The Joe-Clayton and symmetrised Joe-Clayton density functions

Andrew Patton

6 November, 2003

The Joe-Clayton copula is called the family BB7 in Joe (*Multivariate Models and Dependence Concepts*, Chapman & Hall, 1997). In Joe it is defined using two parameters κ and γ, but I parameterise it using the coefficients of upper and lower tail dependence. The copula density, c, is obtained from the copula, C, (a distribution function)

$$c(u,v) \equiv \frac{\partial^2 C(u,v)}{\partial u \partial v}$$

The Joe-Clayton copula is:

$$\begin{aligned} C\left(u,v;\tau^U,\tau^L\right) &= 1-\left(1-\left[(1-(1-u)^\kappa)^{-\gamma}+(1-(1-v)^\kappa)^{-\gamma}-1\right]^{-1/\gamma}\right)^{1/\kappa} \\ \kappa &= 1/\left(\log_2\left(2-\tau^U\right)\right) \\ \gamma &= -1/\left(\log_2 \tau^L\right) \end{aligned}$$

The Joe-Clayton copula density is:

$$\begin{aligned} c_{JC}\left(u,v;\tau^U,\tau^L\right) &= \frac{A}{B} \\ A &= ((1-(1-u)^\kappa)^{\gamma-1}\cdot(1-u)^{\kappa-1}\cdot(-1+\kappa(\gamma(-1+(-1+ \\ &\quad (1-(1-u)^\kappa)^{-\gamma}+(1-(1-v)^\kappa)^{-\gamma})^{1/\gamma})+(-1+(1- \\ &\quad (1-u)^\kappa)^{-\gamma}+(1-(1-v)^\kappa)^{-\gamma})^{1/\gamma}))\cdot(1-(-1+(1- \\ &\quad (1-u)^\kappa)^{-\gamma}+(1-(1-v)^\kappa)^{-\gamma})^{-1/\gamma})^{1/\kappa}\cdot(1-(1-v)^\kappa)^{\gamma-1}\cdot(1-v)^{\kappa-1}) \\ B &= (((-1+(-1+(1-(1-u)^\kappa)^{-\gamma}+(1-(1-v)^\kappa)^{-\gamma})^{1/\gamma})^2)\cdot \\ &\quad ((1-(1-u)^\kappa)^\gamma+(1-(1-v)^\kappa)^\gamma-(1-(1-u)^\kappa)^\gamma\cdot \\ &\quad (1-(1-v)^\kappa)^\gamma)^2) \end{aligned}$$

The symmetrised Joe-Clayton copula and density are simple functions of the Joe-Clayton copula and density:

$$\begin{aligned} C_{SJC}\left(u,v;\tau^U,\tau^L\right) &= \frac{1}{2}\left\{C_{JC}\left(u,v;\tau^U,\tau^L\right)+C_{JC}\left(1-u,1-v;\tau^L,\tau^U\right)+u+v-1\right\} \\ c_{SJC}\left(u,v;\tau^U,\tau^L\right) &= \frac{1}{2}\left\{c_{JC}\left(u,v;\tau^U,\tau^L\right)+c_{JC}\left(1-u,1-v;\tau^L,\tau^U\right)\right\} \end{aligned}$$