t} = \mathcal{P}_{ij,t} + \mathcal{N}_{ij,t} + \mathcal{M}_{ij,t}$ and so this model can be interpreted as a "decomposed" model for RCOV.

Avg coefficient estimates from the HAR model

Signif at 0.05 level for 75% and 50% of all 1000 pairs indicated by ** and *

	Dependent variable			
	$\mathcal{P}_{ij,t}$	$\mathcal{N}_{ij,t}$	$\mathcal{M}_{ij,t}$	$RCOV_{ij,t}$
$\mathcal{P}_{ij,t-1}$	0.038*	0.050*	-0.035*	**0.052****
$\mathcal{P}_{ij,t-2:t-5}$	0.004	0.057	-0.002	0.059
$\mathcal{P}_{ij,t-6:t-22}$	-0.074	0.023	0.009	0.048
$\mathcal{N}_{ij,t-1}$	**0.248****	**0.192****	**-0.096****	**0.344****
$\mathcal{N}_{ij,t-2:t-5}$	**0.312****	**0.250****	-0.090*	**0.472****
$\mathcal{N}_{ij,t-6:t-22}$	**0.349****	0.206*	-0.021	**0.534****
$\mathcal{M}_{ij,t-1}$	-0.075*	-0.072*	**0.141****	-0.006
$\mathcal{M}_{ij,t-2:t-5}$	-0.044	-0.049	**0.209****	0.116
$\mathcal{M}_{ij,t-6:t-22}$	0.028	-0.020	**0.409****	**0.417****

- The three lags of $\mathcal{N}$ are significant for both $\mathcal{P}$ and $\mathcal{N}$.
- $\mathcal{M}$ seems to be mostly driven by its own lags.
- The coefficients for RCOV are the sum of those for $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{M}$.

Portfolio volatility forecasting using "semi" decompositions

- The variance of a portfolio of assets with weight vector **w** is:

$$RV_t^p = \mathbf{w}'\mathbf{RCOV}_t\mathbf{w}$$

- The portfolio variance can be decomposed using semivariances, following BNKS (2010, book) and Patton and Sheppard (2015, *REStat*):

$$RV_t^p = \mathcal{V}_t^{+p} + \mathcal{V}_t^{-p}$$

- It can alternatively be decomposed using semicovariances:

$$\begin{aligned} RV_t^p &= \mathbf{w}'\mathbf{P}_t\mathbf{w} + \mathbf{w}'\mathbf{N}_t\mathbf{w} + \mathbf{w}'\mathbf{M}_t\mathbf{w} \\ &\equiv \mathcal{P}_t^p + \mathcal{N}_t^p + \mathcal{M}_t^p \end{aligned}$$

  - Note that unlike the semiVariance decomposition, the semiCOVariance decomposition uses returns data on all of the constituent assets.

Models for forecasting portfolio volatility

1. **HAR**, from Corsi (2009, *JFEC*):

$$RV^p_{t+1|t} = \phi_0 + \phi_d RV^p_t + \phi_w RV^p_{t-1:t-4} + \phi_m RV^p_{t-5:t-21}$$

Models for forecasting portfolio volatility

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2. Semivariance HAR (**SHAR**), from Patton and Sheppard (2015, *REStat*):

$$RV^p_{t+1|t} = \phi_0 + \phi_d^+ \mathcal{V}_t^{+p} + \phi_d^- \mathcal{V}_t^{-p} + \phi_w RV^p_{t-1:t-4} + \phi_m RV^p_{t-5:t-21}$$

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3. Semicovariance HAR (**SCHAR**):

$$\begin{aligned} RV^p_{t+1|t} = \; & \phi_0 + \phi_{d,\mathcal{P}} \mathcal{P}^p_t + \phi_{w,\mathcal{P}} \mathcal{P}^p_{t-1:t-4} + \phi_{m,\mathcal{P}} \mathcal{P}^p_{t-5:t-21} \\ & + \phi_{d,\mathcal{N}} \mathcal{N}^p_t + \phi_{w,\mathcal{N}} \mathcal{N}^p_{t-1:t-4} + \phi_{m,\mathcal{N}} \mathcal{N}^p_{t-5:t-21} \\ & + \phi_{d,\mathcal{M}} \mathcal{M}^p_t + \phi_{w,\mathcal{M}} \mathcal{M}^p_{t-1:t-4} + \phi_{m,\mathcal{M}} \mathcal{M}^p_{t-5:t-21} \end{aligned}$$

Models for forecasting portfolio volatility

1. **HAR**, from Corsi (2009, *JFEC*):

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4. Reduced semicovariance HAR (**SCHAR-r**):

$$RV^p_{t+1|t} = \phi_0 + \phi_{d,\mathcal{N}} \mathcal{N}^p_t + \phi_{w,\mathcal{N}} \mathcal{N}^p_{t-1:t-4} + \phi_{m,\mathcal{N}} \mathcal{N}^p_{t-5:t-21} + \phi_{m,\mathcal{M}} \mathcal{M}^p_{t-5:t-21}$$

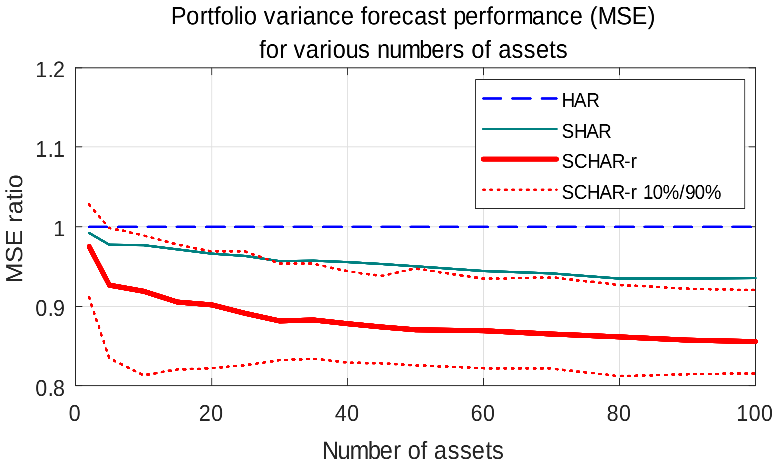
Portfolio volatility out-of-sample forecast results

The SCHAR-r model performs the best, with gains of around 10%

	MSE		*QLIKE*	
	Avg	**Ratio**	**Avg**	**Ratio**
		$N = 1$		
HAR	35.112	1.000	0.239	1.000
SHAR	34.981	0.997	0.238	0.998
		$N = 10$		
HAR	1.849	1.000	0.141	1.000
SHAR	1.671	0.966	0.139	0.986
SCHAR	1.643	0.955	0.210	1.318
SCHAR-r	1.567	0.908	0.139	0.979
		$N = 100$		
HAR	0.048	1.000	0.119	1.000
SHAR	0.045	0.935	0.115	0.957
SCHAR	0.045	0.976	0.236	1.495
SCHAR-r	0.041	0.862	0.111	0.925

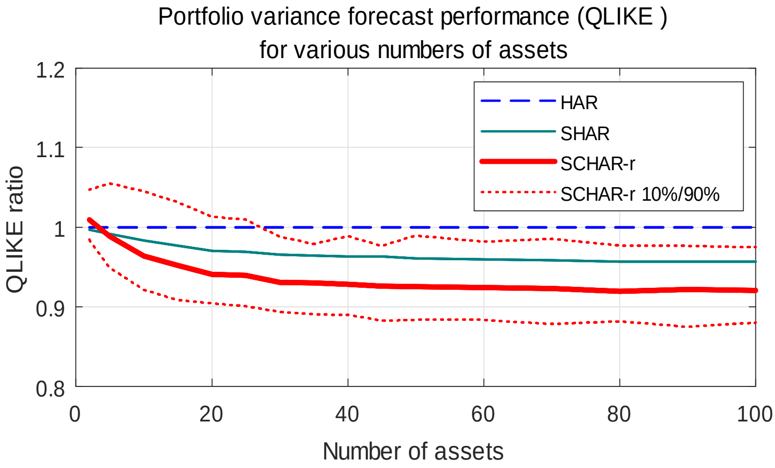
Portfolio volatility forecast results: MSE

The SCHAR-r model performs the best for N>=2



Portfolio volatility forecast results: QLIKE

The SCHAR-r model performs the best for N>=3



Why does the SCHAR model outperform?

- The semicovariance-HAR can be interpreted as a **time-varying parameter** version of the baseline HAR model.

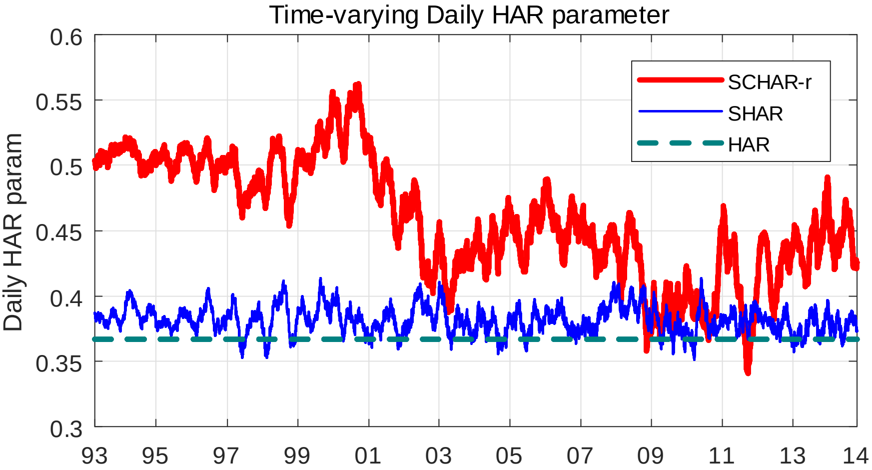
- Consider a simplified case with just one lag:

$$\begin{aligned} RV^{p}_{t+1|t} &= \phi_0 + \phi_{d,\mathcal{P}}\mathcal{P}^{p}_{t} + \phi_{d,\mathcal{N}}\mathcal{N}^{p}_{t} + \phi_{d,\mathcal{M}}\mathcal{M}^{p}_{t} \\ &= \phi_0 + \left(\phi_{d,\mathcal{P}}\frac{\mathcal{P}^{p}_{t}}{RV^{p}_{t}} + \phi_{d,\mathcal{N}}\frac{\mathcal{N}^{p}_{t}}{RV^{p}_{t}} + \phi_{d,\mathcal{M}}\frac{\mathcal{M}^{p}_{t}}{RV^{p}_{t}}\right) RV^{p}_{t} \\ &\equiv \phi_0 + \phi_{1,t} RV^{p}_{t} \end{aligned}$$

- The parameters $\left(\phi_{d,\mathcal{P}}, \phi_{d,\mathcal{N}}, \phi_{d,\mathcal{M}}\right)$ and the values of $\mathcal{P}^{p}_{t}$, $\mathcal{N}^{p}_{t}$, and $\mathcal{M}^{p}_{t}$ determine how much weight is given to the most recent value of RV.
  - The same logic applies to the weekly and monthly variables in the HAR model.

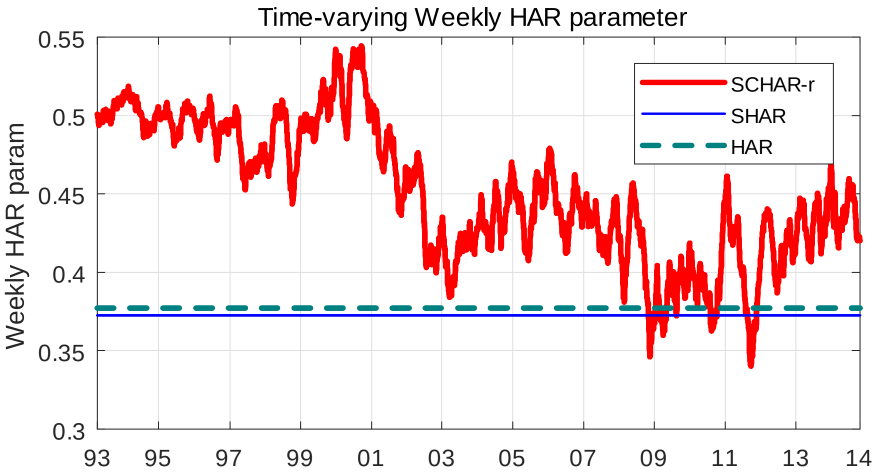
Implied time-varying HAR parameters: Daily

The semicovariance model puts a lot more weight on daily information



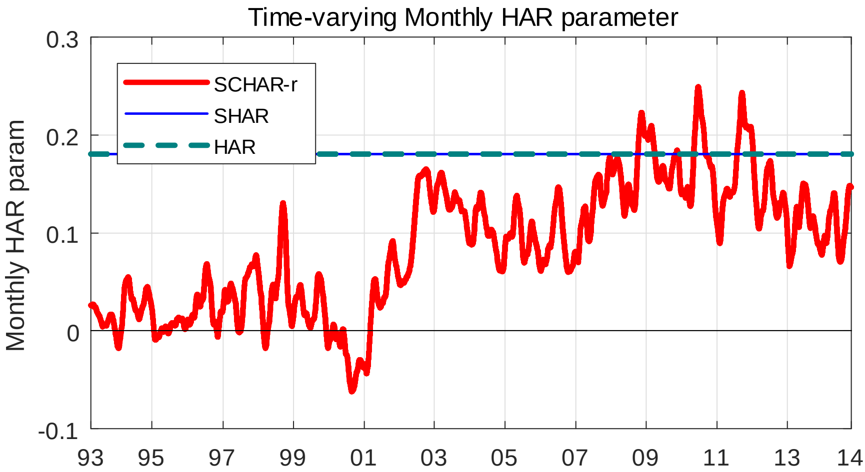
Implied time-varying HAR parameters: Weekly

The semicovariance model puts a lot more weight on weekly information



Implied time-varying HAR parameters: Monthly

The semicovariance model puts little weight on monthly information



Summary

- We propose a new decomposition of the realized covariance matrix into four components, based on the signs of the underlying returns.

$$\mathbf{RCOV}_t = \mathbf{P}_t + \mathbf{N}_t + \mathbf{M}_t^+ + \mathbf{M}_t^-$$

- Under a standard continuous semimartingale assumption we derive the joint limiting behavior of these measures and propose tests for symmetry of semicovariances.
  - We find strong evidence against symmetry, associated with news ann'ments.
- Using data on over 700 US stock returns we find the realized semicovariances have distinct features:
  - Persistence is stronger for "discordant" than "concordant" semicovariances.
  - Negative semicovariances are most important for forecasting (total) covariance.