1 What is Public Goods?

- A good is called a pure public good if "each individual's consumption of such a good leads to no subtraction from any other individual's consumption" (Samuelson 1954, p387)

- This is commonly referred to as *non-rivaly* in use.

- The other property of pure public good is *non-excludability*, that is, it is infeasible to price units of a good in a way that prevents those who do not pay from enjoying its benefits.

- National defense and lighthouse are probably the classical examples of pure public good.

- Public goods in general can vary in the extent of rivalyness and excludability.

2 The Model (Samuelson 1954)

- We will consider the simplest case with a single private good and a single public good.

- n consumers, indexed by $i = 1, ..., n$

- x_i : agent i's consumption of private good and denote $\mathbf{x} = (x_1, ..., x_n)$ as the vector of private consumption

- G: the (common) consumption of public good

- Agent i's preference described by the utility function

$$u_i(x_i, G)$$

  which is differentiable and increasing in both arguments, quasi-concave and satisfies Inada Condition;

- w_i : agent i's endowment of private good and

$$W = \sum_{i=1}^{n} w_i$$

is the total endowment of private good; and public good endowment is taken to be zero

- Public good may be produced from the private good according to a production function $f : R_+ \to R_+$ where $f' > 0$ and $f'' < 0$. That is, if z is the total units of private goods that are used as inputs to produce the public good, the level of public good produced will be

$$G = f(z).$$

3 Optimal Provision of Pure Public Good

- Normative question: What is the optimal level of pure public good?

- Assume that the government of a fully controlled economy chooses the level of G, and the allocation of private goods $\mathbf{x} = (x_1, ..., x_n)$ to agents according to the Pareto criterion.

Definition 1 *An allocation* $(\mathbf{x},G) \in R_{+}^{n+1}$ *is feasible if there exists some* $z \geq 0$ *s.t.*

- $\sum_{i=1}^{n} x_i + z \leq W$;
- $G \leq f(z)$.

Definition 2 *A feasible allocation* $(\mathbf{x}, G)$ *is Pareto optimal if there exists no other feasible allocation* $(\mathbf{x}', G')$ *s.t.*

$$u_i\left(x_i', G'\right) \geq u_i(x_i, G) \, \forall i = 1, ..., n$$

and for some $i \in \{1, ..., n\}$,

$$u_i\left(x_i', G'\right) > u_i(x_i, G).$$

That is, a feasible allocation $(\mathbf{x}, G)$ is Pareto optimal if there is no way of making an agent strictly better off without making someone else worse off.

4 Characterizing P.O. Allocations

- It is the solution to the following problem:

$$\max_{\{x,G,z\}} u_1(x_1, G)$$

$$\text{s.t. } u_i(x_i, G) - \underline{u}_i \geq 0 \text{ for } i = 2, 3, ..., n, \qquad (\gamma_i)$$

$$W - \sum_{i=1}^{n} x_i - z \geq 0 \qquad (\lambda)$$

$$f(z) - G \geq 0 \qquad (\mu)$$

$$G \geq 0, z \geq 0$$

$$x_i \geq 0 \text{ for all } i = 1, ..., n$$

where $\underline{u}_i$ are treated as parameters.

- Inada conditions on the utility function implies that the non-negativity constraints can be ignored.

5 Kuhn-Tucker Optimality Condition

- The necessary and sufficient (sufficiency due to quasi-concavity assumption on u and f) Kuhn-Tucker conditions are:

$$
\begin{aligned}
(x_i:) \qquad & \gamma_i \frac{\partial u_i(x_i, G)}{\partial x_i} - \lambda = 0 \qquad (1) \\
(G:) \qquad & \sum_{i=1}^{n} \gamma_i \frac{\partial u_i(x_i, G)}{\partial G} - \mu = 0 \\
(z:) \qquad & -\lambda + \mu f'(z) = 0
\end{aligned}
$$

where we have set $\gamma_1 = 1$ by convention.

- From the fist n equalities, we obtain

$$\gamma_i = \frac{\lambda}{\partial u_i (x_i, G) / \partial x_i}.$$

- From the last equality, we obtain

$$\mu = \frac{\lambda}{f' (z)}$$

- Plugging these $n+1$ equalities into the the middle condition regarding G, we get

$$\sum_{i=1}^{n} \frac{\partial u_i (x_i, G) / \partial G}{\partial u_i (x_i, G) / \partial x_i} = \frac{1}{f' (z)}. \tag{2}$$

- This condition is referred to as the Samuelson condition (Lindahl-Samuelson condition, or Bowen-Lindahl-Samuelson Condition).

- Interpretations: The left hand of equation (2) is the sum of the marginal rates of substitutions of the n agents. To see this, note that from agent i's indifference curve, the term
$$\frac{\partial u_i(x_i, G)/\partial G}{\partial u_i(x_i, G)/\partial x_i}$$
denotes the quantity of private good agent i is willing to give up for a small unit increase in the level of the public good. The right hand of equation (2) is the amount of private good required to produce an additional unit of public good (also known as the marginal rate of transformation).

- Hence the Samuelson condition says the following: Any optimal allocation is such that the sum of the quantity of private goods consumers would be willing to give up for an additional unit of public good must equal to the quantity of private good that is actually required to produce the additional unit of public good.

- If there are more than one private goods, say k private goods; and the public good is produced according to

$$f(z_1, ..., z_k),$$

then the corresponding Samuelson condition for the optimal level of public goods is given by

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial u_i \left(x_{ij}, G\right) / \partial G}{\partial u_i \left(x_{ij}, G\right) / \partial x_{ij}} &= \frac{1}{\partial f(z_1, ..., z_k) / \partial z_j} \\ \text{for all } j &= 1, ..., k. \end{aligned}$$

6 A Diagrammatic Illustration

- Consider the case of two individuals and two goods is given in Figure 1.

- In Figure 1, the upper part shows the indifference curves for citizen I and the production constraint AB. Suppose that we fix citizen I on the indifference curve $\underline{u}_I$, then the possibilities for citizen II are shown in the lower part of Figure 1 by CD (which is the difference between AB and $\underline{u}_I$).

- Clearly Pareto efficiency requires the marginal rate of substitution of the second individual be equal to the slope of the curve CD (i.e. at point E). But this is just the difference between the marginal rate of transformation (the slope of the production possibilities schedule) and the marginal rate of substitution of the first individual (the slope of his indifference curve). Thus we have

$$MRS^{II} = MRT - MRS^{I}.$$

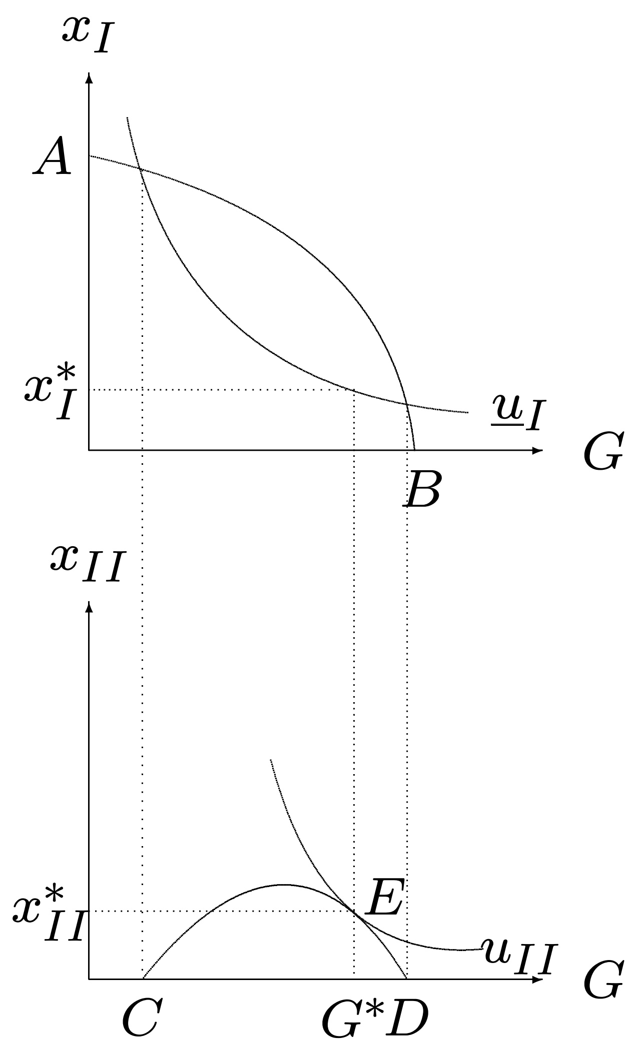

Figure 1: Optimal Provision of Public Goods - The Two Person Example

7 Can the Optimal Allocation be Decentralized?

Imagine that competitive markets exist for both the private and the public goods. Let the private good be the numeraire.

- Let p denote the price of the public good (in terms of the private good);

- Let g_i denote the quantity of public good purchased by agent i;

- Without loss of generality, we assume that there is a single price-taking profit maximizing firm that operates on the market.

- We will make the following (somewhat sloppy) assumption: we assume that every agent are price-takers (i.e. their choice does not affect the price level), but they do feel that their purchase can affect the aggregate level of public goods.

- Given the public good purchases by other agents $\vec{g}_{-i} = (g_1, ..., g_{i-1}, g_{i+1}, ..., g_n)$, agent i's best response to $\vec{g}_{-i}$ given a price p is defined as

$$\begin{aligned}\beta_i(\vec{g}_{-i}, p) &= \arg\max_{\{g_i\}} u_i\left(w_i - pg_i, g_i + \sum_{j \neq i} g_j\right) \\ \text{s.t. } g_i &\geq 0 \\ w_i - pg_i &\geq 0\end{aligned}$$

- Assuming that u_i is strictly quasi-concave, there is a unique solution to the maximization problem for the agent given $\vec{g}_{-i}$ and p which is fully characterized by

$$-\frac{\partial u_i}{\partial x_i} p + \frac{\partial u_i}{\partial G} + \lambda - \mu p = 0$$

$$\begin{aligned}\lambda g_i &= 0 \\ \mu\left(w_i - pg_i\right) &= 0\end{aligned}$$

Since u_i satisfies Inada condition, $\mu = 0$. Hence we have

$$p \geq \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i}.$$

The profit maximizing supplier of the public good solves, for a given price p, the following problem

$$\max_{z \geq 0} pf\left(z\right) - z$$

which yields the condition that

$$p = \frac{1}{f'\left(z\right)}.$$

Definition 3 *A competitive equilibrium consists of* $p^*, G^* = \left(g_1^*, ..., g_n^*\right)$ *such that*

1. *For each* i, *given* p^* *and* $\vec{\mathbf{g}}_{-i}^* = \left(g_1^*, ..., g_{i-1}^*, g_{i+1}^*, ..., g_n^*\right)$,
$$g_i^* \in \beta_i\left(\vec{\mathbf{g}}_{-i}^*, p^*\right)$$

2. *The firm optimizes, i.e.*
$$p^* = \frac{1}{f'\left(f^{-1}\left(\sum_{i=1}^n g_i^*\right)\right)}.$$

- Because of the Inada condition on u_i regarding G, we must have that for some $j \in \{1, ..., n\}$,
$$p = \frac{\partial u_j / \partial G}{\partial u_j / \partial x_i}.$$
Thus for that j,
$$\frac{1}{f'(z)} = \frac{\partial u_j / \partial G}{\partial u_j / \partial x_j}.$$

Hence in competitive equilibirum it must be the case that

$$\sum_{i=1}^{n} \frac{\partial u_i / \partial G}{\partial u_i / \partial x_i} > \frac{1}{f'(z)}.$$

Hence there is under-provision of the public good relative to the level prescribed by the Samuelson condition. Intuition.

Example 1 *Suppose* $u_i(x_i, g) = \gamma \ln g + \ln x_i$, *and* $w_i = W/n$, *and* $f(z) = z$. *Find the egalitarian Pareto optimal allocation; and the competitive equilibrium allocation.*

- It can be shown that the egalitarian Pareto optimal allocation is given by

$$\hat{G} = \frac{\gamma W}{1+\gamma}, \hat{x}_i = \frac{W}{n(1+\gamma)}, i = 1, ..., n$$

- and the competitive equilibrium allocation is

$$G^* = \frac{\gamma W}{n+\gamma}, x_i^* = \frac{W}{n+\gamma}, i = 1, ..., n.$$

  It is clear that as n gets larger, the under-provision of the public good gets more severe.

8 Lindahl Equilibria (Lindahl 1958)

- While the competitive equilibrium with a fixed price of the public good will yield an inefficient allocation, Lindahl introduced the idea of think of the amount purchased by each agent as a distinct commodity and have each agent to face a *personalized price* p_i and to have these price chosen in a way such that all agents agree on the level of the public good.

- Let $s_i \in [0, 1]$ be agent i's share of the firm's profit with $\sum_{i=1}^{n} s_i = 1$.

Definition 4 *A Lindahl equilibrium is a vector* $\mathbf{p}^* = \left(p_1^*, ..., p_n^*\right)$ *and an allocation* $\left(x_1^*, ..., x_n^*, G^*\right)$ *such that*

- *The firm maximizes profits, that is,*

$$G^* = \arg\max_{G \geq 0} \left(\sum_i p_i^*\right) G - f^{-1}(G)$$

- *Each consumer maximizes utility, that is,*

$$(x_i^*, G^*) = \arg\max_{x_i, G} u_i(x_i, G)$$

s.t.

$$w_i + s_i \left(\sum_i p_i^* G^* - f^{-1}(G^*)\right) - x_i - p_i^* G \geq 0$$

- *(Redundant by Walras Law) Market clears, i.e.*

$$\sum_{i=1}^n x_i^* + f^{-1}(G^*) \leq \sum_{i=1}^n w_i.$$

- The Lindahl equilibrium is a competitive equilibrium in a fictitious economy where the space of goods has been expanded to $(n+1)$ goods, the private goods and n personalized public goods, that is, the public goods of agent 1 through agent n.

- These n goods are produced "jointly", so that we must find a vector of prices for which all agents demand equal quantities of the public good.

Example 2 *Find the Lindahl equilibrium of the economy described by Example 1: Suppose* $u_i(x_i, g) = \gamma \ln g + \ln x_i$, *and* $w_i = W/n$, *and* $f(z) = z$.

- Suppose that agent i's personalized price for the public good is p_i. It is easy to solve for i's demand for the public good will be given by

$$g_i(p_i) = \frac{1}{p_i} \frac{\gamma W}{(\gamma + 1) n}.$$

Since the demand of public good must be equal for all the agents in a Lindahl equilibrium it must be the case that in a Lindahl equilibrium, $p_i^* = p_j^*$ for all $i, j \in \{1, ..., n\}$. The firm's profit maximization requires that

$$\sum_{i=1}^{n} p_i^* = 1$$

Hence $p_i^* = 1/n$ for all i.

- Plugging this individualized price, we obtain that

$$g_i(p_i^*) = \frac{\gamma W}{\gamma + 1} \text{ for all } i$$

which is the public good level in the egalitarian Pareto-optimal allocation.

Proposition 1 *Any Lindahl equilibrium is Pareto optimal.*

It can be established using an argument which is more or less a copy of the textbook proof of the first welfare theorem. More intuitively, note that FOC for the firm's profit maximization gives

$$\sum_{i=1}^{n} p_i^* = \frac{1}{f'\left(f^{-1}\left(G^*\right)\right)}$$

and FOC for individual i's utility maximization is

$$\frac{\partial u_i\left(x_i^*, G^*\right)}{\partial x_i} p_i^* = \frac{\partial u_i\left(x_i^*, G^*\right)}{\partial G} \text{ for all } i = 1, ..., n.$$

Hence

$$\sum_{i=1}^{n} \frac{\partial u_i\left(x_i^*, G^*\right)/\partial G}{\partial u_i\left(x_i^*, G^*\right)/\partial x_i} = \frac{1}{f'\left(f^{-1}\left(G^*\right)\right)}$$

which satisfies the by-now familiar Samuelson condition.

9 Is Lindahl Equilibrium a Reasonable Market Mechanism?

- The Lindahl equilibrium is more a normative prescription for the allocation of public goods than a positive description of the market mechanism. The reason is simple: by the definition of the personalized price in the Lindahl equilibrium, an agent will quickly lean that he should *not* behave competitively (an assumption which has always been justified by the existence of a large number of market participants). He will have incentive to mis-report her desire for the public good.

- Contrary to the case of private goods, where the incentive to reveal false demand functions decreases with the number of agents, an increase in the number of agents in the case of public good only aggregates the problem. We demonstrate this problem by the following example.

Example 3 *Consider n agents with utility function*

$$u_i(x_i, G) = \ln x_i + \alpha_i \ln G.$$

We suppose that each agent has an endowment of the private good $w_i = 1$ and no public good. Suppose that the technology is linear, i.e. $f(z) = z$ for all $z \geq 0$.

- Facing a personalized price p_i, it is clear that agent i will demand public good

  $$p_i g_i(p_i) = \frac{\alpha_i}{1 + \alpha_i}$$

  Since in a Lindahl equilibrium

  $$g_i(p_i) = G \, \forall i$$

  we have

  $$G \sum p_i = \sum_i \frac{\alpha_i}{1 + \alpha_i}$$

  For the firm's profit maximization problem to have a solution, it must be that

  $$\sum p_i = 1$$

hence

$$G = \sum_i \frac{\alpha_i}{1+\alpha_i}$$

and

$$p_i = \frac{\alpha_i / (1+\alpha_i)}{\sum_j \alpha_j / \left(1+\alpha_j\right)}$$

Agent i's consumption of private good is

$$x_i = \frac{1}{1+\alpha_i}$$

- Suppose that $n = 3$, and $\alpha_i = 1$ for $i = 1, 2, 3$. The Lindahl equilibrium is then

$$p_i^* = \frac{1}{3}, x_i^* = \frac{1}{2}, G^* = \frac{3}{2}$$

so the equilibrium utility level for agent i is

$$\ln x_i^* + \alpha_i \ln G^* = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{3}{2}\right)$$

- Now make the following thought experiment: suppose Mr. 2 and 3 report truthfully that their types are $\alpha_i = 1$, but that Mr. 1 lies and claim that $\alpha_1 = 0$. If the planner computes the Lindahl price believing all the agents, the corresponding Lindahl prices and allocations will be

  $$p_1 = 0, p_2 = p_3 = \frac{1}{2}, x_1 = 1, x_2 = x_3 = \frac{1}{2}, G = 1.$$

  Mr. 1's utility would then be

  $$2 \ln(1) = 0.$$

  It is easy to see that

  $$\ln(1) > \frac{1}{2} \ln\left(\frac{1}{2}\right) + \frac{1}{2} \ln\left(\frac{3}{2}\right)$$

  since logarithm is strictly concave. Hence truth telling is not an equilibrium of this game.

- While the above example is special, the logic is perfectly general. If agents have to report preferences (or wealth) they will take into consideration that under-reporting means a lower personalized price, so the free-riding problem applies. This does not mean, however, that one can not design more complicated and somewhat contrived mechanisms to implement Lindahl equilibrium allocation.