# Non-Linearities and Stress Testing*

Mathias Drehmann† Bank of England

Andrew J. Patton‡ London School of Economics and Bank of England

Steffen Sorensen § Bank of England

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**Abstract**

We explore the impact of possible non-linearities on aggregate credit risk in a vector autoregression framework. By using aggregate data on corporate credit in the UK we investigate the non-linear transmission of macroeconomic shocks to aggregate corporate default probability. We show two important results: firstly, we find that non-linearities matter for the level and shape of impulse response functions of credit risk following small as well as large shocks to systematic risk factors. Secondly, we show that ignoring estimation uncertainty in stress tests can lead to a substantial underestimation of credit risk, particularly in extreme conditions.

Keywords: credit risk, impulse response functions, stress testing, nonlinear time series, VAR models

J.E.L. Codes: G33, C32.

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†Corresponding author. Contact address: Financial Stability, Bank of England, Threadneedle Street, London EC2R 8AH. Email: mathias.drehmann@bankofengland.co.uk.

‡Contact address: Financial Markets Group, London School of Economics, Houghton Street, London WC2A 2AE. Email: a.patton@lse.ac.uk.

§Contact address: Financial Stability, Bank of England, Threadneedle Street, London EC2R 8AH. Email: steffen.sorensen@bankofengland.co.uk.

## 1 Introduction

Stress tests have become a well-established risk management tool to assess the impact of severe but plausible events on banks' exposures. A "stress test" is an estimate of the impact of a (large) shock to a systematic risk factor on a given set of exposures, an estimate that is closely related to traditional impulse response analyses. For market risk, stress tests are routinely undertaken and are used to complement Value-at-Risk (VaR) measures, see BIS (2005). However, quantitative stress tests for credit risk are not yet as well-developed, though a lot of banks extensively undertake qualitative stress tests. In the future, this is likely to change as stress tests have to be undertaken for banks to be eligible for the internal ratings approach under Basel II. Stress tests have also recently gained increased prominence as a tool to assess the financial stability of banking systems. For example, more than 90 "stress tests" have been currently completed/in progress as part of the IMF's Financial Stability Assessment Programme.

One of the main challenges stress tests face is that the models used are generally specified in (log-) linear form. Of course, if the underlying data generating process (DGP) is linear, then this assumption is correct. Alternatively, if interest lies in studying the impact of small shocks around the equilibrium of the process, then a standard linear model may produce adequate forecasts even if the true DGP is non-linear. In such a case the linear model may be interpreted as a first-order Taylor series approximation to the true DGP. However, stress tests do not consider small shocks, and it is not likely that the relevant data generating processes are all log-linear. In Drehmann *et al.* (2006) we explore the generally made assumption of log-linearity and the impact of large macro shocks on firm specific PDs. This companion paper looks at the same questions, but investigates the impact of large macro shocks on aggregate liquidation rates often used to proxy aggregate credit risk in stress testing models. We show that allowing for non-linearities leads to substantially different predictions of losses in scenarios typically considered for stress testing.

Sorge (2004) provides an excellent overview of the current state of the macro stress testing literature. Broadly speaking there are three strands of macro stress testing models, reduced form models, portfolio credit risk models and structural models. Reduced form models are often based on time series or panel-analysis which link write-offs or provisions to macroeconomic factors. These reduced form equations are then used to assess how severe macro scenarios impact on provisions or write-offs of banks. Pain (2004) constructs such a model for the UK and shows that in particular real GDP growth, real interest rates and lagged aggregate lending growth have a strong impact on banks' provisioning.

Another class of models which is extensively used is based on the idea of CreditPortfolioView (see Wilson, 1997a and1997b). Here, the default process it modelled using a Probit model which relates macroeconomic factors to the probability of default of companies. In this spirit Boss (2002) develops a stress testing model for the aggregate Austrian banking sector, whereas Virolainen (2004) applies such a model to the Finnish banking system.

So far, few structural models for stress testing have been developed. One such model is at the core of the Bank of England's stress testing agenda (see Bunn *et al.* (2005)). This model starts by feeding shocks through the Bank's structural macroeconomic model, then through a structural "satellite" model linking macroeconomic variables to arrears and liquidation rates, and then finally to a reduced-form model assessing the impact of liquidations rates and arrears on banks' write-offs and profits. DeBandt and Oung (2004) describe a structural model for France. Generally, structural models are very useful from a central bank's perspective as they are tractable and conform to the way central bankers communicate. Hence, they provide an ideal framework to discuss financial stability risks. By design these models assume a linear relation between macro factors and credit risk. Even in a Probit specification, as used in applications of Credit PortfolioView, the underlying relationship is modelled in a linear fashion. But as discussed above, it is not clear whether the DGP is linear, especially when focusing on extreme down side risks.

This paper studies the aggregate corporate liquidation rate, often used in macro stress tests. We start by estimating a non-linear vector autoregression model (VAR) of the underlying macroeconomic drivers of risk. We concentrate on the three key macroeconomic factors: GDP growth, inflation and the interest rate. We then investigate how macroeconomic shocks feed through to the aggregate liquidation rate.

Nonlinear models for macroeconomic variables have been studied by Koop *et al.* (1996) and Jorda (2005), *inter alia.* We employ the methodology of Jorda (2005) in this paper. Jorda's approach builds on the fact that a standard VAR can be interpreted as a first-order approximation to the true unknown DGP. Thus, a more flexible approximation may be obtained by considering, for example, a quadratic or cubic approximation. An important implication of considering a standard linear VAR as a linear approximation to the true DGP is that it is no longer clear that forecasts or stress tests of horizons greater than one period should be obtained by iterating the one-period model forward, which is the standard practice when deriving impulse response functions for VAR models. As Jorda (2005) points out, if the one-period model is mis-specified, then iterating it forward may well lead to a compounding of mis-specification error. He suggests an alternative approach, namely to estimate a different model for each horizon of interest. If the DGP is truly a VAR then this approach is consistent but not efficient, while if the DGP is not a VAR then this approach offers the best approximation at each horizon, rather than just at the one-quarter horizon. This modelling approach has its roots in the direct multi-step versus iterated forecasting approaches (see for example Stock and Watson, 1999). A benefit of this approach is that simple ordinary least squares (OLS) techniques can be used to obtain the impulse response functions from the non-linear VAR used in the stress tests.

In this paper we show that the results of the non-linear VAR are significantly different to results using standard linear models, especially when considering large shocks. This can be seen in the simple three variable macro model of inflation, GDP growth and a short term interest rate. More importantly, we show that accounting for non-linearities in the underlying macroeconomic environment leads to substantially different conclusions for credit risk projections in stressed conditions.

The remainder of the paper is as follows. In Section 2 we briefly discuss the more formal motivation for the consideration of non-linear multivariate models when studying the impact of large shocks as presented in Drehmann *et al.* (2006). In Section 3 we discuss the estimation of the macro model and the resulting impulse response functions. Further we introduce the model for the corporate liquidation rate and present the results of our analysis of large macroeconomic shocks on default probabilities. Section 4 concludes. Technical details and estimation results are presented in the Appendix.

## 2 Why non-linearities matter

Suppose we are interested in a scalar variable $y_t$, which is follows the following general process:

$$y_t = h\left(y_{t-1}, \varepsilon_t; \theta\right), \quad t = 1, 2, ... \tag{1}$$

where the residual $\varepsilon_t$ is independent of $y_{t-1}$, $h$ is some (possibly non-linear) function, and $\theta$ is a parameter vector. In the standard linear setting we would have

$$h\left(y_{t-1}, \varepsilon_t; \theta\right) = \phi_0 + \phi_1 y_{t-1} + \varepsilon_t \tag{2}$$

and so $y_t$ would follow a first-order autoregressive process. In this case the conditional mean function,

$$\mu\left(y\right) \equiv E\left[y_t | y_{t-1} = y\right] = \phi_0 + \phi_1 y \tag{3}$$

is affine in $y_{t-1}$ and so a first-order Taylor series approximation of $\mu$ corresponds exactly to $\mu$. For other data generating processes the conditional mean function need not be affine in $y_{t-1}$. For example, in the Appendix we describe a simple non-linear specification for $h$, under which $y_t$ is stationary and unconditionally normally distributed, with first-order autocorrelation coefficient of 0.5, and a nonlinear conditional mean function. The conditional mean function is plotted in Figure 1.

Figure 1 illustrates why linear approximations my be inadequate for stress testing: such approximations may work satisfactorily in the middle of the distribution, but can perform poorly in the tails. In this example the linear approximation matches the true conditional mean function reasonably well for $|Y_{t-1}| \leq 1.5$, but deviates outside this region[1]. For the study of "small" shocks to $Y_{t-1}$ the linear approximation may be acceptable, but for the study of large shocks (two or more standard deviations) it is not. For example, if $Y_{t-1} = -3$, corresponding to a three standard deviation shock in this setting, the linear approximation would predict $Y_t = -1.5$, while the true conditional mean of $Y_t$ is $-2.7$. Consider now a quadratic approximation to the conditional mean function, also plotted in Figure 1. This approximation is very close to the true conditional mean function for $|Y_{t-1}| \leq 2$. More importantly, for our interest in studying large shocks, the quadratic approximation also does a lot better in the tails. Continuing our previous example, if $Y_{t-1} = -3$ the quadratic approximation predicts $Y_t = -2.8$, close to the true value of $-2.7$. This simple example highlights the potential for more flexible models to provide better estimates of the impact of "large" shocks.

## 2.1 A nonlinear VAR(p) model

Consider a $(k \times 1)$ vector of macroeconomic variables, $\mathbf{Y}_t$. The most widely used model for the dynamics of macroeconomic time series is the linear vector autoregression.

$$\mathbf{Y}_t = B_0 + B_1 \mathbf{Y}_{t-1} + ... B_p \mathbf{Y}_{t-p} + \mathbf{e}_t \tag{4}$$

More generally, we can think about the mapping between $\mathbf{Y}_{t-1}^p \equiv \left[\mathbf{Y}'_{t-1}, ..., \mathbf{Y}'_{t-p}\right]'$ and $\mathbf{Y}_t$ as some general unknown function, $\mathbf{g} : \mathbb{R}^{pk} \rightarrow \mathbb{R}^k$

$$\mathbf{Y}_t = \mathbf{g}\left(\mathbf{Y}_{t-1}^p\right) + \mathbf{e}_t \tag{5}$$

and interpret the standard VAR above as a simple first-order Taylor series approximation to the

[1]The linear approximation to the true conditional mean function is obtained by noting that the optimal mean squared error approximation is a line through zero with slope equal to one-half. This follows from the fact that $E[Y_t] = 0$, $V[Y_t] = 1$ and $Cov[Y_t, Y_{t-1}] = 0.5$. The quadratic approximation can similarly be derived analytically from the properties of the joint distribution of $(Y_t, Y_{t-1})$.

unknown function $\mathbf{g}$. For notational simplicity let us assume that all variables have mean zero.

$$\begin{aligned}\mathbf{g}\left(\mathbf{Y}_{t-1}^{p}\right) &\approx \mathbf{g}\left(\mathbf{0}\right)+\nabla\mathbf{g}\left(\mathbf{0}\right)\mathbf{Y}_{t-1}^{p}\\ &\equiv B_0+B_1^p\mathbf{Y}_{t-1}^{p}\\ &\equiv B_0+B_1\mathbf{Y}_{t-1}+...+B_p\mathbf{Y}_{t-p} \end{aligned}\tag{6}$$

If we are primarily interested in studying the dynamics of $\mathbf{Y}_t$ "near" its unconditional mean, then the first-order approximation of $\mathbf{g}$ provided by a standard VAR may be sufficiently accurate. By convention, VAR studies show impulse response functions to one standard deviation shocks. But it is well known (see for example Koop *et al.* (1996)) that for standard linear VAR models the magnitude of the shock has no impact on the shape of the impulse response function; it merely affects the scale. As discussed above, in stress testing studies interest lies not in small- or medium-sized shocks, but extreme shocks. Considering three or five standard deviation shocks means considering the dynamics of the variables "far" from their unconditional mean. In such case the first-order Taylor series approximation may be a poor approximation to the true, unknown, data generating process.

An obvious extension is to expand to a second or third-order Taylor series approximation of $\mathbf{g}$. As shown in the Appendix, the mean function, up to a second order approximation[2], will be given by:

$$\mathbf{g}\left(\mathbf{Y}_{t-1}^{p}\right)=B_0+\sum_{m=1}^{p}B_{1m}\mathbf{Y}_{t-m}+\sum_{i=1}^{p}\sum_{j=i}^{p}B_{2ij}vech\left(\mathbf{Y}_{t-i}\mathbf{Y}_{t-j}^{\prime}\right)\tag{7}$$

where $vech\left(X\right)$ stacks only the lower triangle of the matrix $X$. We use the $vech$ function rather than the $vec$ function as $\mathbf{Y}_{t-1}^{p}\mathbf{Y}_{t-1}^{p\prime}$ includes both $Y_{1,t-1}Y_{2,t-1}$ and $Y_{2,t-1}Y_{1,t-1}$, for example, and we can collect such terms. The number of unknown parameters is larger for the second-order Taylor series approximation relative to the first-order approximation. The first-order approximation has $k+pk^2$ free parameters while the more flexible model has $k+pk^2+pk^2\left(p+1\right)\left(k+1\right)/4$ free parameters. Several possibilities exists for reducing the number of free parameters. First, one could restrict all second-order effects in equation $i$ to include $Y_{t-m,i}$. Alternatively, we could restrict the second-order terms to only include lagged squared terms. Possibly due to the high parameterisation of non-linear VAR models, these are not widely-used. In the next section we describe a method, proposed by Jorda (2005), to estimate non-linear VAR models.

[2] The same analysis can easily be repeated for the third-order Taylor series approximation, adding greater flexibility at the cost of additional parameters.

# 3 Estimation results for the non-linear macro VAR

## 3.1 Estimation of flexible non-linear approximations

We employ a third-order approximation in our models of the relationships between the macroeconomic variables and the measures of corporate default. In the interests of parsimony we drop all cross-product terms from this approximation, and consider only one lag of the higher-order terms. Thus, our model for the macroeconomic variables at the one-quarter horizon is:

$$Y_{jt} = \beta_{0j} + \sum_{m=1}^{p} \boldsymbol{\beta}'_{1jm} \mathbf{Y}_{t-m} + \boldsymbol{\gamma}'_{2j} \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} + \boldsymbol{\gamma}'_{3j} \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} + e_{jt} \tag{8}$$

for $j = 1, 2, 3$, where $\odot$ is the Hadamard product. This model is estimable via OLS, and thus is very simple to implement.

As discussed in the introduction, an implication of considering a standard linear VAR as an *approximation* to the true DGP, rather than as the DGP itself, is that it is no longer clear that forecasts or stress tests of horizons greater than one period should be obtained by iterating the one-period model forward. Jorda (2005) proposes an alternative approach, namely to estimate a different approximation model for each horizon of interest. Following this argument, one can estimate a set of models for the three macroeconomic variables and eight horizons:

$$Y_{j,t+h-1} = \beta^{h}_{0j} + \sum_{m=1}^{p} \boldsymbol{\beta}^{h\prime}_{1jm} \mathbf{Y}_{t-m} + \boldsymbol{\gamma}^{h\prime}_{2j} \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} + \boldsymbol{\gamma}^{h\prime}_{3j} \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} \odot \mathbf{Y}_{t-1} + e^{h}_{jt} \tag{9}$$

for $j = 1, 2, 3$ and $h = 1, 2, ..., 8$. For each variable and each horizon this model is estimable via OLS.

Estimating the model for each horizon of interest it is simple to obtain the confidence intervals on the impulse responses, or stress tests: they come directly from the covariance matrix of the parameters estimated for each horizon. This is in contrast with standard linear VAR approach, where the confidence intervals for horizons greater than one period must be obtained either via the "delta" rule, or a bootstrap procedure.

## 3.2 Data

We use quarterly data on three key UK macroeconomic variables, GDP growth, the three-month Treasury bill rate, and inflation, to summarise the state of the macro economy. To proxy for corporate credit risk we use the corporate liquidation rate defined as the number of defaulting companies in a given quarter relative to the total number of companies. Our macro model is small relative to some of the macroeconomic models used in the analysis of credit risk, Pesaran *et al.* (2005) being a prominent example. But it is large enough to convey the main ideas of this paper.

The sample period is 1992Q4 to 2004Q3. These series are available for a much longer period, but we focus on data after 1992Q4, as at this point the UK adopted an inflation targeting regime and it has been recognized that inflation targeting in the UK and other countries lead to a significant reduction in the volatility of macroeconomic series (see for example Kuttner and Posen, 1999 or Benati 2004). It is, therefore, reasonable to assume that the introduction of inflation targeting induced a structural break in the UK macroeconomic time series in 1992Q4. In Figure 3 we plot the macroeconomic variables and some descriptive statistics of the data set can be found in Table 1.

### 3.3 The estimated macroeconomic non-linear VAR

One of the properties of a standard linear VAR is that the size and the sign of the shock, as well as the starting values of the variables, do not change the shape of the impulse response function[3] (IRF), see Koop *et al.* (1996). In the non-linear VAR, however, the size, the sign and starting values are important. In all cases we evaluate the IRF holding all non-shocked variables at their unconditional averages[4]. Consistent with the existing macroeconomic literature, we order the variables as GDP growth, inflation, interest rate.

In line with Jorda (2005) and much of the VAR literature, we use a Cholesky factorisation of the covariance matrix of errors to obtain the scenarios. The use of a simple Cholesky factorisation may be restrictive, but for the purpose of this paper this method is sufficient to illustrate our results. The methodology outlined below, however, is general enough to consider any type of scenario. To distinguish between small and large shocks we consider the impact on variables in the VAR from unexpected one and three standard deviation shocks to GDP growth, inflation and the interest rate.

In Figures 4 to 7 we plot the impulse response functions (IRFs) of the three-variable macroeconomic VAR to shocks of various sizes and signs. Figure 4 reveals that in most one-standard deviation IRFs the cubic and the linear models yield similar results. But the response of interest rates to GDP growth shocks and interest rate shocks do differ substantially: the response of inter-

[3] Given the correspondence between impulse response functions and stress tests in our study, we will use these terms interchangeably.

[4] For details on the computation of IRFs in this setting see Jordà (2005).

est rates to a GDP growth shock and an interest rate shock is significantly greater, for horizons 1 through 4 quarters, if cubic terms are considered than if these are ignored.

Figure 5 shows the IRFs for a -1 standard deviation shock. For the linear model this figure is just a sign change of the plots in Figure 4, whereas this is not necessarily so for the cubic model. For example, the response of interest rates to a positive GDP shock was significantly greater using the cubic model than the linear model, whereas this difference between the two models essentially disappears for a negative GDP shock.

In Figure 6 we present the IRFs for a positive 3 standard deviation shock. For the linear model these IRFs are just 3 times the IRFs from Figure 4, while this is not so for the cubic model. Some interesting differences appear comparing Figures 4 and 6. For example, the response of interest rates to a 1 standard deviation inflation shock was small and positive (negative) for the linear (cubic) model, slowly increasing as the horizon approached eight quarters. However, for a three standard deviation shock the cubic model suggests a large positive response of interest rates for the first 5 quarters, followed by a decrease in interest rates at the 8th quarter. This indicates a difference between small shocks to inflation, which lead to modest changes in interest rates, and very large shocks to inflation, which lead to much different interest rate reactions.

### 3.4 Impulse response function for the liquidation rate

To estimate the impact of macroeconomic shocks on the aggregate liquidation rate we estimate the following Logit model:

$$\Delta^h \Lambda^{-1}\left(P_{t+h-1}\right) = \beta_0^h + \alpha_1^h \Delta\Lambda^{-1}\left(P_{t-1}\right) + \boldsymbol{\beta}_{1j}^{h\prime}\mathbf{Y}_{t-1} + \boldsymbol{\gamma}_{2j}^{h\prime}\mathbf{Y}_{t-1}\odot\mathbf{Y}_{t-1} + \boldsymbol{\gamma}_{3j}^{h\prime}\mathbf{Y}_{t-1}\odot\mathbf{Y}_{t-1}\odot\mathbf{Y}_{t-1} + e_{t+h-1} \tag{10}$$

for $h = 1, 2, ..., 8$, where $\Delta^h X_t \equiv X_t - X_{t-h}$, $\Lambda(x) = 1/\left(1 + e^{-x}\right)$ is the standard logistic function and $\mathbf{Y}_t = [Y_{1t}, Y_{2t}, Y_{3t}]'$ is the vector of the three macroeconomic variables. Since the model above involves a nonlinear transformation of the liquidation rate, to obtain the results from a stress test, or impulse response function, we use a simulation-based method[5]. Although we estimate the model in differences (of the transformed rates), we present the impulse responses for the liquidation rate in levels, as this is the object of economic interest.

[5] The stress test results and confidence intervals are obtained by estimating the above model and then simulating 10,000 draws from the asymptotic distribution of the parameter estimates, and a Normal distribution for the regression residual, to obtain an estimated impact of a shock above what would be observed when all variables are held at their unconditional values.

In Figure 8-11 we present the results of the stress tests for the liquidation rate. Again, in all cases we evaluate the IRF holding all non-shocked variables at their unconditional averages. Since the variable of interest is a probability, it is more natural to present the results as a proportion of the base case, i.e. as $E\left[\tilde{P}^{(j)}_{t+h-1}\left(\tilde{\mathbf{Y}}_{t-1}\right)/\bar{P}^{(j)}_{t+h-1}\right]$, rather than as a difference from the base case, $E\left[\tilde{P}^{(j)}_{t+h-1}\left(\tilde{\mathbf{Y}}_{t-1}\right)-\bar{P}^{(j)}_{t+h-1}\right]$. Hence to interpret the impulse response function, a value of 2 corresponds a doubling, relative to the sample average, of the probability of default.

Figure 8 presents the results of a 1 standard deviation shock to each of the three macroeconomic variables, using either the linear or the cubic model. This figure reveals that the mean impact on the corporate liquidation rate from a macroeconomic shock is relatively small: the largest impact on the liquidation rate occurs 8-quarters after a shock to GDP (as estimated from the cubic model) where the liquidation rate increases by almost 10% from its sample average. In Figure 9 we show the results of the 3 standard deviation macroeconomic shocks. In comparison to Figure 8 it is clear that large shocks have a significantly different impact on the liquidation rate than small shocks. For a positive 3 standard deviation shock to GDP the non-linear model predicts a significant fall in the liquidation rate 2 quarters ahead - based on the linear VAR one would falsely conclude that the liquidation rate does not change in the quarters following large GDP shocks.

In all cases, the confidence bounds on the impulse responses are wide. In some applications the fact that the confidence intervals always include 1, the base case, would be taken as evidence that corporate liquidations are independent of business cycle shocks. But the estimation uncertainty is important for regulators as well as banks to set capital at a sufficiently conservative level. Therefore, we should pay more attention to the upper confidence interval bound which is the upper bound on the mean impact of a shock to each of these variables at the 95% confidence level in our figures. Following a 1 standard deviation shock the upper bound is approximately 1.1 for all three shocks at the one-quarter horizon, and is between 1.3 and 1.4 at the eight-quarter horizon. Thus it is plausible that the mean impact of a one standard deviation shock is a 30% to 40% increase in the probability of default, meaning that the liquidation rate could move from 1.32% to as high as 1.85%, a substantial increase. This effect is even more significant if the shocks are large, where the impact from macro shocks can lead to a more than a 100% increase of PDs at a 95% confidence level.

Of course, the upper confidence interval bound could be made arbitrarily close to 100% by including more and more, potentially irrelevant, variables in the model for the liquidation rate. Doing so would increase the estimation error, increasing the uncertainty surrounding the estimated impact of a shock, and thus increase the upper confidence bound. For this reason it is important to carefully consider which variables to include in the model and the degree of flexibility to allow. By including numerous irrelevant variables we will likely obtain an upper confidence bound that is too conservative; by excluding the possibility of non-linearities and other effects we may instead obtain an upper confidence bound that is too low that does not reflect the true uncertainty.

Our conclusions are confirmed when looking at the impact of small and large negative shocks (Figure 10 and 11 in the Appendix). Following a negative 3 standard deviation shock to GDP the liquidation rate increases significantly (borderline) in the second quarter by roughly 50% relative to the sample average. Independent of the forecast horizon we find that the liquidation rate falls strongly, and significantly, in the quarters after large negative shocks to the interest rate. The maximum fall occurs after 8 quarters when the liquidation rate is roughly 30% lower than on average. Overall, our results suggest that large positive as well as negative unexpected changes in interest rates have some impact on the corporate liquidation rate both in the very short and in the more intermediate term (up to 2 years).

### 3.5 Estimating the non-linear VAR over different samples

Looking at the write-off ratio of UK banks, Hoggarth *et al.* (2005) find that the sample period is important for their conclusions on the link between credit risk and macroeconomic shocks. In particular, they find that the impact of a shock to output, relative to potential, is stronger in the years after the UK adopted an inflation targeting regime. As discussed in Section 3.2, this may be due to a structural break in the relation between the macroeconomic variables. Obviously, if a structural break is present in 1992 then it would be better to focus on the post 1992 estimations as we have done so far. However, we extend our sample back to 1985Q1 for comparison. This sample period includes the recession of the early 1990s, which may contain useful information for stress testing. The IRFs from the estimated VAR based on the longer sample shocks can be found in Figure 12 and 13. We only show the impact on the liquidation rate following large macroeconomic shocks as small shocks have an insignificant impact on the liquidation rate at all horizons.

This robustness check leads to some interesting conclusions. First, we find that large positive GDP shocks imply a significant fall in the corporate liquidation rate, using the cubic model, up to 1 year following the shock. Although negative GDP shocks are found to increase the corporate liquidation rate in the short run the impact is insignificant - the maximum impact from a large negative GDP shock is after 3 quarters, rather than 2 quarters using the 1992Q4-2004Q3 sample. Second, we find a substantially different impact from interest rate shocks on the corporate liquidation rate in comparison with the VAR estimated on the 1992Q2-2004Q3 sample. Large positive shocks to the interest rate increases the corporate liquidation rate significantly at all horizons with a maximum impact after 8 quarters, which is 8 times as high as its sample average. Large negative interest rate shocks, on the other hand, decrease the corporate liquidation rate significantly, in particular 2-6 quarters following the interest rate shock. The largest fall in the liquidation rate after large negative interest rate shocks is in the fifth quarter when the level of the liquidation rate is around 90% lower relative to its sample average.

The results from this alternative sample period indicate that the corporate liquidation rate is much more strongly related to the interest rate once the recession of the early 1990s is included in the sample. Large negative shocks to GDP do not lead to a significant increase in the corporate liquidation rate whereas both large positive GDP and large negative interest rate shocks imply a fall in the corporate liquidation rate. These results emphasise that some care must be taken in the choice of sample period: in our case this involved trading off valuable information from the recession of the early 1990s against the use of data from a different statistical regime.

## 4 Conclusion

In this paper we investigate the impact of possible non-linearities on credit risk in a VAR setup. As standard VAR models are unable to deal with non-linearities we use the methodology proposed by Jorda (2005). The key insight of Jorda was to interpret a general VAR as a first order Taylor series approximation of an unknown data generating process. His approach allows to estimate more flexible approximations, which capture possible non-linearities in the data. We apply this methodology to a small model of the macro economy and extend it to analyse the interaction between the aggregate corporate liquidation rate and macroeconomic variables. We show that the results of the non-linear VAR are different to results using standard linear models, especially when considering large shocks. This was illustrated using a simple three variable macro model. Most importantly, we show that accounting for non-linearities in the underlying macroeconomic environment leads to substantially different conclusions for aggregate credit risk projections. In contrast to most other papers we explicitly account for the underlying estimation uncertainty of the models. We show that this can have significant implications for the estimated level of credit risk, especially when looking at the tails of the credit risk distribution.

Overall, our analysis confirms the findings of previous papers (see for example Benito *et al.* (2001)) which suggest that large increases in interest rates are a key driver of credit risk, and that large positive shocks to GDP tend to reduce risk significantly. Our analysis also points to a stronger relation between the liquidation rate and the macroeconomic variables, in particular interest rates, once the early 1990s is included in the sample.

# 5 Appendix

## 5.1 A simple non-linear model

To illustrate the importance of different approximations to a non-linear data generating process consider $(y_t, y_{t-1})$ the following joint distribution:

$$(y_t, y_{t-1}) \sim F = C_C(\Phi, \Phi; \kappa)$$

where $F$ is some bivariate distribution with standard normal marginal distributions (denoted $\Phi$) connected with Clayton's copula, $C_C$, with dependence parameter $\kappa$. This type of time series process was first studied in economics by Chen and Fan (2004). This implies that we can write

$$\begin{aligned} y_t &= h(y_{t-1}, \varepsilon_t; \kappa) \\ \text{where } \varepsilon_t | y_{t-1} &\sim N(0,1) \\ C_C(u|v;\kappa) &\equiv \frac{\partial C_C(u,v;\kappa)}{\partial v} \\ h(y,\varepsilon;\kappa) &\equiv C_C^{-1}(\Phi(\varepsilon)|\Phi(y);\kappa) \end{aligned}$$

which is a general, stationary, non-linear data generating process that is simple to simulate. In Figure 4 we show one simulated sample path from this process for $k = 1.1$, which yields $Corr[y_t, y_{t-1}] = 0.5$.

## 5.2 A second order Taylor expansion

Consider the second order expansion for the first element of $\mathbf{g}$, denoted $g_1$, where $g_1 : \mathbb{R}^{pk} \rightarrow \mathbb{R}$

$$g_1\left(\mathbf{Y}_{t-1}^p\right) \approx g_1(\mathbf{0}) + \nabla g_1(\mathbf{0})\,\mathbf{Y}_{t-1}^p + \frac{1}{2}\mathbf{Y}_{t-1}^{p\prime}\nabla^2 g_1(\mathbf{0})\,\mathbf{Y}_{t-1}^p \tag{11}$$

We can re-write this expression in a more convenient form by making use of the *vec* operator and the Kronecker product (denoted $\otimes$):

$$
\begin{aligned}
\underset{(1\times 1)}{g_1\left(\mathbf{Y}_{t-1}\right)} &\approx \underset{(1\times 1)}{g_1\left(\mathbf{0}\right)} + \underset{(1\times pk)}{\nabla g_1\left(\mathbf{0}\right)} \underset{(pk\times 1)}{\mathbf{Y}_{t-1}^{p}} + \frac{1}{2}\underbrace{vec(\underbrace{\nabla^2 g_1\left(\mathbf{0}\right)}_{(pk\times pk)})'}_{(1\times p^2k^2)} \underbrace{(\underset{(pk\times 1)}{\mathbf{Y}_{t-1}^{p}} \otimes \underset{(pk\times 1)}{\mathbf{Y}_{t-1}^{p}})}_{(p^2k^2\times 1)} \\
&= g_1\left(\mathbf{0}\right) + \nabla g_1\left(\mathbf{0}\right)\mathbf{Y}_{t-1}^{p} + \frac{1}{2} vec\left(\nabla^2 g_1\left(\mathbf{0}\right)\right)' vec\left(\mathbf{Y}_{t-1}^{p}\mathbf{Y}_{t-1}^{p\prime}\right)
\end{aligned}
$$

We can stack the equations to obtain:

$$
\begin{aligned}
\mathbf{g}\left(\mathbf{Y}_{t-1}\right) &\approx \mathbf{g}\left(\mathbf{0}\right) + \nabla \mathbf{g}\left(\mathbf{0}\right)\mathbf{Y}_{t-1}^{p} + \frac{1}{2}\begin{bmatrix} vec\left(\nabla^2 g_1\left(\mathbf{0}\right)\right)' \\ vec\left(\nabla^2 g_2\left(\mathbf{0}\right)\right)' \\ \vdots \\ vec\left(\nabla^2 g_k\left(\mathbf{0}\right)\right)' \end{bmatrix} vec\left(\mathbf{Y}_{t-1}^{p}\mathbf{Y}_{t-1}^{p\prime}\right) \\
\text{Let } \nabla^2 \mathbf{g}\left(\mathbf{0}\right) &\equiv \begin{bmatrix} vec\left(\nabla^2 g_1\left(\mathbf{0}\right)\right)' \\ vec\left(\nabla^2 g_2\left(\mathbf{0}\right)\right)' \\ \vdots \\ vec\left(\nabla^2 g_k\left(\mathbf{0}\right)\right)' \end{bmatrix} \\
\text{Then } \underset{(k\times 1)}{\mathbf{g}\left(\mathbf{Y}_{t-1}^{p}\right)} &= \underset{(k\times 1)}{\mathbf{g}\left(\mathbf{0}\right)} + \underset{(k\times pk)}{\nabla \mathbf{g}\left(\mathbf{0}\right)} \underset{(pk\times 1)}{\mathbf{Y}_{t-1}^{p}} + \frac{1}{2}\underset{(k\times p^2k^2)}{\nabla^2 \mathbf{g}\left(\mathbf{0}\right)} \underbrace{\left(\mathbf{Y}_{t-1}^{p} \otimes \mathbf{Y}_{t-1}^{p}\right)}_{(p^2k^2\times 1)} \\
&\equiv \underset{(k\times 1)}{B_0} + \underset{(k\times pk)}{B_1^p} \underset{(pk\times 1)}{\mathbf{Y}_{t-1}^{p}} + \underset{(k\times pk(pk+1)/2)}{B_2^p} \underbrace{vech\left(\mathbf{Y}_{t-1}^{p}\mathbf{Y}_{t-1}^{p\prime}\right)}_{(pk(pk+1)/2\times 1)}, \text{ collecting terms} \\
&\equiv \underset{(k\times 1)}{B_0} + \sum_{m=1}^{p} \underset{(k\times k)}{B_{1m}} \underset{(k\times 1)}{\mathbf{Y}_{t-m}} + \sum_{i=1}^{p}\sum_{j=i}^{p} \underset{(k\times k(k+1)/2)}{B_{2ij}} \underbrace{vech\left(\mathbf{Y}_{t-i}\mathbf{Y}_{t-j}^{\prime}\right)}_{(k(k+1)/2\times 1)}, \text{ collecting terms}
\end{aligned}
$$

where $vech\left(X\right)$ stacks only the lower triangle of the matrix $X$. We use the $vech$ function rather than the $vec$ function as $\mathbf{Y}_{t-1}^{p}\mathbf{Y}_{t-1}^{p'}$ includes both $Y_{1,t-1}Y_{2,t-1}$ and $Y_{2,t-1}Y_{1,t-1}$, for example, and we can collect such terms. Moving from the penultimate to the final line above also follows from a collection of terms, further reducing the number of free parameters.

## 5.3 Tables

Table 1

| Descriptive statistics | Mean | Standard deviation | Skewness | Kurtosis |
|---|---|---|---|---|
| Corporate liquidation rate | 0.0132 | 0.0046 | 1.2919 | 4.3303 |
| GDP growth | 0.0072 | 0.0032 | 0.4080 | 1.9562 |
| Interest rate | 0.0130 | 0.0025 | -0.1474 | 2.1483 |
| Inflation | 0.0062 | 0.0046 | 0.2076 | 2.7698 |

Descriptive statistics for data in VAR. 1992Q4-2004Q4.

## 5.4 Charts

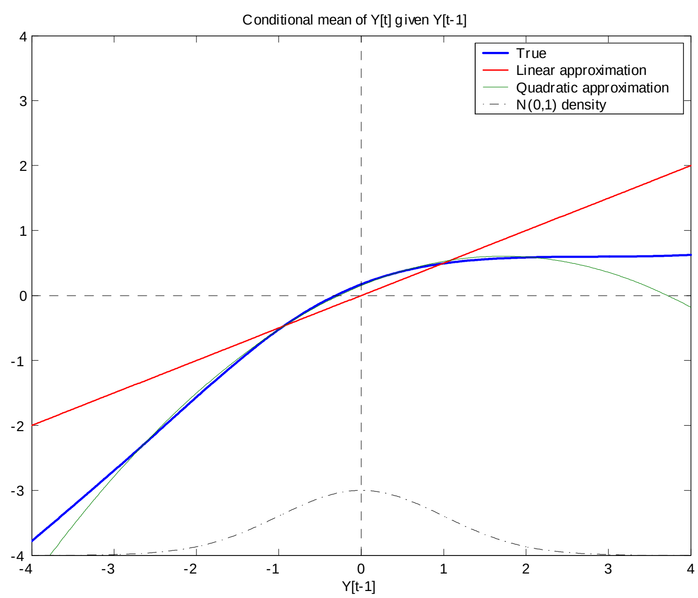


Figure 1: The conditional mean of $Y_t$ given $Y_{t-1}$ assuming $Y_t \sim N(0,1)$ and using a Clayton copula implying first-order autocorrelation of 0.5.

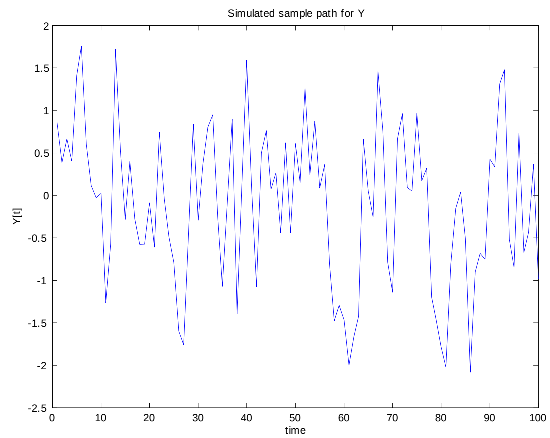


Figure 2: One sample path for $Y_t$ using a nonlinear DGP.

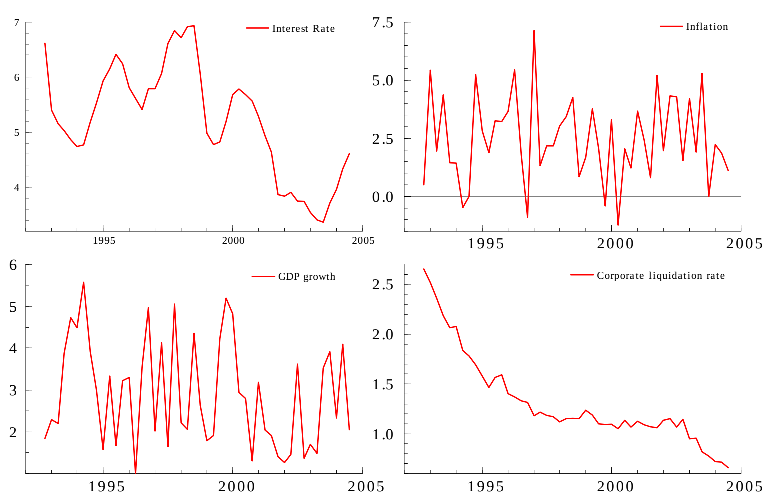


Figure 3: The Corporate liquidation rate and macroeconomic variables included in the VAR.

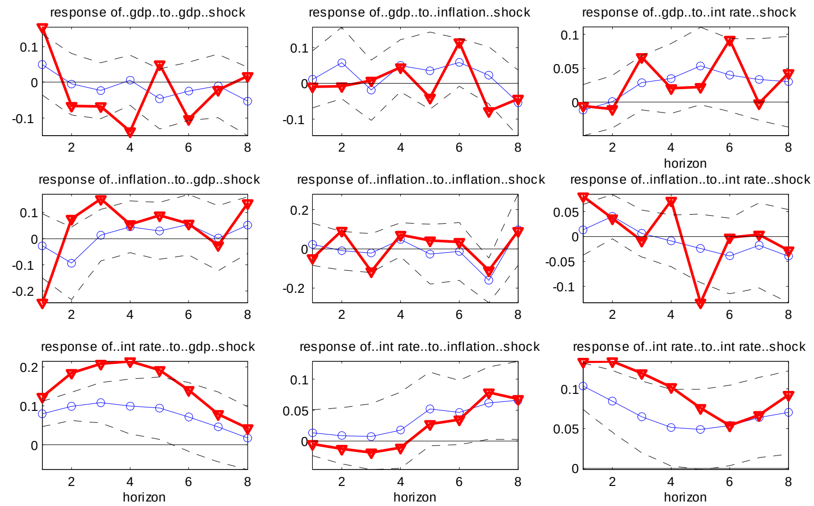


Figure 4: The thick line marked with triangles is the impulse response for a 1 standard deviation shock from the cubic projection; the thin line marked with circles is the impulse response from the linear projection; the dashed lines are the 95% confidence bounds on the impulse response from the linear projection.

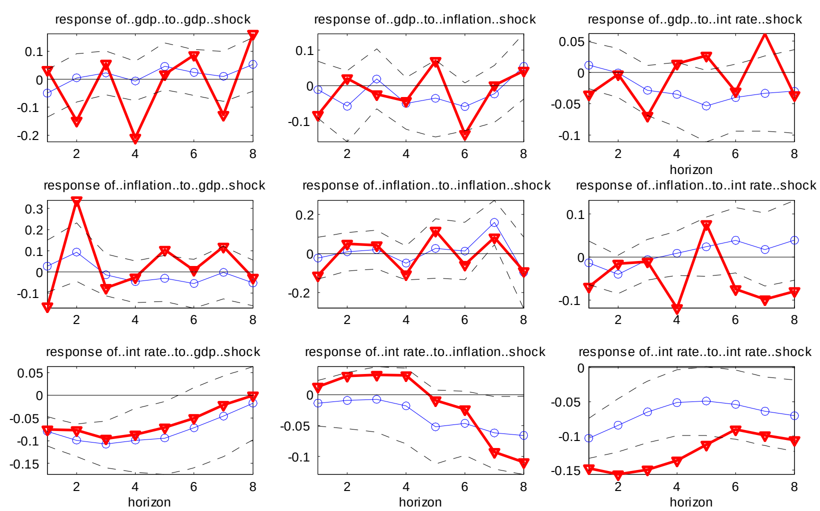


Figure 5: The thick line marked with triangles is the impulse response for a -1 standard deviation shock from the cubic projection; the thin line marked with circles is the impulse response from the linear projection; the dashed lines are the 95% confidence bounds on the impulse response from the linear projection.

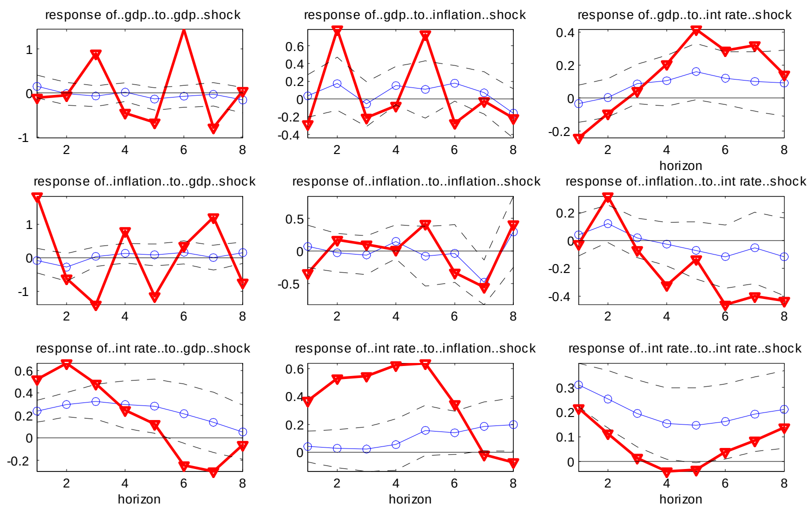


Figure 6: The thick line marked with triangles is the impulse response for a 3 standard deviation shock from the cubic projection; the thin line marked with circles is the impulse response from the linear projection; the dashed lines are the 95% confidence bounds on the impulse response from the linear projection.

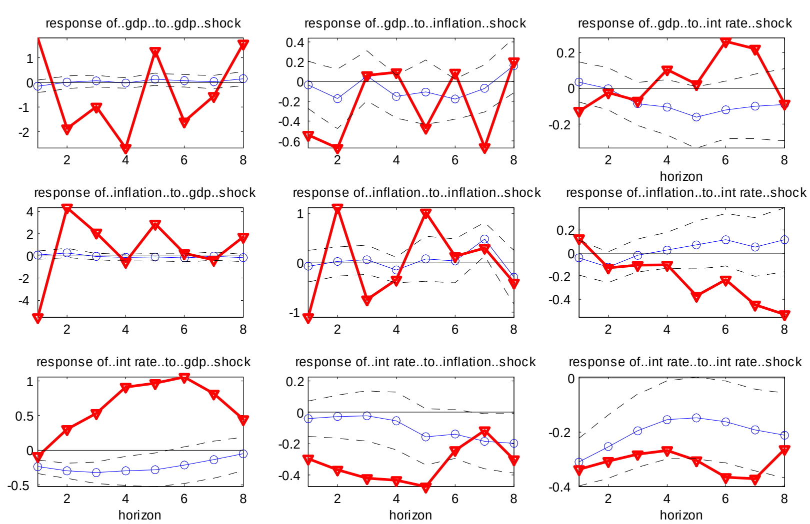


Figure 7: The thick line marked with triangles is the impulse response for a -3 standard deviation shock from the cubic projection; the thin line marked with circles is the impulse response from the linear projection; the dashed lines are the 95% confidence bounds on the impulse response from the linear projection.

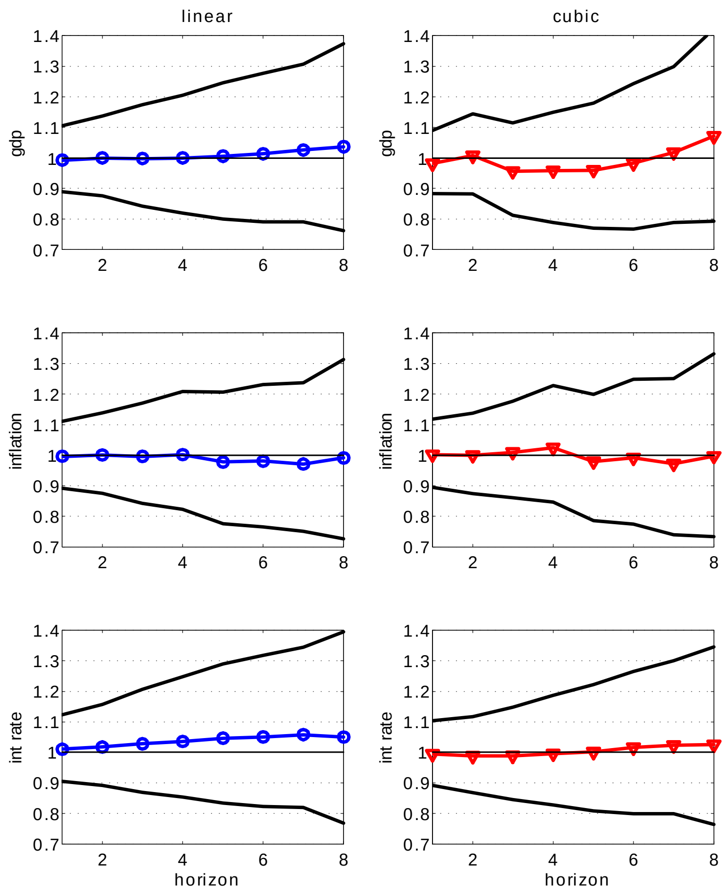


Figure 8: These figures show the response of the liquidation ratio to a 1 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line.

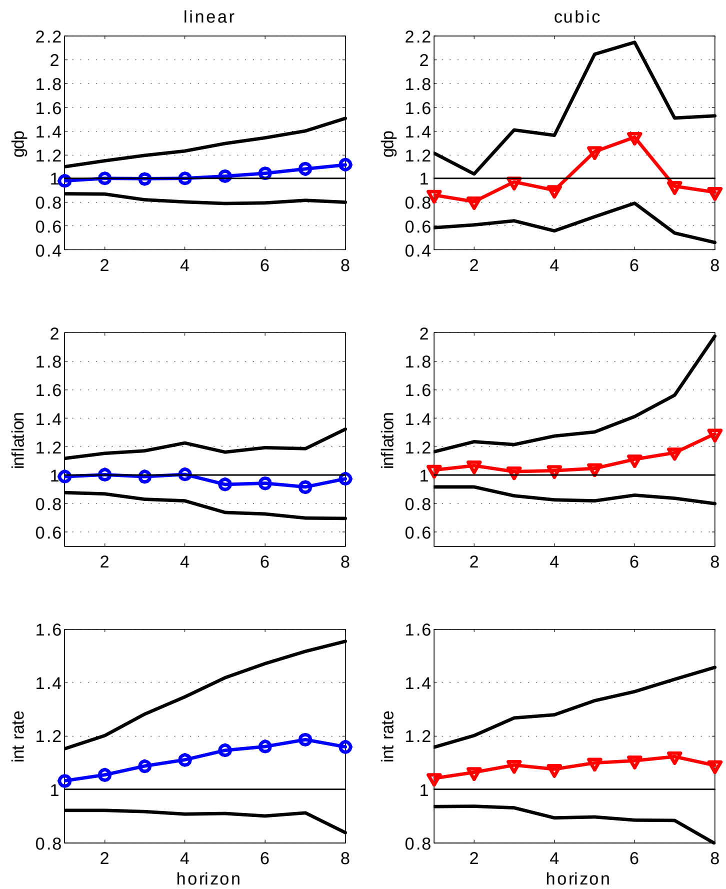


Figure 9: These figures show the response of the liquidation ratio to a 3 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line.

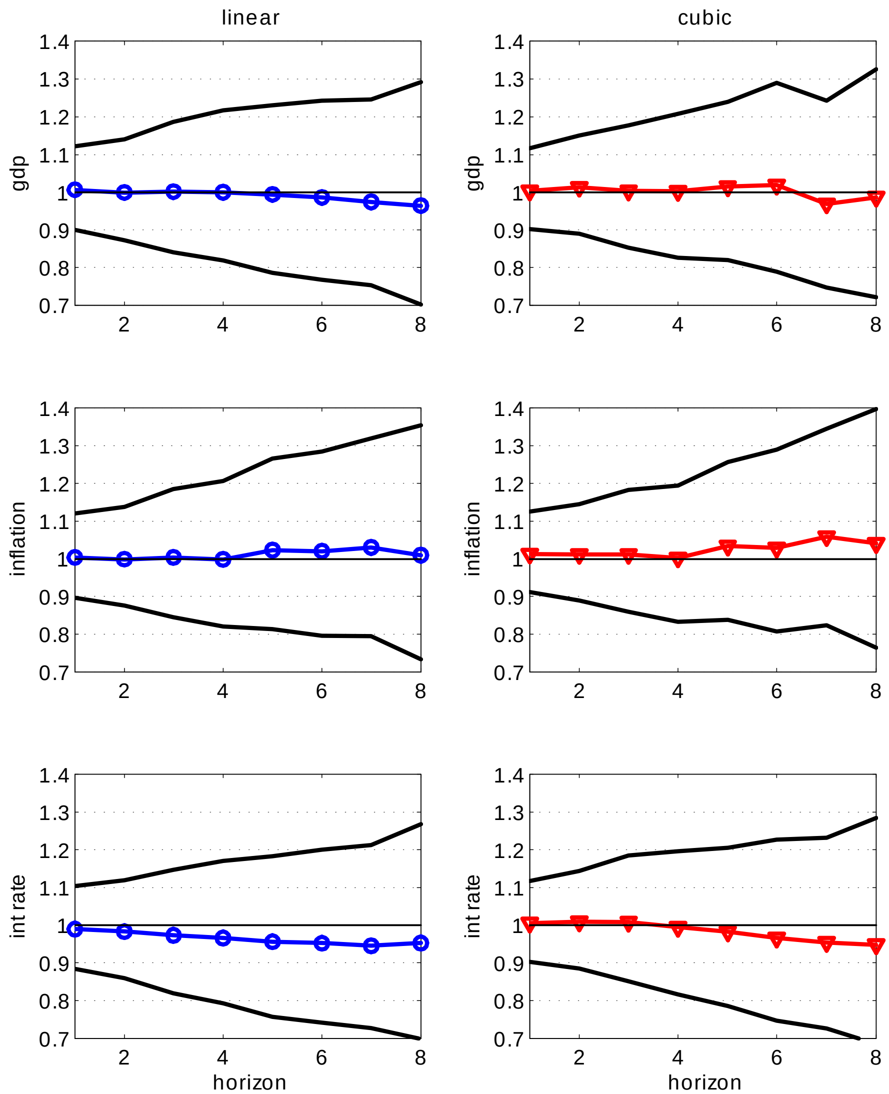


Figure 10: These figures show the response of the liquidation ratio to a -1 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line.

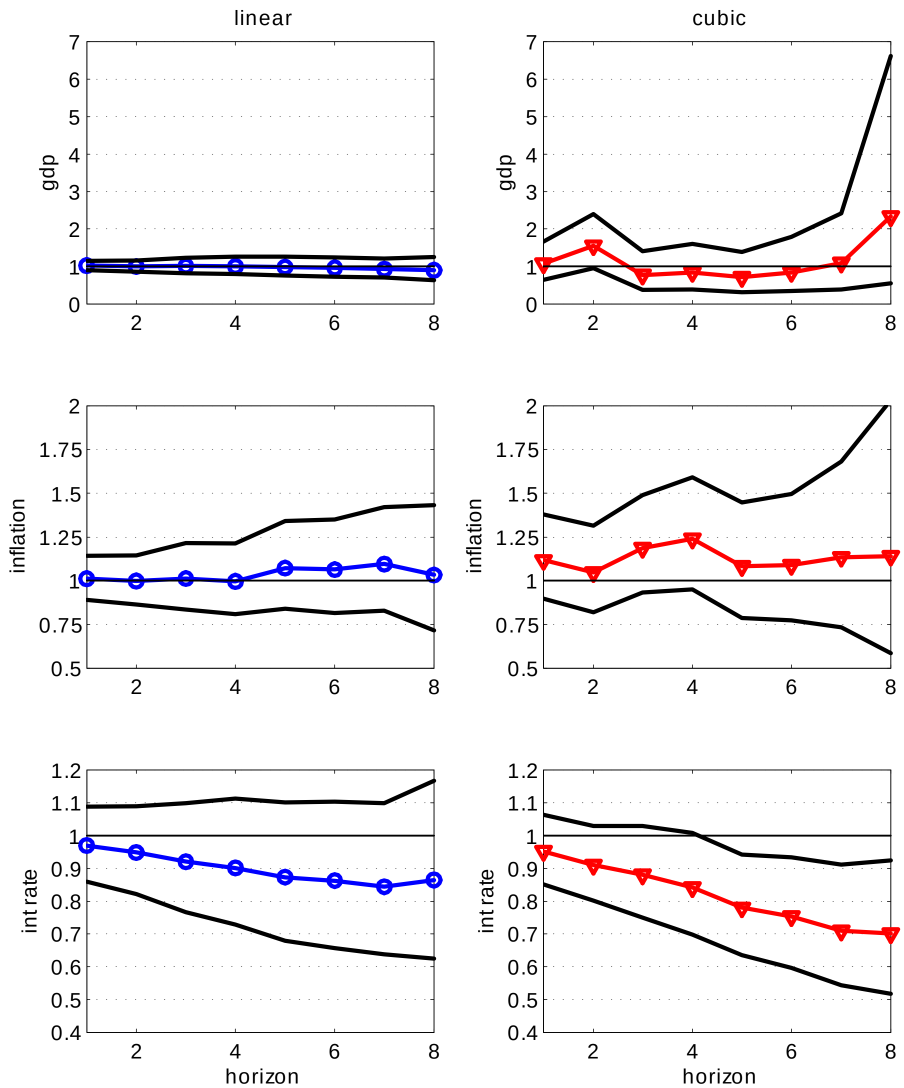


Figure 11: These figures show the response of the liquidation ratio to a -3 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line.

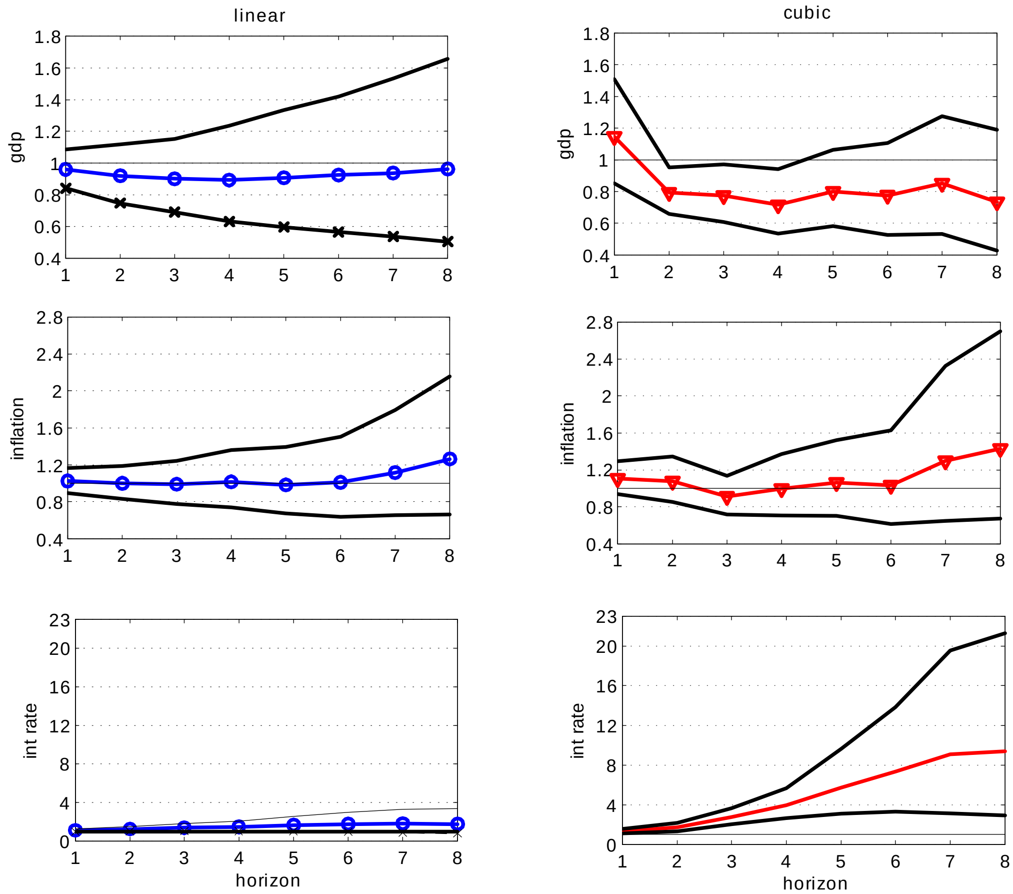


Figure 12: These figures show the response of the liquidation ratio to a +3 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line. Using sample from 1985Q1-2004Q3.

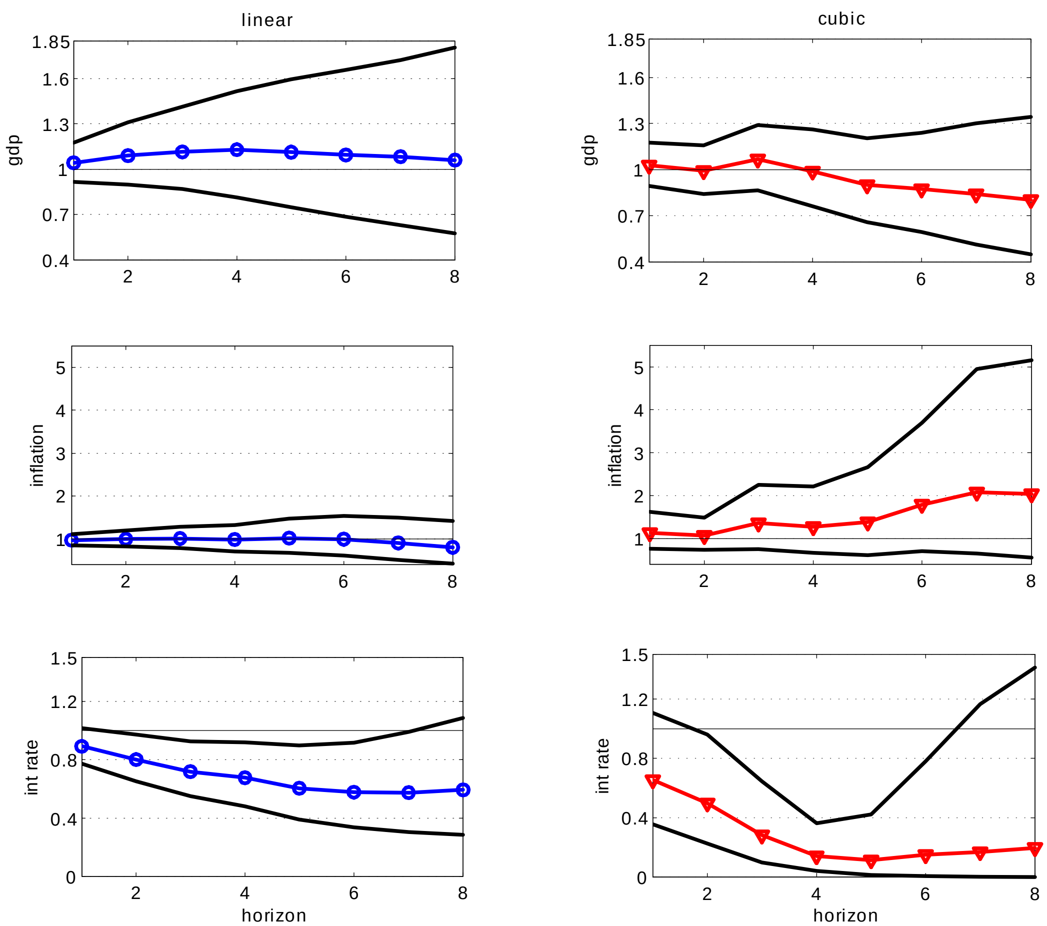


Figure 13: These figures show the response of the liquidation ratio to a -3 standard deviation shock, relative to the baseline liquidation ratio of 1.32% per year. The rows indicate the shocked variables; the columns show the model used, either a linear projection or a cubic projection. 95% confidence intervals are denoted with a thick line. Using sample from 1985Q1-2004Q3.

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