

Positive Models of Private Provision of Public Goods: A Static Model

(Bergstrom, Blume and Varian 1986)

- Public goods will in general be under-supplied by voluntary contributions.
- Still, voluntary contributions of public goods constitutes a large fraction of available resources in the economy (approximately 2% of GDP are private donations to charity).
- Hence it seems to be of importance to have a reasonable positive theory of private provision of public goods from which one can derive policy implications.
 - For instance, we may be interested in how private provisions change due to changes in the income distribution and how private donations are affected by public provisions.

1 A Model of Private Contributions

- From previous analysis, we learn that assuming a concave production function $G = f(z)$ does not add any new insights - so we will here assume linear production technology;
- Also, as long as we assume that firms are competitive, decentralizing production does not add any further distortion - so we will assume that private good can be turned into a public good by any agent.

Notations:

- A set of consumers $N = \{1, \dots, n\}$
- w_i : i 's (exogenous) wealth
- x_i : i 's consumption of private goods
- g_i : i 's contribution toward the public good. For ease of notation, write

$$G = \sum_{i=1}^n g_i$$
$$G_{-i} = \sum_{j \neq i} g_j$$

- Consumer i has utility function $u_i(x_i, G)$, increasing in both arguments.
- Timing of the voluntary contribution game: each agent simultaneously chooses $g_i \in [0, w_i]$.

2 Nash Equilibrium

- A *Nash equilibrium* of the voluntary contribution game is a vector of contributions (g_1^*, \dots, g_n^*) such that, for all $i \in N$,

$$g_i^* \in \arg \max_{g_i \in [0, w_i]} u_i(w_i - g_i, g_i + G_{-i}^*)$$

where

$$G_{-i}^* = \sum_{j \neq i} g_j^*.$$

- In the above definition, we used the fact that the budget constraint must hold with equality since the utility function is increasing in both arguments.
- Alternatively we can write agent i 's problem as

$$\begin{aligned} & \max_{x_i, G} u_i(x_i, G) & (1) \\ \text{s.t.} & x_i + G = w_i + G_{-i}^* \\ & G \geq G_{-i}^* \end{aligned}$$

3 A Neutrality Theorem

- The first result is that, *if all agents contribute*, then small changes in the distribution of income will leave the allocation unchanged.
- This result was first obtained by Warr (1983) who used an implicit differentiation argument of the first order condition.
- BBV extended this argument by a non-calculus approach and could then give a more complete description of how equilibria are affected by changes in the wealth distribution.

Proposition 1 (Neutrality Theorem) *Suppose u_i is quasi-concave for all $i \in N$, and let (g_1^*, \dots, g_n^*) be the initial equilibrium. Consider a redistribution of income among contributing agents $C \equiv \{i : g_i^* > 0\}$ such that no agent loses more income than his original contribution. Let w'_i be i 's post-redistribution wealth. Then the post-redistribution NE $\{g_1'^*, \dots, g_n'^*\}$ satisfies*

$$g_i'^* - g_i^* = w'_i - w_i$$

Hence,

$$G'^* \equiv \sum_{i=1}^n g_i'^* = G^* \equiv \sum_{i=1}^n g_i^*.$$

Sketch of the Proof.

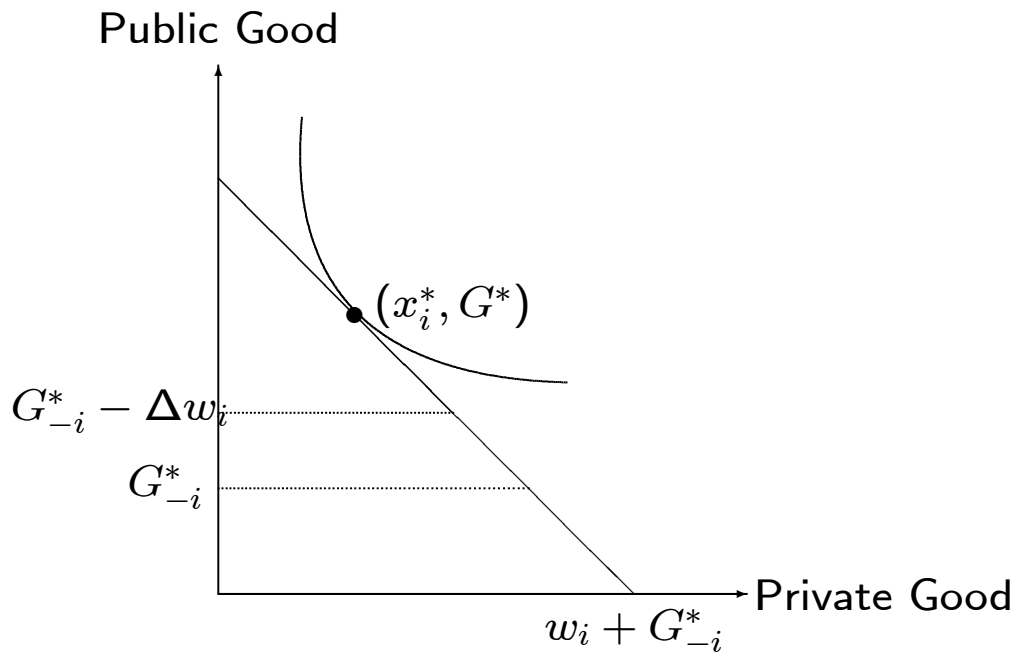
- Suppose that Δw_i is the change in agent i 's wealth caused by the redistribution.
- Suppose that in a post-redistribution equilibrium, every agent other than agent i changes his contribution by the exact amount of his change in wealth. This implies that

$$G'_{-i} = G^*_{-i} - \Delta w_i.$$

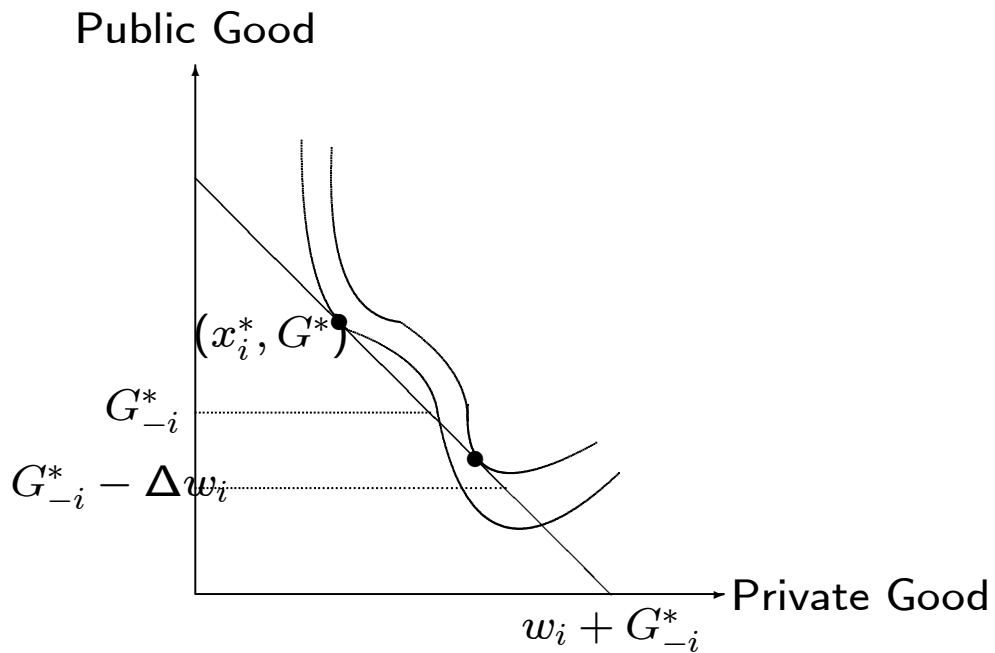
- Agent i 's best response is the solution to the following problem

$$\begin{aligned} & \max_{\{x_i, G\}} u_i(x_i, G) && (2) \\ \text{s.t.} \quad & x_i + G = \underbrace{(w_i + \Delta w_i)}_{\text{new wealth}} + \underbrace{(G^*_{-i} - \Delta w_i)}_{\text{others' total contribution}} \\ & G \geq G'_{-i} - \Delta w_i \end{aligned}$$





(a) Case I: $\Delta w_i < 0$



(b) Case II: $\Delta w_i > 0$: Convexity Rules this Out

Figure 1: Graph in the Proof of the Neutrality Theorem

4 A General Characterization of the Set of Nash Equilibria

- Consider the problem

$$\begin{aligned} & \max u_i(x_i, G) \\ \text{s.t.} \quad & x_i + G = W \end{aligned}$$

This is a standard consumer optimization problem. Assuming that u_i is strictly quasi-concave, we have a unique solution $f_i(W)$, which in consumer theory language is the demand for the public good G . We make the following assumption:

- Assumption: $f_i(\cdot)$ is single-valued, differentiable and satisfy

$$f'_i(W) \in (0, 1).$$

- This assumption requires that both the private and the public goods are normal goods.

- The problem that determines i 's best response is of course the problem (1). It is clear that the best response function from the problem (1) is (taking i 's contribution to the public good, g_i , as the strategic variable), is

$$\beta_i(G_{-i}) = \max \{f_i(w_i + G_{-i}) - G_{-i}, 0\}. \quad (3)$$

Proposition 2 (Existence of Equilibrium) *A Nash Equilibrium exists.*

Proof. Let $\mathcal{W} = \{z \in R^n : z_i \in [0, w_i] \forall i\}$. This is a compact and convex set. Note that

$$(\beta_1(G_{-1}), \dots, \beta_n(G_{-n}))$$

is a continuous function from \mathcal{W} to itself. By Brouwer's fixed point theorem, there must exist a fixed point, which is the Nash equilibrium of the voluntary contribution game. ■

- To know more of the equilibrium of the voluntary contribution game, consider an equilibrium (g_1^*, \dots, g_n^*) .
- As before, define the set of positive contributors as

$$C^* = \{i \in N : g_i^* > 0\}.$$

- FACT 1:

$$G^* = \sum_i g_i^* = f_i(w_i + G_{-i}^*) \quad \forall i \in C^* \quad (4)$$

$$G^* \geq f_j(w_j + G_{-j}^*) = f_j(w_j + G^*) \quad \forall j \notin C^*.$$

- Implications:

- Once we know G^* , then the set of positive contributors are unique.
- Once we know G^* , we know g_i^* for every i :

$$g_i^* = w_i + G^* - f_i^{-1}(G^*) \quad \text{for } i \in C^*$$

and zero otherwise.

FACT 2: There exists a real valued function $F(G^*, C^*)$, differentiable and increasing in G^* such that in a Nash equilibrium,

$$F(G^*, C^*) = \sum_{i \in C^*} w_i.$$

Proof. By assumption $f'_i \in (0, 1)$, it has a strictly increasing inverse ϕ_i with $\phi'_i > 1$. Applying ϕ_i on both sides of the equality in (4), we obtain

$$\forall i \in C^*, \phi_i(G^*) = w_i + G^*_{-i} = w_i + G^* - g_i^*$$

Summing over all $i \in C^*$, we have

$$\sum_{i \in C^*} \phi_i(G^*) = \sum_{i \in C^*} w_i + (|C^*| - 1) G^*$$

where $|C^*|$ is the cardinality of the set C^* . Now we can define

$$F(G^*, C^*) = \sum_{i \in C^*} \phi_i(G^*) - (|C^*| - 1) G^*. \blacksquare$$

- The monotonicity of $F(\cdot, C^*)$ immediately implies that for a fixed set of contributors, there is a unique solution G^* to the equation $F(G^*, C^*) = \sum_{i \in C^*} w_i$.
- Combining FACTS 1 and 2, we have the following important conclusion: characterizing the Nash equilibrium allocation is equivalent to characterizing C^* or G^* .

Proposition 3 [*Uniqueness of Equilibrium*] Under the assumption that $f'_i \in (0, 1)$, there is a unique Nash equilibrium given any distribution of wealth.

Proposition 3 suggests the following algorithm to calculate the Nash equilibrium for any finite set N and wealth distribution:

- Choose an arbitrary subset $C \subseteq N$;
- Use $F(G, C) = \sum_{i \in C} w_i$ to calculate G [a unique G exists since F is monotonic in G];
- If $G \geq f_j(w_j + G)$ for all $j \notin C$ is satisfied, DONE, WE HAVE FOUND THE UNIQUE EQUILIBRIUM; otherwise, continue with a different set C .

5 Comparative Statics

Proposition 4 *Let $\{g_i\}$ and $\{g'_i\}$, $i = 1, \dots, n$ be Nash equilibria given the wealth distributions $\{w_i\}$ and $\{w'_i\}$ respectively. Let C and C' be the corresponding sets of contributing agents. Then*

$$F(G', C) - F(G, C) \geq \sum_{i \in C} (w'_i - w_i).$$

Proof. From FACT 1, we have

$$G' \geq f_i(w'_i + G'_{-i}) \quad \forall i \in C$$

Hence

$$w'_i + G'_{-i} \leq \phi_i(G') \quad \forall i \in C$$

Sum over $i \in C$,

$$\begin{aligned} \sum_{i \in C} w'_i &\leq \sum_{i \in C} \phi_i(G') - |C| G' + \sum_{i \in C} g'_i \\ &\leq \sum_{i \in C} \phi_i(G') - |C| G' + G' = F(G', C) \end{aligned}$$

Since $\sum_{i \in C} w_i = F(G, C)$, we have the desired conclusion. ■

Proposition. The following results follow from the above Proposition:

1. Any change in the wealth distribution that leaves unchanged the aggregate wealth of current contributors will either increase or leave unchanged the equilibrium supply of the public good;
2. Any change in the wealth distribution that increases the aggregate wealth of current contributors will necessarily increase the equilibrium supply of the public good;
3. If a redistribution of wealth among current contributors increase the equilibrium supply of the public good, then the set of contributing agents after the redistribution must be a proper subset of the original set of contributors;

4. Any simple transfer of wealth from another agent to a currently contributing agent will either increase or leave constant the equilibrium supply of the public good.

Proof of claim 3: From FACT 1, we know that for all $j \notin C$, $G - f_j(w_j + G) \geq 0$. Since $f'_j \in (0, 1)$, we know that if $G' > G$, we have $G' - f_j(w_j + G') \geq 0$ for all $j \notin C$. Hence $C' \subseteq C$. But if $C' = C$, then if we consider a redistribution from $\{w'_i\}$ to $\{w_i\}$, we will have, by claim 1, $G \geq G'$. A contradiction.

6 Does Public Provision Crowd Out Private Provision?

- It seems intuitively rather obvious that it should be possible to translate the above results regarding income redistribution to results about what happens when the government provides the good publicly, financed by taxes on the citizens. Let
 - g_0 be the public provision of the good;
 - t_i be the (lump sum) tax on citizen i .
 - Budget balance: $g_0 = \sum_{i=1}^n t_i$.
- From the single consumer's point of view, it is immaterial whether the good is provided by the government or other consumers, so g_0 affects best responses just like any other voluntary contribution. Also, the lump sum tax t_i is just a reduction in the endowment for i .

Proposition 5 *Let (g_1^*, \dots, g_n^*) be the unique equilibrium levels of contribution in the model with no public intervention. Consider a policy (g_0, t_1, \dots, t_n) with $\sum_{i=1}^n t_i = g_0$. Then*

- 1. If $t_i \leq g_i^*$ for all $i \in N$, then there is a unique equilibrium in the model with policy, given by (g'_1, \dots, g'_n) such that $g'_i = g_i^* - t_i \forall i \in N$;*
- 2. If $t_j > 0$ for some $j \notin C^*$, then although private contribution may decrease, the equilibrium total supply of the public good must increase;*
- 3. If for some $i \in C^*$, $t_i > g_i^*$, then the equilibrium total supply of the public good must increase.*

SKETCH OF PROOF. As before, we can show that under policy (g_0, t_1, \dots, t_n) , the equilibrium (G, C) must satisfy

$$F(G, C) = \sum_{i \in C} w_i - \sum_{i \in C} t_i + g_0$$

where

$$F(G, C) = \sum_{i \in C} \phi_i(G) - (|C| - 1)G.$$

Claim 1: If $\sum_{i \in C^*} t_i = g_0$, then by revealed preference argument as in the proof of Proposition 1 shows that $G = G^*$. For claim 2, if $g_0 - \sum_{i \in C^*} t_i > 0$, then $F(G, C^*) - F(G^*, C^*) \geq g_0 - \sum_{i \in C^*} t_i > 0$. Hence $G > G^*$. For claim 3: think as follows, first taxing the agent by the exact amount of his contribution, and then taxing him for the extra amount. In the first step, he will reduce his contribution to zero, and leave the total public good unchanged by assertion 1; the second step increases the public good supply by assertion 2.