

2 Admati and Perry (ReStud, 1991)

THE CONTRIBUTION GAME

- Two identical players, 1 and 2;
- The value of a completed project is V to both players;
- The total cost of the completing the project is K . [Note that the project is either completed or not completed];
- Players take turns in making contributions, starting with player 1 in period 1. The project is completed as soon as the total contributions made by both players reach the total cost K .

PRELIMINARIES

- Let c_i^t be the amount of player i 's contribution at period t ;
- A history at time τ is a sequence of contributions made by agents prior to time τ : $\{c_1^t, c_2^t\}_{t=1}^{\tau-1}$. If it is not i 's turn to move in period t , then $c_i^t = 0$;
- A *strategy* for player i specifies the size of his contribution for each history after which it is i 's turn to move;
- Players are impatient and discount rates are both $\delta \in (0, 1)$. Let T be the first time at which the total contributions reach K , where $T = \infty$ if it is not completed. An outcome of the game is

$$\left\{ T, \{c_1^t\}_{t=1}^T, \{c_2^t\}_{t=1}^T \right\}.$$

Player i 's payoff from such an outcome is

$$U_i \left(T, \{c_1^t\}_{t=1}^T, \{c_2^t\}_{t=1}^T \right) = \delta^{T-1} V - \sum_{t=1}^T \delta^{t-1} W(c_i^t)$$

where $W(\cdot)$ is the function which measures the disutility from contributing, $W' > 0$, $W'' > 0$ and $W(0) = 0$.

The main result of the paper relies on the following variables:

- After a history $\left(\{c_1^t\}_{t=1}^\tau, \{c_2^t\}_{t=1}^\tau \right)$, denote X as the total amount of contribution still required for the completion of the project, i.e.,

$$X \left(\{c_1^t\}_{t=1}^\tau, \{c_2^t\}_{t=1}^\tau \right) = K - \sum_{t=1}^\tau c_1^t - \sum_{t=1}^\tau c_2^t.$$

Now we recursively define the amount of contribution that would be made in the equilibrium path if

the project is completed in equilibrium, denoted by $\{R_q\}_{q=1}^{\infty}$

STEP 1: Define R_1 as the maximum amount that a player is willing to contribute now if by doing so he completes the project ; while if he contributes zero then the project will be completed next period by the other player. That is, R_1 solves

$$V - W(R_1) = \delta V.$$

It is clear that if in a subgame $X < R_1$, then it is a dominant strategy for the player whose turn it is to complete the project. For any $X \in [0, R_1]$, let

$$U_a^*(X) = V - W(X)$$

$$U_b^*(X) = V.$$

$U_a^*(X)$ and $U_b^*(X)$ are respectively the payoffs of the first and second player to move if the remaining size of the project is X and the first player completes the project in his turn.

STEP 2: Let R_2 be the contribution level that makes a player indifferent between (i) contributing R_2 right now under the assumption that the project will be completed in the next period by the other player; and (ii). contributing zero right now and completing the project in two periods by contributing R_1 then (with the rest contributed by the other player in the next move). That is, R_2 solves

$$\delta V - W(R_2) = \delta^2 V - \delta^2 W(R_1),$$

which can be rewritten as

$$\delta U_b^*(S_1) - W(R_2) = \delta^2 U_a^*(S_1)$$

where $S_n = \sum_{q=1}^n R_q$ [hence $S_1 = R_1$]. It is clear that if in a subgame $R_1 < X < R_1 + R_2$, it is an iteratively dominant strategy for the player with the current move to contribute $X - R_1$ assuming that the other player will follow his dominant strategy to complete the project (with contribution R_1). For $X \in [R_1, R_1 + R_2]$, or equivalently, for $X \in [S_1, S_2]$, define

$$\begin{aligned} U_a^*(X) &= \delta U_b^*(S_1) - W(X - R_1) \\ U_b^*(X) &= \delta U_a^*(S_1) \end{aligned}$$

$U_a^*(X)$ is the current mover's payoff if he contributes just enough, $X - R_1$, so that after his move the required contribution is R_1 and the second mover completes the project in his turn.

STEP n : Define R_n recursively as the amount that makes player i whose turn it is indifferent between (i) contributing R_n now, and obtaining $U_b^*(S_{n-1})$ in the next period; and (ii) contributing zero now and obtaining $U_a^*(S_{n-1})$ in two periods. That is, R_n solves

$$\delta U_b^*(S_{n-1}) - W(R_n) = \delta^2 U_a^*(S_{n-1}).$$

And for every $X \in [S_{n-1}, S_n]$, define

$$\begin{aligned} U_a^*(X) &= \delta U_b^*(S_{n-1}) - W(X - S_{n-1}) \\ U_b^*(X) &= \delta U_a^*(S_{n-1}). \end{aligned}$$

The essentially unique equilibrium path of the contribution game is described as follows:

Proposition 1 *Let $S_\infty = \sum_{q=1}^{\infty} R_q$.*

1. Suppose $S_\infty > K$.

(a) If there exists $N < \infty$ such that $S_{N-1} < K < S_N$, then the unique equilibrium path is: player 1 contributes $K - S_{N-1}$ in period 1, and for $1 < t \leq N$, the amount contributed in period t is R_{N-t+1} . Thus the project is completed in N rounds.

(b) If there exists $N < \infty$ such that $S_N = K$, then there are two equilibrium paths. In addition to the path described above, the other equilibrium path is as follows: player 1 contributes zero in period 1, and for $2 \leq t \leq N$, the amount contributed in period t is R_{N-t+2} . Thus the project is completed in $N + 1$ periods.

2. If $S_\infty \leq K$, then the unique equilibrium path is $c_i^t = 0$ for all i and all t .

Now we consider a special linear contribution cost function $W(c) = bc$. The equilibrium path described

in the above proposition remains an equilibrium path (though uniqueness is lost). For the linear case, we can explicitly solve for S_n , which turns out to be

$$S_n = \frac{V(1 - \delta^{2n-1})}{b},$$

hence $S_\infty = V/b$. Hence Proposition 1 implies that the project is completed in that equilibrium if and only if $K < V/b$, which is equivalent to $V > bK$. That is, in a linear case a necessary and sufficient condition for the completion of the project in the above equilibrium is that each player would complete the project immediately if he was the only player. The inefficiency due to delay is stark. However, there *may* be other more efficient equilibria.

More generally, Admati and Perry show that:

Proposition 2 If $V \leq W'(0)K$, then the project is not completed in equilibrium.