B Appendix: Proofs for Results in Section 2

Proof of Proposition 1.

Claim B1 *For any incentive feasible mechanism $\mathcal{G}$ of the form (3), there exist an incentive feasible mechanism*

$$G = \left(\left(\rho^j, \eta_1^j, ..., \eta_n^j \right)_{j \in \mathcal{J}}, (t_i)_{i \in \mathcal{I}} \right), \tag{B1}$$

that generates the same social surplus, where $\rho^j : \Theta^n \to [0,1]$ is the provision rule for good j, $\eta_i^j : \Theta \to [0,1]$ is the inclusion rule for agent i and good j,and $t_i : \Theta \to R$ is the transfer rule for agent i.

Proof. Consider an incentive feasible mechanism $\mathcal{G}$. Pick $k \in [0,1]$ arbitrarily and define,

$$\begin{aligned}
\rho^j(\theta) &= \mathrm{E}_\Xi \zeta^j(\theta, \vartheta) = \int_0^1 \zeta^j(\theta, \vartheta)\, d\vartheta && \text{(B2)} \\
\eta_i^j(\theta_i) &= \begin{cases} \frac{\mathrm{E}_{-i}\zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)}{\mathrm{E}_{-i}\zeta^j(\theta,\vartheta)} = \frac{\int_{\Theta_{-i}^n}\int_0^1 \zeta^j(\theta,\vartheta)\omega_i^j(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})}{\int_{\Theta_{-i}^n}\int_0^1 \zeta^j(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})} & \text{if } \int_{\Theta_{-i}^n}\int_0^1 \zeta^j(\theta,\vartheta)\, d\vartheta d\mathbf{F}(\theta_{-i}) > 0 \\ k & \text{if } \int_{\Theta_{-i}^n}\int_0^1 \zeta^j(\theta,\vartheta)\, d\vartheta d\mathbf{F}(\theta_{-i}) = 0 \end{cases} \\
t_i(\theta_i) &= \mathrm{E}_{-i}\tau(\theta) = \int_{\Theta_{-i}^n} \tau(\theta)\, d\mathbf{F}(\theta_{-i}),
\end{aligned}$$

for each $\theta \in \Theta^n, j \in \mathcal{J}$ and $i \in \mathcal{I}$. This is a mechanism of the form in (B1), and we will call it G. Use of the law of iterated expectations on $\rho^j(\theta)$ and $t_i(\theta_i)$ shows that (BB) is unaffected when switching from $\mathcal{G}$ to G. It remains to show that the surplus is unchanged, and that (IC) and (IR) continue to hold under G. The utility of agent i of type $\theta_i \in \Theta$ who announces $\hat{\theta}_i \in \Theta$ is

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\omega_i^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\theta_i - \tau\left(\hat{\theta}_i, \theta_{-i}\right)\right] \text{ in mechanism } \mathcal{G} \tag{B3}$$

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \rho^j\left(\hat{\theta}_i, \theta_{-i}\right)\eta_i^j\left(\hat{\theta}_i\right)\theta_i - t_i\left(\hat{\theta}_i\right)\right] \text{ in mechanism } G. \tag{B4}$$

If $\int_{\Theta_{-i}^n}\int_0^1 \zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right) d\vartheta d\mathbf{F}(\theta_{-i}) = 0$, we trivially have that the payoffs in (B3) and (B4) are identical, whereas if $\int_{\Theta_{-i}^n}\int_0^1 \zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right) d\vartheta d\mathbf{F}(\theta_{-i}) > 0$, we have that

$$\begin{aligned}
\mathrm{E}_{-i}\rho^j\left(\hat{\theta}_i, \theta_{-i}\right)\eta_i^j\left(\hat{\theta}_i\right)\theta_i &= \mathrm{E}_{-i}\zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\frac{\mathrm{E}_{-i}\omega_i^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)}{\mathrm{E}_{-i}\zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)} && \text{(B5)} \\
&= \mathrm{E}_{-i}\omega_i^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\zeta^j\left(\hat{\theta}_i, \theta_{-i}, \vartheta\right)\theta_i.
\end{aligned}$$

Trivially, $\mathrm{E}_{-i}t_i(\theta_i) = t_i(\theta_i) = \mathrm{E}_{-i}\tau(\theta)$, which combined with (B5) implies that the payoffs in (B3) and (B4) are identical. Since the equality between (B3) and (B4) were established for any i, θ_i and $\hat{\theta}_i$, it follows that all incentive and participation constraints (IC) and (IR) hold for mechanism G given that they are satisfied in mechanism $\mathcal{G}$. Moreover, [again by (B5)]

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \rho^j(\theta)\,\eta_i^j(\theta_i)\,\theta_i\right] = \mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}} \omega_i^j(\theta,\vartheta)\,\zeta^j(\theta,\vartheta)\,\theta_i\right], \tag{B6}$$

so it follows by integration over Θ and summation over i that

$$\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\rho^{j}(\theta)\,\eta_{i}^{j}(\theta_{i})\,\theta_{i}\right]=\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\zeta^{j}(\theta,\vartheta)\,\omega_{i}^{j}(\theta,\vartheta)\,\theta_{i}\right], \tag{B7}$$

By construction, we also have that $\rho^{j}(\theta)=\mathrm{E}_{\Xi}\zeta^{j}(\theta,\vartheta)$ for every θ. Thus $\mathrm{E}\left[\rho^{j}(\theta)\,C^{j}(n)\right]=\mathrm{E}\left[\zeta^{j}(\theta,\vartheta)\,C^{j}(n)\right]$, implying that

$$\sum_{j\in\mathcal{J}}\mathrm{E}\rho^{j}(\theta)\left[\sum_{i\in\mathcal{I}}\eta_{i}^{j}(\theta_{i})\,\theta_{i}-C^{j}(n)\right]=\sum_{j\in\mathcal{J}}\mathrm{E}\zeta^{j}(\theta,\vartheta)\left[\sum_{i\in\mathcal{I}}\omega_{i}^{j}(\theta,\vartheta)\,\theta_{i}-C^{j}(n)\right]. \tag{B8}$$

Hence, $\mathcal{G}$ and G generate the same social surplus. ■

Claim B2 *For every incentive feasible mechanism of the form (B1), there exists an anonymous simple incentive feasible mechanism g of the form (5) that generates the same surplus.*

Proof. Consider an incentive feasible simple mechanism G on form (B1). For $k\in\{1,....,n!\}$, let $P_k:\mathcal{I}\to\mathcal{I}$ denote the k-th permutation of the set of agents $\mathcal{I}$. Note that $P_k^{-1}(i)$ gives the index of the agent who takes agent i's position in permutation P_k. Moreover, for any given $\theta\in\Theta^n$, let $\theta^{P_K}=\left(\theta_{P_k^{-1}(1)},...,\theta_{P_k^{-1}(n)}\right)\in\Theta^n$ denote the corresponding k-th permutation of θ.[18] For each $k\in\{1,...,n!\}$, let $G_k=\left(\left(\rho_k^{j},\eta_{k1}^{j},...,\eta_{kn}^{j}\right)_{j=1,2},t_{k1},...,t_{kn}\right)$ be given by

$$\begin{aligned}\rho_k^{j}(\theta)&=\rho^{j}\left(\theta^{P_k}\right)\quad\forall\,\theta\in\Theta^n,\ j\in\mathcal{J},\\ \eta_{ki}^{j}(\theta_i)&=\eta_{P_k^{-1}(i)}^{j}(\theta_i)\quad\forall\,\theta_i\in\Theta,\ j\in\mathcal{J},i\in\mathcal{I},\\ t_{ki}(\theta_i)&=t_{P_k^{-1}(i)}(\theta_i)\quad\forall\,\theta_i\in\Theta,i\in\mathcal{I},\end{aligned}\tag{B9}$$

and let $g=\left(\left(\widetilde{\rho}^{j},\widetilde{\eta}_1^{j},...,\widetilde{\eta}_n^{j}\right)_{j=1,2},\widetilde{t}_1,...,\widetilde{t}_n\right)$ be given by

$$\begin{aligned}&\widetilde{\rho}^{j}(\theta)=\tfrac{1}{n!}\textstyle\sum_{k=1}^{n!}\rho_k^{j}(\theta)\quad\forall\,\theta\in\Theta^n,\ j\in\mathcal{J}\\ &\widetilde{\eta}_i^{j}(\theta_i)=\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^{j}(\theta)\right]\eta_{ki}^{j}(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^{j}(\theta)\right]}\quad\forall\,\theta_i\in\Theta,i\in\mathcal{I},\ j\in\mathcal{J}\\ &\widetilde{t}_i(\theta_i)=\tfrac{1}{n!}\textstyle\sum_{k=1}^{n!}t_{ki}(\theta_i)\quad\forall\,\theta_i\in\Theta,i\in\mathcal{I}.\end{aligned}$$

We now note that: (1) for each $j\in\mathcal{J}$, $\widetilde{\rho}^{j}(\theta)=\widetilde{\rho}^{j}(\theta')$ if θ' is a permutation of θ. This is immediate since the sets $\left\{\rho_k^{j}(\theta)\right\}_{k=1}^{n!}=\left\{\rho^{j}(P_k(\theta))\right\}_{k=1}^{n!}$ and $\left\{\rho_k^{j}(\theta')\right\}_{k=1}^{n!}=\left\{\rho^{j}(P_k(\theta'))\right\}_{k=1}^{n!}$ are the same; (2) for $j\in\mathcal{J}$ and each pair $i,i'\in\mathcal{I},\widetilde{\eta}_i^{j}(\cdot)=\widetilde{\eta}_{i'}^{j}(\cdot)$. That is, the inclusion rules are the same for all agents. To see this, consider agent i and i', and suppose that $\theta_i=\theta_{i'}$. We then have that $\left\{\mathrm{E}_{-i}\left[\rho_k^{j}(\theta)\right]\eta_{ki}^{j}(\theta_i)\right\}_{k=1}^{n!}$ and $\left\{\mathrm{E}_{-i'}\left[\rho_k^{j}(\theta)\right]\eta_{ki'}^{j}(\theta_{i'})\right\}_{k=1}^{n!}$ are identical and that $\mathrm{E}_{-i}\left[\widetilde{\rho}^{j}(\theta)\right]=$

[18] To illustrate, suppose $n=3,M=2,\theta=(\theta_1,\theta_2,\theta_3)=((1,2),(3,2),(2,1))$. Consider, for example, purmutation k given by $P_k(1)=2,P_k(2)=1,P_k(3)=3$. Then $P_k^{-1}(1)=2,P_k^{-1}(2)=1,P_k^{-1}(3)=3$ and $\theta^{P_k}=\left(\theta_{P_k^{-1}(1)},\theta_{P_k^{-1}(2)},\theta_{P_k^{-1}(3)}\right)=(\theta_2,\theta_1,\theta_3)=((3,2),(1,2),(2,1))$.

$\mathrm{E}_{-i'}\left[\widetilde{\rho}^j(\theta)\right]$; and (3) for each pair $i,i'\in\mathcal{I}, \widetilde{t}_i(\cdot)=\widetilde{t}_{i'}(\cdot)$, which is obvious since the sets $\{t_{ki}(\theta_i)\}_{k=1}^{n!}$ and $\left\{t_{ki}\left(\theta_i'\right)\right\}_{k=1}^{n!}$ are identical. Together, (1), (2) and (3) establishes that g is anonymous and simple.

Now we show that g is incentive feasible and generates the same expected surplus as G. First, since G and G_k are identical except for the permutation of the agents, we have, for $k=1,...,n!$,

$$\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_{ki}^j(\theta_i)\,\theta_i^j-C^j(n)\right]\right\}=\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_i^j(\theta_i)\,\theta_i^j-C^j(n)\right]\right\}. \tag{B10}$$

Hence,

$$\begin{aligned}\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\widetilde{\rho}^j(\theta)\left[\sum_{i\in\mathcal{I}}\widetilde{\eta}^j(\theta_i)\,\theta_i^j-C^j(n)\right]\right\} &= \sum_{j\in\mathcal{J}}\mathrm{E}\left\{\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j-C^j(n)\right]\right\}\\ =\ \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\mathrm{E}_{\theta_i}\left\{\frac{1}{n!}\sum_{k=1}^{n!}\mathrm{E}_{-i}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)\,\theta_i^j\right\}-\mathrm{E}\left[\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\right]C^j(n)&\\ =\ \frac{1}{n!}\sum_{k=1}^{n!}\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_{ki}^j(\theta_i)\,\theta_i^j-C^j(n)\right]\right\}=\sum_{j\in\mathcal{J}}\mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}}\eta_i^j(\theta_i)\,\theta_i^j-C^j(n)\right]\right\},&\end{aligned}$$

where the last equality follows from (B10). Hence the surplus generated by g is identical to that by original mechanism G. To show that g is incentive feasible we first note that $\mathrm{E}\rho_k^j(\theta)=\mathrm{E}\rho^j(\theta)$ and $\mathrm{E}\sum_{i\in\mathcal{I}}t_{ki}(\theta_i)=\mathrm{E}\sum_{i\in\mathcal{I}}t_i(\theta_i)$ for all k, since the agents' valuations are drawn from identical distributions and G_k and G only differ in the index of the agents. Thus

$$\begin{aligned}\mathrm{E}\textstyle\sum_{i\in\mathcal{I}}\widetilde{t}_i(\theta_i)-\sum_{j\in\mathcal{J}}\mathrm{E}\widetilde{\rho}^j(\theta)\,C^j(n) &= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}}\frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\theta_i)-\sum_{j\in\mathcal{J}}\mathrm{E}\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\,C^j(n)\\ &= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}}t_i(\theta_i)-\sum_{j\in\mathcal{J}}\mathrm{E}\rho^j(\theta)\,C^j(n)\ ,\end{aligned}$$

so g satisfies (BB) if G does. Second, (IC) holds for any permuted mechanism, that is,

$$\mathrm{E}_{-i}\textstyle\sum_{j\in\mathcal{J}}\rho_k^j(\theta)\eta_{ki}^j(\theta_i)\,\theta_i^j-t_{ki}(\theta_i)\geq\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j-t_{ki}(\widehat{\theta}_i,\theta_{-i}) \tag{B11}$$

for all $i\in\mathcal{I}$, and $\theta_i,\widehat{\theta}_i\in\Theta$. Hence,

$$\begin{aligned}\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\widetilde{\rho}^j(\theta)\widetilde{\eta}^j(\theta_i)\,\theta_i^j-\widetilde{t}(\theta_i)=\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\left[\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\right]\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]}\theta_i^j-\frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\theta_i)&\\ =\ \frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)\,\theta_i^j-t_{ki}(\theta_i)\right]\geq\frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j-t_{ki}(\widehat{\theta}_i,\theta_{-i})\right]&\\ =\ \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j-\frac{1}{n!}\sum_{k=1}^{n!}t_{ki}(\widehat{\theta}_i,\theta_{-i})=\sum_{j\in\mathcal{J}}\mathrm{E}_{-i}\widetilde{\rho}^j(\widehat{\theta}_i,\theta_{-i})\widetilde{\eta}_i^j\left(\widehat{\theta}_i\right)\theta_i^j-\widetilde{t}(\widehat{\theta}_i),&\end{aligned} \tag{B12}$$

where the inequality follows from (B11). Hence g is satisfies (IC). Finally, g also satisfies the (IR) because (see the second line in (B12)) all the permuted mechanisms satisfy participation constraints. Proposition 1 follows by combining Claims B1 and B2. ■

Proof of Proposition 2.

Notation: This proof requires us to be explicit about the coordinates of the vector θ when permuting $\mathcal{J}$. We therefore need some extra notation for this proof (only). We will, with some abuse of notation, write $\mathbf{F}(\theta) \equiv \Pi_{i\in\mathcal{I}} F(\theta_i)$ and $\mathbf{F}(\theta_{-i}) \equiv \Pi_{k\in\mathcal{I}\setminus i} F(\theta_k)$ as the joint distribution of θ and θ_{-i} respectively. We write $\theta_i^{-j} = \left(\theta_i^1, ..., \theta_i^{j-1}, \theta_i^{j+1}, ..., \theta_i^M\right)$ for a type vector where good j has been removed. Analogously, $\theta^{-j} = \left(\theta_1^{-j}, ..., \theta_n^{-j}\right)$ stands for the type profile with good j coordinate removed for all agents and $\theta^j = \left(\theta_1^j, ..., \theta_n^j\right)$ is the vector collecting the valuations for good j for all agents. Furthermore, $\theta_{-i}^{-j} = \left(\theta_1^{-j}, ..., \theta_{i-1}^{-j}, \theta_{i+1}^{-j}, ..., \theta_n^{-j}\right)$ and $\theta_{-i}^j = \left(\theta_1^j, ..., \theta_{i-1}^j, \theta_{i+1}^j ..., \theta_n^j\right)$ are used for the vectors obtained respectively from θ^{-j} and θ^j by removing agent i. These conventions are used also on the distributions, so, for example, $\mathbf{F}_{-i}^{-j}$ denotes the cumulative distribution of θ_{-i}^{-j}. Conditional distributions are denoted in the natural way: for example $\mathbf{F}_{-i}^{-j}\left(\cdot| \,\theta_i^j\right)$ denotes the joint distribution of θ_{-i}^{-j} conditional on θ_i^j. Since no integrals are taken over subsets of the range of integration, we also conserve space and write $\int_\theta h(\theta)\, d\mathbf{F}(\theta)$ rather than $\int_{\theta\in\Theta^n} h(\theta)\, d\mathbf{F}(\theta)$ when integrating a function h over θ and similarly for integrals over various components of θ.

Proof. Consider a simple anonymous incentive feasible mechanism g. For $k \in \{1, ..., M!\}$ let $P_k : \mathcal{J} \to \mathcal{J}$ be the k-th permutation of $\mathcal{J}$ and $\theta_i^{P_k} = \left(\theta_i^{P_k^{-1}(1)}, ..., \theta_i^{P_k^{-1}(M)}\right) \in \Theta$ be the permutation of θ_i when the goods are permuted according to P_k. Let $\theta^{P_k} = \left(\theta_1^{P_k}, ..., \theta_n^{P_k}\right) \in \Theta^n$ denote the corresponding permutation of θ.[19] For each $k \in \{1, ..., M!\}$ define mechanism $g_k = (\left\{\rho_k^j\right\}_{j\in\mathcal{J}}, \left\{\eta_k^j\right\}_{j\in\mathcal{J}}, t_k)$, where for every $\theta \in \Theta^n$;

1. $\rho_k^j(\theta) = \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)$ for every $j \in \mathcal{J}$;[20]
2. $\eta_k^j(\theta_i) = \eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)$ for every $j \in \mathcal{J}$;[21]
3. $t_k(\theta_i) = t\left(\theta_i^{P_k}\right)$.

By construction, each g_k is simple. Each g_k is also anonymous by the anonymity of g. Using the definition of g_k and manipulating the result by observing that the labeling of the variables is

[19] To illustrate, suppose $n = 2, M = 3$, and $\theta = (\theta_1, \theta_2) = ((1,2,0), (3,2,1))$. Consider, for example, permutation k given by $P_k(1) = 2, P_k(2) = 1, P_k(3) = 3$. Then $P_k^{-1}(1) = 2, P_k^{-1}(2) = 1, P_k^{-1}(3) = 3$ and $\theta_1^{P_k} = \left(\theta_1^{P_k^{-1}(1)}, \theta_1^{P_k^{-1}(2)}, \theta_1^{P_k^{-1}(3)}\right) = (2,1,0), \theta_2^{P_k} = \left(\theta_2^{P_k^{-1}(1)}, \theta_2^{P_k^{-1}(2)}, \theta_2^{P_k^{-1}(3)}\right) = (2,3,1), \theta^{P_k} = \left(\theta_1^{P_k}, \theta_2^{P_k}\right) = ((2,1,0),(2,3,1))$.

[20] This implies that $\rho_k^{P_k^{-1}(j)}\left(\theta^{P_k}\right) = \rho^j(\theta)$ for every $j \in \mathcal{J}$.

[21] This implies that $\eta_k^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right) = \eta^j(\theta_i)$ for every $j \in \mathcal{J}$.

irrelevant, we get:[22]

$$\begin{aligned}
\mathrm{E}\rho_k^j(\theta)\eta_k^j(\theta_i)\theta_i^j &= \int_\theta \rho_k^j(\theta)\eta_k^j(\theta_i)\theta_i^j d\mathbf{F}(\theta) \,/\text{def of } g_k/ = \int_{\theta\in\Theta^n} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}(\theta) \\
&= \int_{\theta^j}\left[\int_{\theta^{-j}} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}^{-j}\left(\theta^{-j}\middle|\theta^j\right)\right] d\mathbf{F}^j\left(\theta^j\right) \qquad \text{(B13)} \\
/\text{relabel}/ &= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right] d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right)
\end{aligned}$$

where we recall,

$$\left(\theta^{-j}\right)^{P_k} \equiv \left(\theta^{P_k^{-1}(1)},...,\theta^{P_k^{-1}(j-1)},\theta^{P_k^{-1}(j+1)},...,\theta^{P_k^{-1}(n)}\right). \qquad \text{(B14)}$$

By exchangeability, we have

$$\begin{aligned}
& d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th (vector) argument}}\right) \qquad \text{(B15)} \\
&= d\mathbf{F}^{-j}\left(\theta^{P_k^{-1}(1)},...,\theta^{P_k^{-1}(j-1)},\theta^{P_k^{-1}(j+1)},...,\theta^{P_k^{-1}(n)} | j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right) \\
&= d\mathbf{F}^{-j}\left(\theta^{-j} | j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right) \\
&= d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)} | P_k^{-1}(j)\text{-th (vector) argument } = \theta^{P_k^{-1}(j)}\right);
\end{aligned}$$

and

$$d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right) = d\mathbf{F}^{\theta^{P_k^{-1}(j)}}\left(\theta^{P_k^{-1}(j)}\right). \qquad \text{(B16)}$$

Using (B13), (B15) and (B16), we have that

$$\begin{aligned}
& \mathrm{E}\rho_k^j(\theta)\eta_k^j(\theta_i)\theta_i^j \qquad \text{(B17)} \\
&= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle| \underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right] d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right) \\
&= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{(\theta^{-j})^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)}| \underbrace{\theta^{P_k^{-1}(j)}}_{P_k^{-1}(j)\text{-th argument}}\right)\right] d\mathbf{F}^{P_k^{-1}(j)}\left(\theta^{P_k^{-1}(j)}\right) \\
&= \int_\theta \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}.
\end{aligned}$$

Moreover, exchangeability implies that $\mathrm{E}t_k(\theta_i) = \mathrm{E}t\left(\theta_i^{P_k}\right) = \mathrm{E}t(\theta_i)$. The ex ante utility,

$$\begin{aligned}
\mathrm{E}\left[\sum_{j=1}^M \rho_k^j(\theta)\eta_k^j(\theta_i)\theta_i^j - t_k(\theta_i)\right] &= \left[\sum_{j=1}^M \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}\right] - \mathrm{E}t(\theta_i) \\
\Big/ \begin{matrix}\text{Same elements in } \mathcal{J} \text{ and} \\ \left\{P_k^{-1}(1),...,P_k^{-1}(M)\right\}\end{matrix} \Big/ &= \left[\sum_{j=1}^M \mathrm{E}\rho^j(\theta)\,\eta^j(\theta_i)\theta_i^j\right] - \mathrm{E}t(\theta_i), \qquad \text{(B18)}
\end{aligned}$$

[22] It is important to point out that, in reaching the fourth equality in (B13), we can relabel the integrating varibles (since they are dummies) but not the integrating functions.

is thus unchanged when changing from g to g_k. The same steps as in (B13) through (B17) (only somewhat simpler) establishes that $\mathrm{E}\rho_k^j(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}$ for every j, implying that

$$\begin{aligned} \mathrm{E}\left[\sum_{j=1}^M \rho_k^j(\theta) C^j(n) - \sum_i t_k(\theta_i)\right] &= \left[C(n)\,\mathrm{E}\sum_{j=1}^M \rho_k^j(\theta) - \sum_i \mathrm{E}t_k(\theta_i)\right] \qquad \text{(B19)} \\ = \left[C(n)\,\mathrm{E}\sum_{j=1}^M \rho^j(\theta) - \sum_i \mathrm{E}t(\theta_i)\right] &= \mathrm{E}\left[\sum_{j=1}^M \rho^j(\theta) C(n) - \sum_i t(\theta_i)\right], \end{aligned}$$

so the feasibility constraint is unaffected when changing from g to g_k. Next, write Write $U(\theta_i, \theta_i'; g)$ and $U(\theta_i, \theta_i'; g_k)$ for the expected utility from announcing θ_i' when the true type is θ_i in mechanisms g and g_k respectively. Next, by a calculation in the same spirit as (B13) through (B17):

$$\begin{aligned} \mathrm{E}_{-i}\rho_k^j(\theta_{-i}, \theta_i') &= \int_{\theta_{-i}} \rho_k^j(\theta_{-i}, \theta_i')\, d\mathbf{F}_{-i}(\theta_{-i}) \text{ /def of } g_k/ = \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}\left((\theta_{-i}, \theta_i')^{P_k}\right) d\mathbf{F}_{-i}(\theta_{-i}) \\ &= \int_{\theta_{-i}^j}\left[\int_{\theta_{-i}^{-j}} \rho^{P_k^{-1}(j)}\left((\theta_{-i}, \theta_i')^{P_k}\right) d\mathbf{F}_{-i}^{-j}\left(\theta_{-i}^{-j}\middle|\,\theta_{-i}^j\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^j\right) \qquad \text{(B20)} \\ \text{/relabel/} &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}^{-j}\left(\left(\theta_{-i}^{-j}\right)^{P_k}\middle|\,\theta_{-i}^{P_k^{-1}(j)}\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\ \text{/exchangeability/} &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}^{-P_k^{-1}(j)}\left(\theta_{-i}^{-P_k^{-1}(j)}\middle|\,\theta_{-i}^{P_k^{-1}(j)}\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\ &= \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}(\theta_{-i}) = \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) \end{aligned}$$

That is, the perceived probability of getting j when announcing θ_i' in mechanism g_k is the same as the perceived probability of getting good $P_k^{-1}(j)$ when announcing $(\theta_i')^{P_k}$, so that

$$\begin{aligned} U(\theta_i, \theta_i'; g_k) &= \mathrm{E}_{-i}\sum_{j=1}^M \rho_k^j(\theta_{-i}, \theta_i')\, \eta_k^j(\theta_i')\theta_i^j - t_k(\theta_i') \qquad \text{(B21)} \\ &= \sum_{j=1}^M \eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left((\theta_i')^{P_k}\right), \end{aligned}$$

whereas

$$\begin{aligned} U(\theta_i, \theta_i'; g) &= \sum_{j=1}^M \eta_k^j(\theta_i')\theta_i^j \mathrm{E}_{-i}\rho_k^j(\theta_{-i}, \theta_i') - t(\theta_i') \Rightarrow \qquad \text{(B22)} \\ & U(\theta_i, \theta_i'; g)\Big|_{\substack{\theta_i = \theta_i^{P_k} \\ \theta_i' = \theta_i'^{P_k}}} = \sum_{j=1}^M \eta_k^j((\theta_i')^{P_k})\theta_i^{P_k^{-1}(j)} \mathrm{E}_{-i}\rho_k^j\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left((\theta_i')^{P_k}\right) \\ & = \sum_{j=1}^M \eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j \mathrm{E}_{-i}\rho_k^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left((\theta_i')^{P_k}\right) = U(\theta_i, \theta_i'; g_k), \end{aligned}$$

which establishes that type θ_i who announces θ_i' in mechanism g_k gets the same utility as type $\theta_i^{P_k}$ who announces $(\theta_i')^{P_k}$ in mechanism g. Hence incentive compatibility and individual rationality of g_k follows from incentive compatibility and individual rationality of g. Now, construct a new mechanism $\widetilde{g}=(\{\widetilde{\rho}^j\}_{j\in\mathcal{J}},\{\widetilde{\eta}^j\}_{j\in\mathcal{J}},\widetilde{t})$ by letting

$$\begin{aligned}\widetilde{\rho}^j(\theta) &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\rho_k^j(\theta)=\tfrac{1}{M!}\sum_{k=1}^{M!}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\\ \widetilde{\eta}^j(\theta_i) &= \frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\,\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}=\frac{\sum_{k=1}^{M!}\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}\\ \widetilde{t}(\theta_i) &= \tfrac{1}{M!}t_k(\theta_i)=\tfrac{1}{M!}t\left(\theta_i^{P_k}\right)\end{aligned}\tag{B23}$$

let $P:\mathcal{J}\to\mathcal{J}$ be an arbitrary perturbation of the set of goods. Then,

$$\widetilde{\rho}^{P^{-1}(j)}\left(\theta^P\right)=\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)=\tfrac{1}{M!}\sum_{k=1}^{M!}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)=\widetilde{\rho}^j(\theta)\ ,\tag{B24}$$

since the sets $\left\{\rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)\right\}_{k=1}^{M!}$ and $\left\{\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\right\}_{k=1}^{M!}$ are identical. Furthermore

$$\begin{aligned}\widetilde{\eta}^{P^{-1}(j)}\left(\theta_i^P\right) &= \frac{\sum_{k=1}^{M!}\eta_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta_i^P\right)^{P_k}\right)\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)}\\ &= \frac{\sum_{k=1}^{M!}\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}=\widetilde{\eta}^j(\theta_i)\end{aligned}\tag{B25}$$

for the same reason. It is obvious that $\widetilde{t}\left(\theta_i^P\right)=\widetilde{t}(\theta_i)$, which together with (B24) and (B25) establishes that $\widetilde{g}$ is symmetric. To complete the proof we need to show that $\widetilde{g}$ is incentive feasible and generates the same surplus as g. We note that

$$\begin{aligned}&\mathrm{E}\widetilde{\rho}^j(\theta)\,\widetilde{\eta}^j(\theta_i)\,\theta_i^j=\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\rho_k^j(\theta)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j\\ &=\tfrac{1}{M!}\mathrm{E}_{\theta_i}\textstyle\sum_{k=1}^{M!}\left[\mathrm{E}_{-i}\rho_k^j(\theta)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j\right]=\tfrac{1}{M!}\mathrm{E}\left[\sum_{k=1}^{M!}\eta_k^j(\theta_i)\,\rho_k^j(\theta)\,\theta_i^j\right]\\ &\Rightarrow \mathrm{E}\textstyle\sum_{j=1}^{M}\left[\widetilde{\rho}^j(\theta)\,\widetilde{\eta}^j(\theta_i)\,\theta_i^j-\widetilde{t}(\theta_i)\right]=\tfrac{1}{M!}\sum_{k=1}^{M!}\mathrm{E}\left[\sum_{j=1}^{M}\eta_k^j(\theta_i)\,\rho_k^j(\theta)\,\theta_i^j-t_k(\theta_i)\right]\\ &/\text{(B17) \& (B18)}/=\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\eta^j(\theta_i)\,\rho^j(\theta)\,\theta_i^j-t(\theta_i)\right],\end{aligned}$$

which establishes that the ex ante utility from $\widetilde{g}$ and g are the same for all agents. Moreover,

$$\begin{aligned}&\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\widetilde{\rho}^j(\theta)\,C^j(n)-\sum_{i=1}^{n}\widetilde{t}(\theta_i)\right]=\mathrm{E}\left[C(n)\sum_{j=1}^{M}\tfrac{1}{M!}\sum_{k=1}^{M!}\rho_k^j(\theta)-\sum_{i=1}^{n}\sum_{k=1}^{M!}\tfrac{1}{M!}t_k(\theta_i)\right]\\ &=\textstyle\sum_{k=1}^{M!}\mathrm{E}\left[C(n)\sum_{j=1}^{M}\rho_k^j(\theta)-\sum_{i=1}^{n}t_k(\theta_i)\right]\ /\ \text{(B19)}/\\ &=\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\left[\sum_{j=1}^{M}\rho^j(\theta)\,C(n)-\sum_{i=1}^{n}t(\theta_i)\right]=\mathrm{E}\left[\sum_{j=1}^{M}\rho^j(\theta)\,C^j(n)-\sum_{i=1}^{n}t(\theta_i)\right],\end{aligned}$$

so the budget balance constraint is unaffected. All incentive compatibility constraints hold since,

$$\begin{aligned}U(\theta_i,\theta_i';\widetilde{g}) &= \sum_{j=1}^{M}\widetilde{\eta}^j(\theta_i')\theta_i^j\mathrm{E}_{-i}\widetilde{\rho}^j\left(\theta_{-i},\theta_i'\right)-\widetilde{t}\left(\theta_i'\right)\\ &= \frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i')\mathrm{E}_{-i}\rho_k^j(\theta_{-i},\theta_i')}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta_{-i},\theta_i')}\mathrm{E}_{-i}\left[\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\rho_k^j\left(\theta_{-i},\theta_i'\right)\right]-\tfrac{1}{M!}\sum_{k=1}^{M!}t_k\left(\theta_i'\right)\\ &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\left[\eta_k^j\left(\theta_i'\right)\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right)-t_k\left(\theta_i'\right)\right]\\ /\ \text{(B21)}/ &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}U(\theta_i,\theta_i';g_k)\ \ \le /\ \text{IC for each } k/\ \ \tfrac{1}{M!}\sum_{k=1}^{M!}U(\theta;g_k\ )=U(\theta;\widetilde{g}).\end{aligned}$$

By the same calculation, $U(\theta;\widetilde{g})=\frac{1}{M!}\sum_{k=1}^{M!}U(\theta;g_k)\ge 0$, since all participation constraints hold for each k. This completes the proof. ■