European Economic Review 78 (2015) 1–26

Contents lists available at ScienceDirect

European Economic Review

journal homepage: www.elsevier.com/locate/eer

From Smith to Schumpeter: A theory of take-off and convergence to sustained growth

Pietro F. Peretto *

Department of Economics, Duke University, Durham, NC 27708, United States

ARTICLE INFO

Article history:
Received 9 October 2014
Accepted 10 May 2015
Available online 27 May 2015

JEL classification:
E10
L16
O31
O40

Keywords:
Endogenous growth
Firm size
Market structure
Take-off

ABSTRACT

This paper proposes a theory of the emergence of modern Schumpeterian growth that focuses on the *within-industry* forces that regulate the response of firms and entrepreneurs to Smithian market expansion and thus identifies an amplification mechanism that the literature has neglected. Because it solves the model in closed-form, the paper provides analytical insight on the forces that drive the economy's phase transition and the associated qualitative transformation of industrial activity. The resulting S-shaped path of GDP per capita replicates the key feature of the data: an accelerating phase followed by a deceleration with convergence to a stationary growth rate. The model also yields predictions for grand ratios like consumption/GDP, profits/GDP, and the distribution of income across factors of production.

> Technological creativity seems to be a uniform and ubiquitous feature of the human species, and yet just once in history has it led to a sea change comparable to phase transitions in physics or the rise of *Homo sapiens* in evolutionary biology. The Industrial Revolution and the subsequent developments did not just rise the *level* of technological capabilities; they changed the entire dynamics of how innovation comes about and the speeds of both invention and diffusion. For much of human history, innovation had been primarily a byproduct of normal economic activity, punctuated by periodical flashing insight that produced a macroinvention, such as water mills or the printing press. But sustained and continuous innovation resulting from systematic R&D carried out by professional experts was simply unheard of until the Industrial Revolution. (Mokyr, 2010, p. 37)

1. Introduction

What forces drove the massive acceleration of income per capita growth at the time of the Industrial Revolution? Despite the large literature on the subject a definitive answer to this question is still missing.[1] Current theories have changed the way we think about it but do not yet rise to what Mokyr (2005, 2010), among others, sees as the main challenge: to explain

* Tel.: +1 919 6601807; fax: +1 919 6848974.
E-mail address: peretto@econ.duke.edu

[1] In the last 10, 15 years this literature has flourished and is now immense. For reviews, see Galor (2005, 2011) and Clark (2015). Mokyr and Voth (2010) discuss its contribution to improving the historian's understanding of the issues.

http://dx.doi.org/10.1016/j.euroecorev.2015.05.001
0014-2921/

not only the rise of the growth rate, but also the qualitative transformation of the economy as sustained growth fueled by profit-driven technological change becomes its defining feature.

In this paper I develop a theory of the emergence of modern growth. The main building block is a Schumpeterian model with two types of innovation activity: (1) existing firms invest in-house to improve the quality of the goods they sell; (2) entrepreneurs invest to design new products and set up new firms to serve the market.[2] I refer to these activities as quality (or vertical) and variety (or horizontal) innovation, respectively. The literature has long recognized that the returns to these activities are increasing in the scale of operation of the firm. What it has not fully acknowledged is that because investment in innovation is a sunk cost that is economically justified only when the anticipated revenue flow is sufficiently large, there exist corner solutions where investment in innovation is zero.[3] Taking this property into account produces a theory of *when* and *how* endogenous Schumpeterian innovation becomes the main driver of growth.

I posit exogenous population growth to capture (in reduced form) forces that enlarge the market in the economy's early history. This is a simplification that proves convenient in deriving analytical results and in focusing the paper on the evolution of industrial activity. There exist two thresholds of quality-adjusted firm size: (1) one where variety innovation is zero and (2) another where quality innovation is zero. Because at least one of these two thresholds is finite, as long as the market for industrial goods grows due to aggregate forces, the economy *must* turn on Schumpeterian innovation. The intuition is that the rents earned by incumbent firms grow large and eventually must be competed away either by entry of new firms or by in-house investment by existing firms.

Since innovation must start, the only question is *when* and what specific *sequence of events* unfolds. The model reduces to a pair of piece-wise linear differential equations describing the evolution of quality-adjusted firm size in two scenarios. In one the economy crosses the threshold for variety innovation first, in the other it crosses the threshold for quality innovation first. In each scenario I obtain a closed-form solution, consisting of an S-shaped (i.e., logistic-like) time path of quality-adjusted firm size and a set of equations that express the relevant endogenous variables — GDP, product variety and product quality, consumption, the shares of GDP earned by the factors of production — as functions of quality-adjusted firm size. I also obtain closed-form solutions for the dates of the events that drive the economy's phase transitions as functions of the fundamentals. The transition path of GDP per capita consists of a convex–concave profile replicating the key feature of long-run data: an accelerating phase followed by a deceleration with convergence from below to a stationary growth rate.

The story that these solutions tell is one where the economy starts out in a situation where there is no profit-driven innovation and firms earn rents that grow with the size of the market. They also reap efficiency gains due to static economies of scale (i.e., unit production costs fall with the volume of production). There is no guarantee, however, that such gains translate aggregate growth into per capita growth. Moreover, whatever its sign, the growth rate of GDP per capita in this phase is negligible since it is a fraction of the rate of population growth.

What happens next depends on which type of innovation starts first. If variety innovation starts first, there is a period in which the tension between the exploitation of firm-level static economies of scale, that requires firm growth, and the exploitation of social returns to variety, that requires entry, results in a profile of GDP per capita growth that is always convex but can be increasing, U-shaped or decreasing over time. Hence, the onset of profit-driven horizontal innovation can be, but not necessarily is, associated with a continuation of the slowdown due to the gradual exhaustion of static economies of scale. This intermediate phase ends when the economy crosses the threshold for quality innovation. The solution for the date of this event says that it is not necessarily finite so that the economy may fail to complete the transition to modern growth.

If, instead, quality innovation starts first, there is a period in which the rate of innovation exhibits explosive behavior because firms are still earning escalating rents driven by aggregate market growth. This intermediate phase has *finite duration* because the date when the economy crosses the threshold for variety innovation is necessarily finite. The time profile of the growth rate of GDP per capita is necessarily convex and decreasing at the onset of quality innovation. The reason is that the initial contribution of quality growth cannot overcome the gradual exhaustion of Smithian static economies of scale since initially it follows a very shallow path.

In both cases growth eventually accelerates, as the contribution of Schumpeterian innovation starts dominating over the gradually vanishing contribution of Smithian economies of scale. Modern growth takes hold when quality-adjusted firm size is sufficiently large and the economy turns on both innovation engines. In this final phase the economy exhibits desirable properties, like the sterilization of the scale effect, that have interesting implications for the role of fundamentals and policy interventions (e.g., taxation, public spending).

The closed-form solution for the transition path provides analytical insight on the timing of the key events in the economy's history. It identifies the determinants of the take-off date, defined as the onset of profit-driven innovation, and

[2] Why a Schumpeterian model? In Mokyr's words (emphasis mine): " …favorable institutions explain first and foremost the kind of Smithian growth in which the expansion of commerce, credit, and more labor mobility were the main propulsive forces. But the exact connection between institutional change and the rate of innovation seems worth exploring, precisely because the Industrial Revolution marked the end of the old regime in which economic expansion was driven by commerce and *the beginning of a new Schumpeterian world of innovation*" (2010, pp. 37–38). Another, equally important reason is that this class of models has received substantial empirical support recently, especially as an explanation of long-term historical data. For example, see Ha and Howitt (2007), Madsen (2008, 2010), Ang and Madsen (2011), Madsen et al. (2010), Laincz and Peretto (2006), and Ulku (2007).

[3] Such corner solutions exist in all models of endogenous innovation but to date have played no role in the theoretical work on the Industrial Revolution.

the determinants of the duration of the transition to the modern phase with both innovation engines turned on and convergence to sustained (scale-free) growth. The conflicting forces playing out in the intermediate phase result in a rich set of possible shapes of a path that eventually converges to the general S-shaped pattern described above. The comparison of these possible histories suggests that the cross-country variation that the data show in terms of take-off dates and shapes of the paths in the immediate neighborhood of the take-off date can be explained within a unified analytical structure.

To summarize, my analysis uses population growth as the trigger that moves the economy from a state of affairs with no profit-driven innovation to one with it.[4] The Schumpeterian approach, however, shows that it is not market size per se that matters, but its contribution to quality-adjusted firm size trough interactions that thus far have been ignored. In this perspective, the paper focuses on the *within-industry* forces that regulate the response of firms and entrepreneurs to Smithian market expansion and identifies an amplification mechanism that is not specific to a particular driver — population growth or something else — of such expansion. The paper thus puts *firms* and *industry* at the heart of the mechanism that is explored.

The paper is organized as follows. Section 2 discusses the related literature. Section 3 sets up the model. Section 4 derives the closed-form solution of the model. Section 5 interprets the solution and discusses its implications for history and its potential empirical applications. Section 6 discusses features of the theory that although not central to the paper's main message are nevertheless of interest. Section 7 concludes.

2. Related literature and some relevant empirical observations

If one focuses on the model's ability to explain the key feature of the data, namely the S-shaped time path of the growth rate, then two strands of literature stand out for comparison since both propose mechanisms that produce such a time path. The first is Unified Growth Theory (UGT), pioneered by Galor and Weil (2000) and reviewed in Galor (2005), (Galor, 2011; see also Lucas, 2002).[5] The second is the literature on Growth and Structural Transformation recently reviewed in Herrendorf et al. (2015). Both approaches differ from what this paper does in that they downplay within-industry forces and thus develop models of the Industrial Revolution where entrepreneurs and firms play at best a minor role. This paper in contrast minimizes the role of all other components of the process of transformation and makes entrepreneurs and firms the stars of the story it tells. This is a logical and desirable thing to do given that the bulk of the historical evidence brought to bear on discussions of the mechanism behind the S-shaped time-path of the growth rate highlights the importance of productivity growth driven by changes in the menu of industrial goods produced and the processes used to produce them; see Mokyr (2005, 2010) and, in particular, Clark (2015) who stresses the deficiencies of UGT and other prominent theories as explanations of the Industrial Revolution, concluding that what is needed is a story centered on technological change.

The model, however, does more than simply produce an S-shaped time-path of the growth rate because it focuses on a mechanism of qualitative transformation that relates to several other aspects of the historical process discussed by the literature. The first is, of course, the transition from a Smithian to a Schumpeterian mode of growth, where the former is not self-sustaining while the latter is, potentially, self-sustaining. Kelly (1997), for example, shows that Smithian market expansion was essential for Chinese economic development in the medieval period, but that the growth spurt was temporary. Other examples from history are the expansion of canals and turnpikes in the UK and several other European countries in the 18th and the 19th centuries. Similarly, in their overview of growth in the 20th century Crafts and O'Rourke (2015, p. 265) emphasize that the defining feature of the evidence is the qualitative change in the forces driving economic growth in transition from the pre-industrial period to the industrial revolution and beyond. Noting the almost negligible growth rates up to the Industrial Revolution, they ask "What were the underpinnings of this modest pre-industrial growth in England?" and answer that they were "a combination of increases in hours worked per person and Smithian growth, rather than any major contribution from technological change" — thus highlighting once more Mokyr's point that we need to understand the shift to a different mode of growth.

The second strand of historical evidence is exemplified by Allen (2009), who stresses profitability, the fixed cost of innovation and thus the key role of market size. Allen also stresses that his view is reminiscent of Habakkuk's account of nineteenth century America, which holds that high American wages were the result of the abundance of land and natural resources. In particular, he emphasizes the role of coal: "Britain's extensive coal fields played a similar role in the eighteenth century. Cheap energy made it possible for businesses to pay high wages and remain competitive" (p. 15). This is important because it stresses the role of a natural resource traded in a market at an equilibrium price, something that is missing from, for example, the canonical UGT model but that is instead present in the model developed here. Such resources — be they land, coal, or anything else — are important determinants of market size and thus of the incentives to innovate while at the same time they are a source of diminishing returns to labor. It is thus desirable to introduce them in models of the transition

[4] One interpretation is that population growth just stands in for forces that enlarge the market. Alternatives to exogenous population growth that I have explored are: exogenous disembodied technological change; exogenous growth of the resource endowment (e.g., discovery and opening up of new land); growth of embodied knowledge through "natural" curiosity and/or learning by being/doing effects. It is possible, moreover, to augment the model with endogenous fertility and reproduce the main results discussed here. An advantage of such exercise is that it captures additional feedbacks that I leave out of this paper for simplicity. They are nevertheless worth studying, and I am doing so in work in progress.

[5] The interested reader can find a detailed comparison of this model with UGT in the appendix.

from low growth to high growth because they play a crucial role in determining the shape the time-path of the growth rate determined by the underlying process.

The third strand of historical evidence concerns the transition from the 1st to the 2nd Industrial Revolution. As noted by Schumpeter in explaining the shift of focus in his own work, from the inventors-entrepreneurs of his model of competitive capitalism to the professional R&D managers that characterize his model of trustified capitalism, in the period that economic historians label the first Industrial Revolution, innovation was the domain of individuals who devoted resources to the development of new goods and processes. Often, taking the new product and the new process to the market required these entrepreneurs to start up a new firm. In time things changed and innovation became the domain of established firms. As I noted in a previous contribution (Peretto, 1998a): "History, therefore, suggests that a fundamental change in the structure of R&D incentives, and consequently in the nature and organization of the R&D process, occurred at the turn of the century. Three questions arise. What is the nature of this change? What economic forces caused it? What are its implications?" (p. 55).[6] To my knowledge this is the only class of models that speak directly to this feature of the process. Moreover, one can take the evidence of this shift as a criterion for selecting the most relevant sequence of events produced by this paper's model.

3. The model

The economy is closed. All variables are functions of (continuous) time but to simplify the notation I omit the time argument unless necessary to avoid confusion.

3.1. Households

The economy is populated by a representative household that supplies labor and land services and trades assets in competitive markets. The household has preferences

$$U(0)=\int_0^\infty e^{-\rho t}L(t)\log\left(\frac{C(t)}{L(t)}\right)dt, \quad \rho>\lambda>0, \tag{1}$$

where 0 is the point in time when it makes decisions, ρ is the individual discount rate, C is the aggregate consumption, and $L=L_0e^{\lambda t}$, $L_0\equiv 1$, is the population size (the mass of household members). Since each household member is endowed with one unit of time, L is the household's endowment of labor. The household's endowment of land is the constant Ω.

Let w and p denote, respectively, the price of labor and land services. In this setup the household supplies labor and land services inelastically and thus faces the flow budget constraint

$$\dot{A}=rA+wL+p\Omega-C, \tag{2}$$

where A is the assets holding and r is the rate of return on assets. Denoting $c\equiv\dot{C}/C-\lambda$, the consumption plan that maximizes (1) subject to (2) consists of the Euler equation:

$$r=\rho+c, \tag{3}$$

the budget constraint (2) and the usual boundary conditions.

3.2. Final good producers

A competitive representative firm produces a final good Y that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is $P_Y\equiv 1$. To keep things simple, there is no physical capital.[7] The production technology is

$$Y=\int_0^N X_i^\theta\left(Z_i^\alpha Z^{1-\alpha}\left(\frac{L}{N^{\eta_L}}\right)^\gamma\left(\frac{\Omega}{N^{\eta_\Omega}}\right)^{1-\gamma}\right)^{1-\theta}di, \quad 0<\theta, \ \ \alpha,\gamma,\eta_L,\eta_\Omega<1, \tag{4}$$

where N is the mass of non-durable differentiated intermediate goods, X_i is the quantity of good i, and L and Ω are, respectively, the services of labor and land purchased from the household. (This representation implicitly imposes labor and land market clearing.) Quality is the good's ability to raise the productivity of L and Ω. Specifically, the contribution of good i to the productivity of the non-reproducible factors depends on its own quality, Z_i, and on average quality, $Z=\int_0^N (Z_j/N)\,dj$. I show below that the parameter α regulates the private return to quality.[8]

[6] See Peretto (1998), especially pp. 53–56, for a more detailed exposition of this argument and references to the historical and managerial literature documenting this fact.

[7] More precisely, there is no physical capital in the neoclassical sense of a homogeneous, durable, intermediate good accumulated through foregone consumption. Instead, there are differentiated, non-durable, intermediate goods produced through foregone consumption. One can think of these goods as capital, albeit with 100% instantaneous depreciation. Introducing neoclassical physical capital complicates the analysis without adding insight.

The parameters η_L and η_Ω capture the degree of congestion (or rivalry[9]) of the services of labor and land across intermediate goods. For $\eta_L=\eta_\Omega=0$ there is no congestion, meaning that services of labor and land can be shared by all intermediate goods with no loss of productivity. This is a case of extreme economies of scope in the use of the services of the physical factor of production L and Ω that, as I show below, in the reduced-form representation of the production function in equilibrium manifest themselves as strong social increasing returns to product variety. At the opposite end of the spectrum, $\eta_L=\eta_\Omega=1$ yields full congestion, where there are no economies of scope and therefore no social returns to variety.

The first-order conditions for the profit maximization problem of the final good producer yield that each intermediate producer faces the demand curve

$$X_i=\left(\frac{\theta}{P_i}\right)^{1/(1-\theta)} Z_i^\alpha Z^{1-\alpha}\left(\frac{L}{N^{\eta_L}}\right)^\gamma\left(\frac{\Omega}{N^{\eta_\Omega}}\right)^{1-\gamma}, \tag{5}$$

where P_i is the price of good i. Also, the final good producer pays total compensation

$$\int_0^N P_i X_i\, di=\theta Y,\quad wL=\gamma(1-\theta)Y \quad\text{and}\quad p\Omega=(1-\gamma)(1-\theta)Y \tag{6}$$

to intermediate goods, labor and land suppliers, respectively.

3.3. Intermediate producers

The typical intermediate firm operates a technology that requires one unit of final output per unit of intermediate good and a fixed operating cost $\phi Z_i^\alpha Z^{1-\alpha}$, also in units of final output.[10] The firm can increase quality according to the technology

$$\dot{Z}_i=I_i, \tag{7}$$

where I_i is R&D in units of final output. Using (5), the firm's gross profit (i.e., before R&D) is

$$\Pi_i=\left[(P_i-1)\left(\frac{\theta}{P_i}\right)^{1/(1-\theta)}\left(\frac{L}{N^{\eta_L}}\right)^\gamma\left(\frac{\Omega}{N^{\eta_\Omega}}\right)^{1-\gamma}-\phi\right]Z_i^\alpha Z^{1-\alpha}. \tag{8}$$

The firm chooses the time path of price and R&D in order to maximize the value of its shares:

$$V_i(0)=\int_0^\infty e^{-\int_0^t r(s)ds}[\Pi_i(t)-I_i(t)]\,dt, \tag{9}$$

subject to (7) and (8), where r is the interest rate and 0 is the point in time when the firm makes decisions. The firm takes average quality, Z, in (8) as given. The characterization of the firm's decision yields a symmetric equilibrium where

$$r=\alpha\frac{\Pi}{Z}=\alpha\left[(P-1)\frac{X}{Z}-\phi\right] \tag{10}$$

is the return to quality innovation (see the Appendix for the derivation) and α is now intuitively interpreted as the elasticity of the firm's gross profit with respect to its own quality, which regulates the private return to quality.

New products are developed by entrepreneurs that set up new firms to serve the market. To start up activity an entrepreneur must sink βX units of final output.[11] Because of this sunk cost, the new firm cannot supply an existing good in Bertrand competition with the incumbent monopolist but must instead introduce a new good that expands product variety. Entry is positive if the value of the new firm is equal to its setup cost, i.e., if $V_i=\beta X$. Entrepreneurs finance entry by issuing equity and enter at the average quality level. The latter is a simplifying assumption that preserves symmetry of equilibrium

[8] This specification modifies the augmented Schumpeterian model developed by Aghion and Howitt (1998) to make it better suited to my purposes and yet leave the core mechanism essentially unchanged. The first modification is diminishing private returns to own quality, i.e., $\alpha<1$. This allows me to work with symmetric equilibria that feature creative accumulation, whereby all incumbent firms do R&D, as opposed to creative destruction, whereby outsiders do R&D to replace the current incumbent. The second modification is that quality enters with exponent $1-\theta$, instead of 1, because my intermediate producers face a marginal cost of production in units of the final good, instead of a marginal cost in units of (physical) capital proportional to their quality level. Both approaches imply that quality enters the reduced-form version of (4) as augmenting the input in exogenous supply, which here is a Cobb–Douglas composite of labor and land. The interested reader can find a more explicit comparison of the two specifications in the appendix.

[9] Rivalry can be modeled by writing the labor and land inputs with a subscript i to capture that their services are assigned to the specific good i and cannot be shared with the other goods. The approach in the text is simpler.

[10] Modeling the fixed cost as $\phi Z_i^\varphi Z^{1-\varphi}$, $0\le\varphi\le1$ complicates the analysis without changing the results.

[11] An analogous assumption is that the entrant must sink $\beta Y/N$ units of final output (see Barro and Sala-i-Martin 2004, pp. 300–302). I also obtain qualitatively similar results if the entry cost is βZ, but in this case the analysis is much more algebra-intensive.

at all times. The free-entry condition then yields the return to variety innovation (see the Appendix for the derivation):

$$r=\frac{\Pi-I}{\beta X}+\frac{\dot{X}}{X}=\frac{(P-1)\frac{X}{Z}-\phi-\frac{I}{Z}}{\beta\frac{X}{Z}}+\frac{\dot{X}}{X}. \tag{11}$$

4. The economy's dynamics

This section focuses on the key allocation problem of the economy — the allocation of final output Y to consumption, production of intermediates and, when profitable, vertical and horizontal innovation — and derives the reduced-form representation of the resulting equilibrium dynamics. The representation yields an analytical solution for the economy's path.

4.1. General equilibrium

Under conditions discussed in previous contributions (Peretto, 1998, 1999a) models of this class have symmetric equilibria in which all intermediate firms charge the same price and have the same quality level at all times.[12] Specifically, intermediate firms set $P=1/\theta$ and receive $N\cdot PX=\theta Y$ from the final producer. I can then write $X=\theta^2 Y/N$. Next, I impose symmetry in the production function (4) and use the previous result to eliminate X, obtaining

$$Y=\theta^{2\theta/(1-\theta)}\cdot N^{\sigma}ZL^{\gamma}\Omega^{1-\gamma},\quad \sigma\equiv 1-\gamma\eta_L-(1-\gamma)\eta_{\Omega}. \tag{12}$$

Thus, the reduced-form representation of the production function of this economy features social returns to variety equal to σ and social returns to quality equal to 1. Taking logs and time derivatives of (12) and subtracting population growth yield

$$y=\sigma n+z-(1-\gamma)\lambda, \tag{13}$$

where $y\equiv\dot{Y}/Y-\lambda$, $n\equiv\dot{N}/N$ and $z\equiv\dot{Z}/Z$. In other words, final output per capita growth is given by the growth rate of technology minus the *growth drag* due to the presence of land, which over time becomes relatively scarce as population L grows exponentially at rate λ. Of course, if $\gamma=1$ the drag disappears.

The key component of the model is the characterization of the incentives to vertical and horizontal innovation. Eqs. (10) and (11) show that the returns to investment and to entry depend on the *quality-adjusted* gross firm size, measured by its quality-adjusted volume of production, X/Z. This suggests using $x\equiv X/Z$ as the stationary state variable in the analysis of dynamics since in this model steady-state growth is driven by exponential growth in quality, Z . Using (12), I also obtain

$$x\equiv\frac{X}{Z}=\frac{\theta}{P}\frac{Y}{NZ}=\left(\frac{\theta}{P}\right)^{1/(1-\theta)}\frac{L}{N^{1-\sigma}}. \tag{14}$$

This expression shows the equilibrium determinants of quality-adjusted firm size.

The definition of x allows me to rewrite the returns to innovation in (10) and (11) as

$$r=\alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right); \tag{15}$$

$$r=\frac{1}{\beta}\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)+\frac{\dot{x}}{x}+z. \tag{16}$$

Expressions (15) and (16) capture the model's main property: firm-level decisions depend on the quality-adjusted firm size, which is increasing in population and the land endowment because they drive production of the final good Y and thereby demand for intermediate goods. It should be clear, thus, that from the viewpoint of the managers of incumbent firms and of the entrepreneurs that set up new firms the critical market size variable is expenditure on intermediates, θY. Recall, moreover, that consumption, production of intermediates and quality and variety innovation are all in units of the final good so that the resource allocation problem of this economy is the allocation across its alternative uses of the quantity Y produced according to (12). The following property characterizes the consumption flow that results from such allocation.

[12] The conditions for this to be the case are (i) that the firm-specific return to quality innovation is decreasing in Z_i (in this paper's setup this follows from $\alpha<1$) and (ii) that entrants enter at the average level of quality Z. The first condition implies that if one holds constant the mass of firms and starts the model from an asymmetric distribution of firm sizes, then the model converges to a symmetric distribution. The second requirement simply ensures that entrants do not perturb such symmetric distribution.

Proposition 1. *In equilibrium the economy's consumption ratio is*

$$\frac{C}{Y}=\begin{cases}\theta^2\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)+1-\theta, & n=0, \quad z\geq 0\\ (\rho-\lambda)\beta\theta^2+1-\theta, & n>0, \quad z\geq 0\end{cases}. \tag{17}$$

Proof. See the Appendix.

When entry is positive the fraction of final output that is consumed is constant throughout the transition as well as in steady state. When entry is zero, instead, the consumption ratio is increasing in quality-adjusted firm size, x, and decreasing in the R&D intensity of the firm, $z=I/Z$, if positive. The reason is that incumbents earn rents that they distribute to the household as dividends. These rents increase with the size of the market (the numerator of x) because the fixed operating cost ϕZ (under symmetry) implies a falling unit production cost as the scale of operations of the firm rises. When entrants become active, these rents are competed away and the consumption ratio no longer rises with quality-adjusted firm size.[13]

4.2. Horizontal and vertical innovation

Proposition 1 highlights that there exist corner solutions where one or both the R&D activities shut down. When entrants are active and the consumption ratio c is constant, the return to saving (3) reduces to $r=\rho+y$ and thus the return to entry (16) yields

$$n=\begin{cases}\frac{1}{\beta}\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)-\rho+\lambda, & z>0\\ \frac{1}{\beta}\left(\frac{1}{\theta}-1-\frac{\phi}{x}\right)-\rho+\lambda, & z=0.\end{cases} \tag{18}$$

This expression says that there is a threshold of quality-adjusted firm size below which entrants are not active ($n=0$) because the return is too low. The value of the threshold depends on whether entrants anticipate that in the post-entry equilibrium $z>0$ or $z=0$ since it affects the net cash flow that they anticipate to earn. Similarly, the saving schedule (3), the reduced-form production function (12), the return to quality innovation (15) and the top line of (18) yield

$$z=\begin{cases}\alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)-\rho+\lambda-\sigma n-\gamma\lambda, & n>0\\ \alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)-\rho+\lambda-(c-y)-\gamma\lambda, & n=0,\end{cases} \tag{19}$$

which says that there is a threshold of quality-adjusted firm size below which incumbents do not do quality R&D ($z=0$) because the return is too low. The value of the threshold depends on whether $n>0$ or $n=0$ since it affects the return to innovation that they anticipate they must earn to deliver to their stockholders (the savers) their reservation rate of return on assets.

The interdependence of agents' activation decisions implies that the *sequence* in which the economy turns on the two innovation engines determines the shape of its transition path and the timing of the key events. It is useful to begin the analysis with a characterization of the equilibrium where both types of R&D are positive.

Proposition 2. *Let x_N denote the threshold of quality-adjusted firm size for variety innovation and x_Z the threshold of quality-adjusted firm size for quality innovation. Assume $\beta x>\sigma$ $\forall x\geq\phi/(1/\theta-1)$, i.e., $\beta\phi/(1/\theta-1)>\sigma$. Then, for $x>\max\{x_Z,x_N\}$ the equilibrium rates of variety and quality innovation are*:

$$n=\frac{\left((1-\alpha)\left(\frac{1}{\theta}-1\right)-(\rho-\lambda)\beta\right)x-(1-\alpha)\phi+\rho-\lambda+\gamma\lambda}{\beta x-\sigma}>0; \tag{20}$$

$$z=\frac{\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)\left(\alpha-\frac{\sigma}{\beta x}\right)-(1-\sigma)(\rho-\lambda)-\gamma\lambda}{1-\frac{\sigma}{\beta x}}>0. \tag{21}$$

Proof. See the Appendix.

Thus, if quality-adjusted firm size grows sufficiently large, the economy turns on both innovation engines. Setting aside for the moment the issue of stability, it is useful to characterize the steady state associated with such equilibrium.

[13] This property deserves a much more detailed discussion. To preserve this section's focus on solving analytically the model, I postpone it to the next section where I discuss the anatomy of the growth path of the economy.

Proposition 3 (*The modern growth steady state*). *Assume*

$$\beta\phi > \frac{1}{\alpha}\left[\frac{1}{\theta} - 1 - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)\right] > \frac{1}{\theta} - 1.$$

Then in the region $x > \max\{x_Z, x_N\}$ *there exists the steady state*:

$$x^* = \frac{(1-\alpha)\phi - \left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)} > 0; \tag{22}$$

$$n^* = \frac{\gamma\lambda}{1-\sigma} > 0; \tag{23}$$

$$z^* = \left[\frac{\alpha\left(\beta\phi - \frac{1}{\theta} + 1\right)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)} - 1\right]\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right) > 0. \tag{24}$$

This steady state exhibits growth of final output per capita:

$$y^* = \sigma n^* + z^* - (1-\gamma)\lambda = \alpha\left(\left(\frac{1}{\theta} - 1\right)x^* - \phi\right) - \rho,$$

which is positive iff

$$\alpha\frac{\left(\beta\phi - \frac{1}{\theta} + 1\right)\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)} > \rho. \tag{25}$$

Proof. See the Appendix.

This proposition establishes conditions under which the economy possesses a steady state where both types of R&D are positive and the growth rate of gross output per capita is constant and positive. Does the economy converge to such steady state?

4.3. Dynamics

The structure of incentives for innovation discussed above identifies conditions that yield two sequences of events.

Proposition 4. *There exists a combination of values of the parameters such that the thresholds* x_N *and* x_Z *are identical. There are thus two regimes, characterized by the order in which the economy activates the quality and the variety engines of growth*:

- *Dominant incentives for variety innovation*: *For parameters such that*

$$\alpha < \frac{\frac{1}{\theta} - 1 - (\rho-\lambda)\beta}{(\rho-\lambda)\beta\phi}\left[\rho - \lambda + \frac{\theta^2\left[\frac{1}{\theta} - 1 - (\rho-\lambda)\beta\right]}{1 - \theta^2\left(\frac{1}{\theta} - (\rho-\lambda)\beta\right)}\gamma\lambda\right],$$

  the ordering of the thresholds is $x_N < x_Z$, *where*

$$x_N = \frac{\phi}{\frac{1}{\theta} - 1 - (\rho-\lambda)\beta};$$
$$x_Z = \arg\text{solve}\left\{\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right)\left(\alpha - \frac{\sigma}{\beta x}\right) = (1-\sigma)(\rho-\lambda) + \gamma\lambda\right\}.$$

- *Dominant incentives for quality innovation. For parameters such that*

$$\alpha > \frac{\frac{1}{\theta} - 1 - (\rho-\lambda)\beta}{(\rho-\lambda)\beta\phi}\left[\rho - \lambda + \frac{\theta^2\left[\frac{1}{\theta} - 1 - (\rho-\lambda)\beta\right]}{1 - \theta^2\left(\frac{1}{\theta} - (\rho-\lambda)\beta\right)}\gamma\lambda\right],$$

the ordering of the thresholds is $x_Z < x_N$, *where*

$$x_N = \frac{(1-\alpha)\phi - \rho + \lambda - \gamma\lambda}{(1-\alpha)\left(\frac{1}{\theta}-1\right) - (\rho-\lambda)\beta};$$

$$x_Z = \arg\text{solve}\left\{ \alpha\left(\left(\frac{1}{\theta}-1\right)x - \phi\right) = \frac{\theta^2 \frac{\phi}{x}}{1-\theta^2\left(1+\frac{\phi}{x}\right)}\gamma\lambda \right\}.$$

Proof. See the Appendix.

In the variety-first case x_Z is the threshold for quality R&D given that the market *already* supports entry of new firms, whereas in the quality-first case it is the threshold for quality R&D given that the market *does not yet* support entry of new firms. Accordingly, in the variety-first case incumbents undertaking quality R&D compete for resources with entrants and face a constant reservation interest rate demanded by savers. In the quality-first case, instead, they do not compete for resources with entrants that are setting up new firms but, because the free entry condition does not hold and they distribute to shareholders rents that grow with the size of the market, they face a reservation interest rate that reflects the growing consumption ratio. Similar reasoning applies to the threshold x_N. The following proposition provides the paper's main analytical result.

Proposition 5. *Let the economy's initial condition be*

$$x_0 = \theta^{2/(1-\theta)} \frac{L_0^{\gamma}\Omega^{1-\gamma}}{N_0^{1-\sigma}} < \min\{x_N, x_Z\}$$

and recall the steady-state quality-adjusted firm size x^* *characterized in Eq.* (22). *The two regimes then yield the following dynamics*:

- *The variety-first path to modern growth: The equilibrium law of motion of quality-adjusted firm size is piecewise linear*:

$$\dot{x} = \begin{cases} \gamma\lambda x, & \phi/\left(\frac{1}{\theta}-1\right) \le x \le x_N \\ \overline{\nu}(\overline{x}^* - x), & x_N < x \le x_Z \\ \nu(x^* - x), & x > x_Z, \end{cases} \tag{26}$$

with coefficients

$$\overline{\nu} \equiv \frac{1-\sigma}{\beta}\left[\frac{1}{\theta} - 1 - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)\right],$$

$$\overline{x}^* \equiv \frac{\phi}{\frac{1}{\theta} - 1 - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)},$$

$$\nu \equiv \frac{1-\sigma}{\beta}\left[(1-\alpha)\left(\frac{1}{\theta}-1\right) - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)\right],$$

and yields the explicit solution $x(t)$ *illustrated in* Fig. 1.
Quality-adjusted firm size follows an S-shaped path with inflection point at T_N *and convergence from below to* x^*, *where*

$$T_N = \frac{1}{\gamma\lambda}\log\left(\frac{x_N}{x_0}\right) \tag{27}$$

is the date when the economy crosses the threshold x_N *and turns on variety innovation and*

$$T_Z = T_N + \frac{1}{\overline{\nu}}\log\left(\frac{\overline{x}^* - x_N}{\overline{x}^*} - x_Z\right). \tag{28}$$

is the date when the economy crosses the threshold x_Z *and turns on quality innovation.*

- *The quality-first path to modern growth: The equilibrium law of motion of quality-adjusted firm size is piecewise linear:,*

$$\dot{x} = \begin{cases} \gamma\lambda x, & \phi/\left(\frac{1}{\theta}-1\right) \le x \le x_Z \\ \gamma\lambda x, & x_Z < x \le x_N \\ \nu(x^* - x), & x > x_N, \end{cases} \tag{29}$$

and yields the explicit solution $x(t)$ illustrated in Fig. 2.

Quality-adjusted firm size follows an S-shaped path with inflection point at T_N and convergence from below to x^*, where

$$T_Z = \frac{1}{\gamma\lambda}\log\left(\frac{x_Z}{x_0}\right) \tag{30}$$

is the date when the economy crosses the threshold x_Z and turns on quality innovation and

$$T_N = T_Z + \frac{1}{\gamma\lambda}\log\left(\frac{x_N}{x_Z}\right) = \frac{1}{\gamma\lambda}\log\left(\frac{x_N}{x_0}\right). \tag{31}$$

is the date when the economy crosses the threshold x_N and turns on variety innovation.

Proof. See the Appendix.

The difference between the two solutions for the last part of the equilibrium path is only in the time periods over which they hold, which are determined by the starting dates T_Z and T_N. The key difference between the two solutions, however, is that in the variety-first case premature market saturation that prevents the economy from reaching the phase of quality innovation is possible. This outcome is the red (lighter) path in Fig. 1 that converges to the steady state $\bar{x}^*$.

4.4. Discussion

The reduced-form, state-space representation of this model consists of a pair of piece-wise linear differential equations in quality-adjusted firm size x characterizing two possible scenarios. In one the incentives for horizontal innovation dominate and the economy crosses the threshold for variety innovation first. In the other the reverse is true: the incentives for vertical innovation dominate and the economy crosses the threshold for quality innovation first.

Proposition 4 says that a finite threshold of quality-adjusted firm size that activates one or the other innovation engine always exists. This means that as long as population growth is positive the economy *must* turn on Schumpeterian

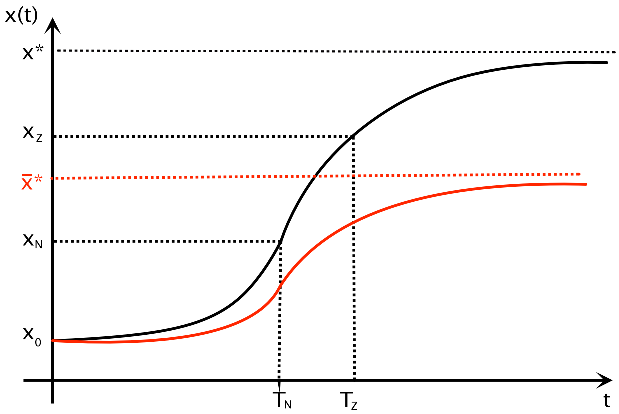

Fig. 1. The time path of firm size in the variety-first case. (For interpretation of the references to color in this figure, the reader is referred to the web version of this paper.)

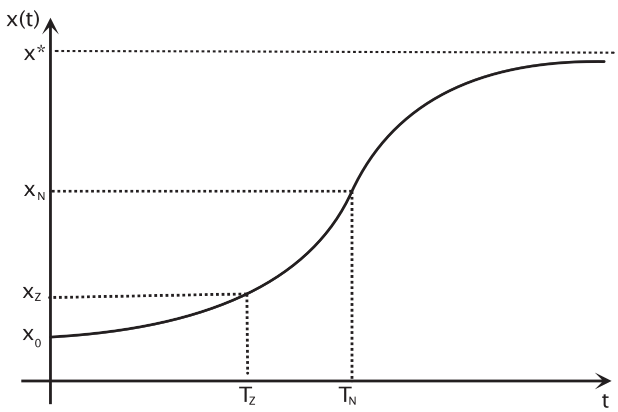

Fig. 2. The path of firm size in the quality-first case.

innovation. The intuition is that as long as the overall market for intermediate goods grows due to exogenous forces, quality-adjusted firm size (i.e., profitability) grows and eventually must cross the threshold where one of the two engines of growth is turned on. This is, in essence, a no-arbitrage argument: as rents escalate the only force that can prevent agents from investing in activities aimed at capturing a share of them is infinite innovation costs or, equivalently, zero productivity of investment of final goods in variety and quality innovation (in the model's notation, $\beta \to \infty$ and $\alpha \to 0$).

Since innovation must start, the only question is *when* and what specific *sequence of events* unfolds. Proposition 4 says that the model's parameters space consists of two thick regions, one where variety innovation starts first, the other where quality innovation starts first. The two paths of quality-adjusted firm size shown in Figs. 1 and 2 generate drastically different economic histories.

5. Interpreting the model: three phases of growth

This section focuses on the model's predictions. It characterizes the economy's path in each case in terms of (a) within-phase behavior of key observables and (b) the timing of the transitions from one phase to the next.

5.1. *Anatomy of the transition: The variety-first case*

Along the paths of the state variable x shown in Figs. 1 and 2 the level of final production is given at any point in time by (12). That expression contains only the levels of the state variables N (product variety), Z (product quality) and L (population). Consequently, the path $Y(t)$ obtains from the paths $N(t)$, $Z(t)$ and $L(t)$. As argued, for simplicity the path of population is exogenous and exponential. Moreover, given initial conditions N_0 and Z_0, and the solution $x(t)$, the paths of variety and quality are fully determined by Eqs. (20) and (21). Although this procedure allows one to solve analytically for the paths $Y(t)$, $N(t)$, $Z(t)$ and $L(t)$, it is more insightful to characterize the evolution of the economy in terms of equations that express the relevant variables as functions of quality-adjusted firm size x.

5.1.1. *Final output, GDP and consumption*

Proposition 1 shows that the allocation of final output across its alternative uses features a ratio C/Y that is increasing in quality-adjusted firm size x when entrants are not active and constant when entrants are active. As argued, such behavior stems from static economies of scale that manifest themselves as efficiency gains in the production of intermediates.

To refine that intuition let G denote this economy's GDP. Subtracting the cost of intermediate production from the value of final production and using the definition of x yield

$$\frac{G}{Y} = 1 - \frac{N(X+\phi Z)}{Y} = 1 - \theta^2 \left(1 + \frac{\phi}{x}\right). \tag{32}$$

The term in brackets is increasing in x because the unit cost of production of the typical intermediate firm falls as its scale of operation rises. Taking logs and time derivatives of (32) yields

$$\frac{\dot{G}}{G} = \frac{\dot{Y}}{Y} + \xi(x)\frac{\dot{x}}{x}, \quad \xi(x) \equiv \frac{\theta^2 \frac{\phi}{x}}{1 - \theta^2 \left(1 + \frac{\phi}{x}\right)}, \tag{33}$$

where $\xi(x)$ is the elasticity of GDP with respect to quality-adjusted firm size. This expression says that GDP growth is given by final output growth plus the contribution from efficiency gains in intermediate production due to quality-adjusted firm size growth.

Eqs. (12)–(13) and (32)–(33) provide a complete characterization of output dynamics for this economy. Initially, final output grows only because of population growth, that is, $\dot{Y}/Y = \gamma\lambda$. Moreover, because consumption equals GDP, $\dot{C}/C = \dot{G}/G = [\xi(x)+1]\gamma\lambda$. Thus, in the early Smithian phase with no Schumpeterian innovation, GDP and consumption growth are due solely to population growth and its amplification through static economies of scale. As the economy crosses the threshold x_N and enters the second phase, final output growth becomes $\dot{Y}/Y = \gamma\lambda + \sigma n(x)$, where the rate of variety growth $n(x)$ is given by the top line of (18). The third and final phase has both variety and quality innovation so that $\dot{Y}/Y = \gamma\lambda + \sigma n(x) + z(x)$, where $n(x)$ and $z(x)$ are given, respectively, by (20) and (21) in Proposition 2.

It is useful to summarize this characterization in terms of the growth rates of per-capita final output, GDP and consumption — y, g, and c, respectively — since these are the objects that the empirical literature typically discusses. Furthermore, it is useful to write these growth rates as the sum of a Schumpeterian innovation component that does not

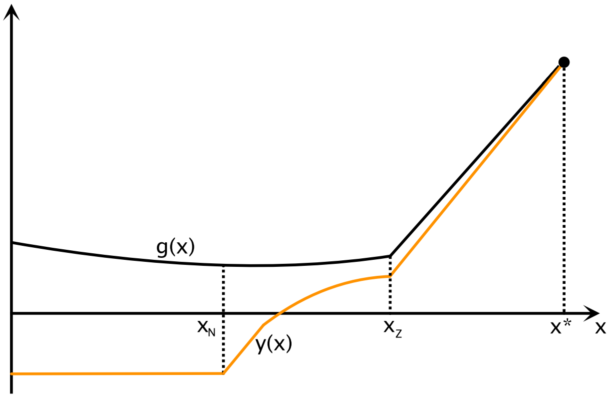

Fig. 3. The growth rates of final output and GDP per capita as functions of firm size in the variety-first case.

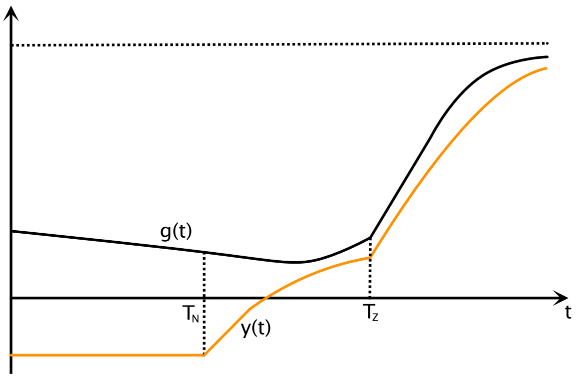

Fig. 4. The growth rates of final output and GDP per capita as functions of time in the variety-first case.

vanish in steady state and a Smithian component, due to static economies of scale, that vanishes in steady state:

$$y(x)=\begin{cases} -(1-\gamma)\lambda, & \phi/\left(\dfrac{1}{\theta}-1\right)\le x\le x_N \\ \left(\dfrac{\gamma}{1-\sigma}-1\right)\lambda-\dfrac{\sigma}{1-\sigma}\overline{\nu}\left(\dfrac{\overline{x}^*}{x}-1\right), & x_N<x\le x_Z \\ \alpha\left(\left(\dfrac{1}{\theta}-1\right)x-\phi\right)-\rho, & x>x_Z; \end{cases}$$

$$g(x)=\begin{cases} ([\xi(x)+1]\gamma-1)\lambda, & \phi/\left(\dfrac{1}{\theta}-1\right)\le x\le x_N \\ \left(\dfrac{\gamma}{1-\sigma}-1\right)\lambda+\left[\xi(x)-\dfrac{\sigma}{1-\sigma}\right]\overline{\nu}\left(\dfrac{\overline{x}^*}{x}-1\right), & x_N<x\le x_Z \\ \alpha\left(\left(\dfrac{1}{\theta}-1\right)x-\phi\right)-\rho+\xi(x)\nu\left(\dfrac{x^*}{x}-1\right), & x>x_Z; \end{cases}$$

$$c(x)=\begin{cases} g(x), & \phi/\left(\dfrac{1}{\theta}-1\right)\le x\le x_N \\ y(x), & x>x_N. \end{cases}$$

Figs. 3 and 4 illustrate these functions and the associated solutions for the growth rates.

The story that these equations tell is one where the economy starts out in a situation where there is no entry and firms earn rents. These rents grow with the size of the market and fuel GDP and consumption growth in excess of final output growth. Consequently, negative growth of final output per capita does not necessarily imply falling GDP and consumption per capita. In fact, it is possible to choose parameter values such that $[\xi(x)+1]\gamma>1$, meaning that GDP per capita grows all the time.[14] More generally, GDP per capita growth can start out negative and stay negative until the economy hits $x=x_N$ and turns on variety innovation. When that happens, the growth rate of GDP per capita starts rising and eventually turns positive if the contribution of product variety to final production is sufficiently strong. The growth rate of consumption per capita c, in contrast, drops to the growth rate of final output per capita, y, and remains below the growth rate of GDP per capita, g, until the end of the transition, where the constant quality-adjusted firm size, $x = x^*$, yields a constant ratio between final output and GDP.

In the intermediate phase the tension between the exploitation of static economies of scale, that requires firm growth, and the exploitation of social returns to variety, that requires entry, results in a profile of GDP per capita growth that is always convex but can be increasing, U-shaped or decreasing in x throughout the interval $x_N < x \le x_Z$. The parameter that drives these cases is the degree of social returns to variety σ. To avoid clutter, the figures illustrate only the second possibility, corresponding to situations where σ is sufficiently large that there exists a value of x where $dg(x)/dx$ becomes positive in the interval $x_N < x \le x_Z$. If such value of x is larger than x_Z, which happens when the degree of social returns to variety σ is small, the function $g(x)$ is decreasing throughout the range $x_N < x \le x_Z$ and the third case arises. This case is remarkable in that it says that the onset of systematic, profit-driven horizontal innovation is associated with a continuation of the slowdown of GDP per capita growth due to the gradual exhaustion of static economies of scale.

5.1.2. *Timing of events: the role of the fundamentals*

The closed-form solution for the transition path provides analytical insight on the determinants of the timing of the key events in the economy's history. The expressions for T_N and T_Z in (27) and (28) reveal the following pattern:

- The activation of horizontal innovation occurs earlier, i.e., T_N is lower, in economies where the ratio x_N/x_0 is lower.
- The activation of vertical innovation occurs if and only if the steady-state quality-adjusted firm size $\bar{x}^*$ associated with the variety-driven phase is smaller than the threshold for quality R&D x_Z.
- Given T_N, and conditional on $\bar{x}^* > x_Z$, the activation of vertical innovation occurs earlier, i.e., T_Z is lower, in economies where
  - the steady-state quality-adjusted firm size $\bar{x}^*$ is larger;
  - convergence in the variety-driven phase is faster, i.e., where $\bar{\nu}$ is higher;
  - the threshold for quality R&D x_Z is smaller.

Checking the expressions for x_0, $\bar{\nu}$, x_N, x_Z, $\bar{x}^*$ provides further detail.

Proposition 6. *The date of activation of variety innovation, T_N, is*

- *decreasing in the initial population L_0, the land endowment Ω, the population growth rate λ and the elasticity of output with respect to labor γ;*
- *increasing in the fixed operating cost ϕ, the sunk entry cost β and the discount rate ρ;*
- *independent of the elasticity of gross profit with respect to own quality α and the degree of social returns to variety σ.*

The duration of the phase with variety innovation only, $T_Z - T_N$, is

- *decreasing in the fixed operating cost ϕ and the elasticity of gross profit with respect to own quality α;*
- *increasing in the degree of social returns to variety σ;*
- *depends ambiguously on the population growth rate λ, the elasticity of output with respect to labor γ, the sunk entry cost β and the discount rate ρ;*
- *independent of the initial population L_0, the land endowment Ω.*

This characterization identifies factors that explain why some economies take off earlier than others — defining the take-off as the onset of systematic, profit-driven innovation — and factors that explain why some economies experience a faster transition than others to the ultimate phase with both innovation engines turned on and convergence to sustained growth. Moreover, it provides a novel insight on why some economies might fail to reach the modern growth phase: they might fail to turn on vertical innovation due to premature market saturation.

[14] Intuitively, this requires a restriction on the elasticity of output with respect to land in final production, i.e., $1-\gamma \le \left[\theta^2/\left(1-\theta^2\right)\right][1-(\rho-\lambda)\beta]$.

5.1.3. A closer look at consumption-saving behavior and factor remuneration

It is useful to examine the behavior of the ratio of consumption to GDP, which can be written as

$$\frac{C}{G}=\frac{C}{Y}\frac{Y}{G}=\begin{cases}\dfrac{\theta^2\left[\left(\dfrac{1}{\theta}-1\right)-\dfrac{\phi+z}{x}\right]+1-\theta}{1-\theta^2\left(1+\dfrac{\phi}{x}\right)}, & n=0,\quad z\geq 0\\[2ex] \dfrac{(\rho-\lambda)\beta\theta^2+1-\theta}{1-\theta^2\left(1+\dfrac{\phi}{x}\right)}, & n>0,\quad z\geq 0.\end{cases}$$

When entrants are not active ($n=0$), the ratio is 1 if there is no vertical innovation ($z=0$) because in that case the economy makes no investment and thus needs no saving. If instead there is vertical innovation ($z>0$), the ratio is less than 1 and decreasing in z, because faster quality growth requires more investment, and increasing in x, because quality-adjusted firm size growth leads to falling unit costs in intermediate production. When entrants are active ($n>0$), the ratio is independent of z and decreasing in x. One can summarize these observations as follows:

$$\frac{C}{Y}=\begin{cases}\theta^2\left[\left(\dfrac{1}{\theta}-1\right)-\dfrac{\phi}{x}\right]+1-\theta, & \phi/\left(\dfrac{1}{\theta}-1\right)\leq x\leq x_N\\ (\rho-\lambda)\beta\theta^2+1-\theta, & x>x_N;\end{cases}$$

$$\frac{C}{G}=\begin{cases}1, & \phi/\left(\dfrac{1}{\theta}-1\right)\leq x\leq x_N\\[2ex] \dfrac{(\rho-\lambda)\beta\theta^2+1-\theta}{1-\theta^2\left(1+\dfrac{\phi}{x}\right)}, & x>x_N.\end{cases}$$

The first expression, reproduced from Proposition 1 for convenience, captures the property already discussed that static efficiency gains drive consumption growth in excess of final output growth in the Smithian phase of the transition. The second expression confirms that such consumption growth comes from efficiency gains in intermediate production that raise GDP. The fact that the ratio of consumption to GDP is decreasing in quality-adjusted firm size x when entrants are active captures the property that after the onset of systematic, profit-driven innovation the economy's investment share rises throughout the transition to the steady state x^*.

Associated with this pattern of consumption-saving, there is a pattern of factor remuneration driven by the following property: factors that earn a flow of payments proportional to final output Y earn a share of GDP that is decreasing in quality-adjusted firm size x and is thus decreasing over time throughout the transition. As shown in Section 2, the three factors that enter the production technology (4) — labor, land and intermediate goods — belong to this category. So, if these factors earn falling shares of GDP over time, what factor earns a rising share of GDP? The answer is that throughout the transformation of this economy what rises is the share of GDP earned by firms in the form of gross profits. Specifically, Eqs. (8) and (32) yield

$$\frac{\text{gross profits}}{\text{GDP}}=\frac{N\Pi}{G}=\frac{\theta Y-N(X+\phi Z)}{Y-N(X+\phi Z)}=\frac{\theta-\theta^2\left(1+\dfrac{\phi}{x}\right)}{1-\theta^2\left(1+\dfrac{\phi}{x}\right)}.$$

In the initial phase with no innovation, these profits are distributed to shareholders and consumed. Only after crossing the threshold for profitable innovation the economy exhibits saving and investment, resulting in a falling consumption share. More importantly, once it kicks in, the free entry condition implies that the rising profits are not escalating pure rents and, more important, that they are reinvested in innovation, thus driving the economy's growth acceleration.

5.2. Anatomy of the transition: The quality-first case

The quality-first case differs from the variety-first case in the intermediate phase and in the timing of its beginning and end. After the economy crosses the threshold x_Z, the growth rate of final output is $\dot{Y}/Y=\gamma\lambda+z$, where z is given by the bottom line of (19). Because that expression contains the growth rate of the consumption output ratio, I cannot just substitute terms to express the growth rates of Y, G and C as functions of x. However, I can use Eq. (17) to construct policy functions that provide the information needed to characterize the equilibrium path. The details are in the Appendix. Here it suffices to note that in the interval $x_Z<x<x_N$ there exists a function $z(x)$ that is increasing, convex and starts out with zero

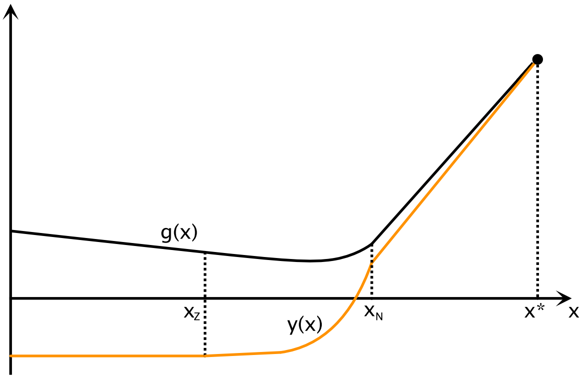

Fig. 5. The growth rates of final output and GDP per capita as functions of firm size in the quality-first case.

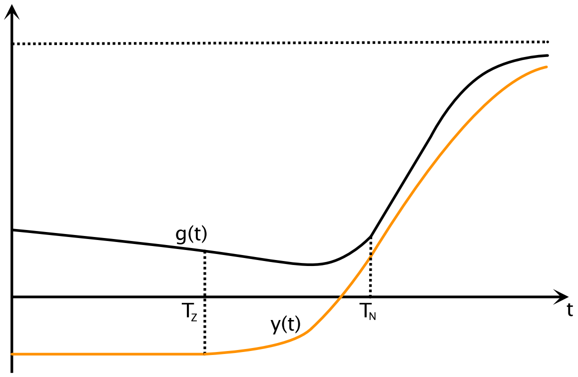

Fig. 6. The growth rates of final output and GDP per capita as functions of time in the quality-first case.

derivative at x_Z. Summarizing, the growth rates of final output, GDP and consumption per capita are

$$y(x)=\begin{cases} -(1-\gamma)\lambda, & \phi/\left(\frac{1}{\theta}-1\right)\le x<x_Z \\ z(x)-(1-\gamma)\lambda, & x_Z<x<x_N \\ \alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)-\rho, & x>x_N; \end{cases}$$

$$g(x)=\begin{cases} ([\xi(x)+1]\gamma-1)\lambda, & \phi/\left(\frac{1}{\theta}-1\right)\le x\le x_Z \\ ([\xi(x)+1]\gamma-1)\lambda+z(x), & x_Z<x\le x_N \\ \alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)-\rho+\xi(x)\nu\left(\frac{x^*}{x}-1\right), & x>x_N; \end{cases}$$

$$c(x)=\begin{cases} g(x), & \phi/\left(\frac{1}{\theta}-1\right)\le x\le x_N \\ y(x), & x>x_N. \end{cases}$$

Figs. 5 and 6 illustrate these functions and the associated solutions for the growth rates.

Along this path the rate of innovation exhibits explosive behavior because firms start undertaking quality R&D when they are still earning escalating rents driven by aggregate market growth due to population growth. As in the previous case, these escalating rents fuel consumption growth in excess of gross output growth. Moreover, since consumption per capita growth hits its minimum at $x=x_Z$, it is possible to choose parameter values such that GDP per capita growth is positive for all $\phi/(1/\theta-1)\le x\le x_Z$ and, consequently, that it grows all the time despite the fact that gross output per capita growth is initially negative. More generally, the economy can experience a period of negative GDP per capita growth until it hits $x=x_Z$. When that happens, the growth rate starts to rise and eventually turns positive. An interesting feature of this case is that, because the function $z(x)$ starts out with zero derivative at x_Z, it must be the case that the time profile of the growth rate of GDP is convex and decreasing at the onset of quality innovation. The reason is that the initial contribution of quality growth cannot overcome the gradual exhaustion of Smithian static economies of scale since it follows a very shallow time path.

The expressions for T_Z and T_N in (27) and (28) yield the following pattern for the timing of events and the role of the fundamentals:

- The activation of vertical innovation occurs earlier, i.e., T_Z is lower, in economies where the ratio x_Z/x_0 is lower.
- Given T_Z, the activation of horizontal innovation occurs earlier, i.e., T_N is lower, in economies where the threshold for variety R&D x_N is smaller.

Checking the expressions for x_0, x_Z, x_N yields further details.

Proposition 7. *The date of activation of quality innovation, T_Z, is*

- *decreasing in the initial population L_0, the land endowment Ω, the fixed operating cost ϕ and the elasticity of gross profit with respect to own quality α;*
- *depends ambiguously on the population growth rate λ and the elasticity of output with respect to labor γ;*
- *increasing in the discount rate ρ;*
- *independent of the sunk entry cost β and the degree of social returns to variety σ.*

The duration of the phase with quality innovation only, $T_N - T_Z$, is

- *increasing in the elasticity of gross profit with respect to own quality α and the sunk entry cost β;*
- *depends ambiguously on the population growth rate λ, the elasticity of output with respect to labor γ, the fixed operating cost ϕ and the discount rate ρ;*
- *independent of the initial population L_0, the land endowment Ω and the degree of social returns to variety σ.*

As in the previous case, this characterization identifies factors that explain why some economies take off earlier than others and factors that explain why some economies experience a faster transition than others to the ultimate phase with sustained, modern growth.

5.3. Bringing it all together: When does the take-off occur?

The initial history of this economy is one of growth of GDP and consumption per capita driven by the amplification of population growth — more generally, aggregate market size growth driven by exogenous forces — through the exploitation of static economies of scale. This process of Smithian growth has been highlighted by many writers (e.g., Jones, 1988; Mokyr, 2005, 2010). The multiplier of population growth in the expressions above, the term $[\xi(x)+1]\gamma-1$, has a theoretical range of $(\gamma(\theta+1)-1, \gamma-1)$ for $x \geq \phi$ so that even if one were to choose parameters that make it positive in the interval $\phi/(1/\theta-1) \leq x \leq \min\{x_N, x_Z\}$, it would yield a growth rate of GDP per capita equal to a fraction of the rate of population growth. Given that historically population growth rates prior to the Industrial Revolution where of the order of 0.1%, the model predicts very low growth rates of GDP and consumption per capita for the period.

How long does this stage of low growth last? Recall that the central message of Proposition 4 is that because population growth is positive at all times the economy *must* turn on Schumpeterian innovation and the only issue is which type it turns on first. Recall also that Eqs. (27) and (30) differ only by the value of the threshold that the economy hits first. Consequently, it is convenient to define a generic value

$$x_T \equiv \min\left\{\frac{\phi}{\frac{1}{\theta}-1-(\rho-\lambda)\beta}, \text{ arg solve}\left\{\alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)=\frac{\theta^2\frac{\phi}{x}}{1-\theta^2\left(1+\frac{\phi}{x}\right)}\gamma\lambda\right\}\right\}$$

and think of the take-off date as $T(1/\gamma\lambda)\ln(x_T/x_0)$. These expressions identify two main forces driving the duration of the initial phase. The first is the contribution of population growth to the growth of gross output, $\gamma\lambda$. The second is the gap between the initial condition and the threshold where the economy activates innovation, x_T/x_0. Using the definition of x_0, the expression for the take-off time becomes

$$T=\frac{1}{\gamma\lambda}\ln\left(\frac{x_T}{x_0}\right)=\frac{1}{\gamma\lambda}\ln\left(\frac{x_T}{\theta^2\frac{L_0^{\gamma}\Omega^{1-\gamma}}{N_0^{1-\sigma}}}\right)$$

and identifies two additional sets of determinants. Technological and preference parameters drive the cost–benefit calculation underlying the activation decision, that is, the term x_T. The land endowment and the initial values of population and of the mass of intermediate firms/products do not enter this calculation; they show up only in the denominator as the determinants of the initial state of the economy.

Suppose that the economy has an initial value $x_0 = x_T/2$, that is, in order to cross the threshold that activates innovation, quality-adjusted firm size has to double. Suppose also that $\gamma = 0.8$ and $\lambda = 0.001 = 0.1\%$. The expression above then shows that starting at time 0, the take-off time is $T = \ln 2/(0.08\%) = (69.3\%)/(0.08\%) = 866.25$, or, using the " rule of 70" that approximates $\ln 2 = 70\%$, $T = \ln 2/(0.08\%) = (70\%)/(0.08\%) = 875$.[15] Thus, an economy with output elasticity with respect to labor of 0.8 and whose population grows at 0.1% per year takes approximately 875 years to double its quality-adjusted firm size. Note that such an economy experiences an increase in population given by $L(T) = 2^{1/\gamma} \cdot L_0 = 2.38 \cdot L_0$.

I have set up this example assuming $x_T/x_0 = 2$ because it simplifies the calculation by exploiting well-known heuristics. The expression for T, however, says much more about this ratio. Recall that the threshold x_T depends on technological and preference parameters and that L_0, Ω and N_0 enter only in the determination of x_0 at the denominator. A key determinant of the take-off time is thus the initial fragmentation of the aggregate market for intermediate goods in submarkets and whether such fragmentation comes with little or large gains in productivity via social returns to variety. The definition $\sigma \equiv 1 - \gamma\eta_L - (1-\gamma)\eta_\Omega$ relates such social returns to variety to the deeper congestion parameters that characterize the model.

There are thus several channels through which institutions and other social factors can enter the determination of the take-off time T. An economy with a larger population takes off faster only in the trivial ceteris paribus sense that the comparative statics effect of L_0 on x_0 is positive. What really matters in the theory, however, is how such an economy differs in terms of quality-adjusted firm size from one with a smaller population. Once this is taken into account, what the model says is that the take-off time depends on a collection of determinants, including the availability of other factors of production (here land, more generally, resources, including exhaustible and/or renewable) and on how the underlying production structure determines congestion in the use of all factors of production across intermediate goods. In this light, it is worth emphasizing the role of the factor Ω. Abundance of this factor reduces the take-off time T while it produces a low resource price p and a high wage w. Without pushing the point too far because it is not central to the paper's goal, this pattern fits the essential features of the Allen (2009) thesis that the Industrial Revolution occurred in England because of cheap energy (low p) and high wages (high w). Of course, his story is much richer that what this model does, but it interesting to note that if one interprets Ω as coal, one sees that this model has the potential to develop into a fuller formalization of the process envisioned by Allen.

6. Other prominent features of the theory

The model has relatively few, standard ingredients and yet produces a rich set of results. Following are some properties worth emphasizing in a separate discussion to bring in even sharper relief what this paper's approach contributes to the literature.

Remark 8. Prior to the onset of profit-driven systematic innovation, static economies of scale in intermediate production deliver income per capita growth in periods of population expansion. Such Smithian growth, however, is not self-sustaining and eventually must vanish.

The best way to see this is to set parameters such that both $x_Z \to \infty$ and $x_N \to \infty$ (i.e., $\beta \to \infty$ and $\alpha \to 0$) so that innovation never takes hold. It then follows that asymptotically $[\xi(x)+1]\gamma\lambda \to \gamma\lambda$ so that GDP per capita and consumption per capita growth converge to $-(1-\gamma)\lambda$. This property is important because the historical record provides abundant evidence of sporadic bursts of income per capita growth, often associated with bursts of population growth. The main characteristics of these episodes are that they all eventually run out of steam and fizzled out. The model's key mechanism fits such pattern: population growth per se cannot give rise to self-sustaining growth of income per capita. As historians have stressed on multiple occasions (e.g., Jones, 1988; Easterlin, 1996; Mokyr, 2005, 2010), the key to the growth acceleration that the world experienced in the 18th and 19th centuries is that it was associated with a qualitative transition to a different mode of growth, one based on sustained profit-driven innovation.[16]

Remark 9. Changes in fundamentals that result in an earlier take-off date do not necessarily result in immediate take-off.

This property sounds obvious but, on reflection, highlights something that the current debate on the timing of the Industrial Revolution seems to ignore: institutional changes that favor innovation do not result in immediate take-off if the economy has not yet matured the other necessary conditions for doing so. Specifically, an economy that at time t experiences an improvement in the business environment that results in lower thresholds x_Z and x_N does not take off at time t if $x(t) < \min\{x_Z, x_N\}$. In other words, an economy that at the time of the favorable institutional change has not yet achieved the required quality-adjusted firm size *has to wait a shorter time to take off but does not take off immediately*. The current debate seems to take for granted that the response should be immediate (see, e.g., Mokyr, 2005, 2010; Mokyr and Voth, 2010; Galor, 2005, 2011), probably because most of the models that deal with the issue postulate economies that need to be shocked out of a steady state with no growth.

Remark 10. When the economy turns on quality innovation first, it exhibits explosive growth that ends in finite time.

[15] The well-known and often used " rule of 72" that approximates $\ln 2 \simeq 72\%$ would yield $T = \ln 2/(0.08\%) = (72\%)/(0.08\%) = 900$.

[16] Jones (1988), in particular, talks about a shift from Smithian to Promethean growth to emphasize the role of knowledge accumulation in the modern era. I follow Mokyr (2005, 2010) and talk about a shift from Smithian to Schumpeterian growth to emphasize the link between the historians' perspective and the economists' modern theory of innovation.

When firms start investing in quality innovation but the free entry condition does not yet apply, the dynamics replicate the special case of endogenous growth models driven by vertical innovation with a fixed number of products and exponential population growth. That is, it replicates the *explosive* growth that has been long considered problematic in first-generation models. This model, however, does not impose arbitrarily that product variety expansion is never operational so that at most a *finite* period of faster than exponential quality growth is possible. Explosive growth due to the scale effect, in other words, is *not* an inherent property of the theory. Rather, in first-generation models it is an artifact of the implausible assumption of fixed product variety — i.e., *infinite* entry costs — that prevents entry from competing away escalating rents.

Remark 11. If the economy turns on variety innovation first, it can fail to cross the threshold for quality innovation.

This property reinforces the previous observation about the importance of entry in competing away incumbents' rents. Not only entry tames explosive quality growth, it can also prevent the economy from reaching the stage where incumbents find profitable to improve their own products. Specifically, if variety innovation starts first and $\overline{x}^* \le x_Z$, then $T_Z \to \infty$, which means that the dissipation of rents due to product proliferation is so strong that the economy stabilizes the value of quality-adjusted firm size before it crosses the threshold for quality innovation.

Remark 12. The steady-state mass of firms is not proportional to population but, rather, is a generic power function of population.

Recall that in steady state quality-adjusted firm size is constant. Accordingly, the definition of x yields

$$N^* = \left[\theta^2 \frac{L^\gamma \Omega^{1-\gamma}}{x^*}\right]^{1/(1-\sigma)},$$

where x^* is independent of L and Ω; see (22). Eliminating the scale effect through product proliferation, therefore, does *not* require the knife-edge assumption:

$$N = (\text{constant}) \cdot L^{\varkappa}, \quad \varkappa = 1,$$

as often claimed (see, e.g., Jones, 2005). Rather, the theory says

$$\varkappa = \frac{\gamma}{1-\sigma} \lesseqgtr 1.$$

To get $\varkappa = 1$ one needs to assume either (a) $\gamma = 1$ (no land) and $\sigma = 0$ (no love of variety in production) or (b) $\gamma = 1-\sigma \Rightarrow 1-\gamma = \sigma$. Case (a) consists of simplifying assumptions that some of the early models imposed for convenience but that are not necessary features of the theory. Case (b) sets social returns to variety equal to the elasticity of output with respect to land.

Remark 13. If the economy enters the ultimate phase with both variety and quality innovation, population expansion is no longer needed to pull income per capita growth. Indeed, the population growth rate can fall to zero with income per capita growth remaining positive.

As the economy converges to the steady state, GDP per capita and consumption per capita growth converge to $g^* = \alpha((1/\theta - 1)x^* - \phi) - \rho$, which is positive for $\alpha((1/\theta - 1)x^* - \phi) > \rho$. The key to this expression is that meeting the condition for positive GDP per capita growth does not require special assumptions on population growth. Indeed, one can see from expression (22) that population growth λ can be zero (or even negative) without compromising the model's ability to deliver endogenous steady-state growth. This property is more important than it appears: it says that a burst of population growth provides a *window of opportunity* that the economy can exploit to go from its initial state with no innovation to the final state with endogenous, innovation-driven growth that does not require continuous market expansion due to exogenous forces.

The expression reveals something else as well: the effect of population growth on GDP per capita growth depends on the same condition that drives the steady-state relation between the mass of firms N and population size L. Specifically, x^* is increasing in $\rho - \lambda + \gamma\lambda/(1-\sigma)$ and therefore increasing in λ for $\gamma > 1-\sigma$, independent of λ for $\gamma = 1-\sigma$ and decreasing in λ for $\gamma < 1-\sigma$. Thus, what drives the model's predictions about the effect of *exogenous* population growth on economic growth are the assumptions on social returns to variety relative to the shares of the factors of production L and Ω.[17]

[17] In fact, things are even more interesting than this because the effects of population growth depend also on the assumptions one makes on preferences. In this paper I use the usual Benthamite specification that adds up utility across family members. Alternative assumptions are feasible. For example, one could modify (1) as follows:

$$U(0) = \int_0^\infty e^{-\rho t} L^\mu(t) \log\left(\frac{C(t)}{L(t)}\right) dt, \quad 0 \le \mu \le 1$$

resulting into an effective discount rate of $\rho - \mu\lambda$ that captures the range of attitudes going from the case of no preference for family members ($\mu = 0$) to the

7. Conclusion

This paper has proposed a theory of the emergence of modern Schumpeterian growth as the result of firms' and entrepreneurs' response to Smithian market expansion. The theory makes detailed predictions about the transition to innovation-driven growth, especially about the qualitative differences due to the timing and sequence of events.

To keep things simple, the paper takes no stand on the demographic transition. The assumption of constant population growth is a simplification that, while convenient in deriving analytical results, deserves further scrutiny. I do not pursue this point here for reasons of space. (I do so in related work in progress.) Moreover, population growth can be seen as just a stand-in force for exogenous market expansion. Alternatives to exogenous population growth that I have explored are exogenous disembodied technological change; exogenous growth of the resource endowment (e.g., discovery and opening up of new land); growth of embodied knowledge through "natural" curiosity and/or learning by being/doing effects (as in UGT).

Such specifications of the forces that drive market size growth yield qualitatively similar results. They differ from exogenous population growth in that, in the absence of a Malthusian mechanism, they all yield *rising* income per capita throughout the transition. The model thus formulated, therefore, would mask one of the key aspects of the Schumpeterian mechanism that I studied in this paper. Namely that there is a fundamental difference between the forces driving income per capita and the forces driving profit per firm. It is the latter that drive the phase transition from the regime with no profit-driven innovation to that with profit-driven innovation. More importantly, the activation of the Schumpeterian engine of endogenous growth may well require that agents tolerate temporarily falling income and/or consumption per capita as population growth builds up the economy to the point where the scale of operations of firms is sufficiently large.

Acknowledgments

I thank for comments and suggestions: John Seater, Domenico Ferraro, David Weil, Oded Galor, Philippe Aghion, Ufuk Akcigit, Paul Romer, Michael Peters and, especially, Peter Howitt. I also thank two anonymous referees and the Editor Theo Eicher.

Appendix A

A.1. A detailed comparison with UGT

How does this paper relate to the existing literature, in particular UGT? First, the focus is different. Rather than studying the breakdown of the Malthusian regime and the demographic transition, this paper focuses on the incentives to innovate and how they evolve with aggregate market size. This difference in focus drives its different modeling choices. Specifically, in order to focus on the role of the quantity–quality trade-off for children, UGT needs to simplify elsewhere — and it does so by modeling technological change as a black box. This paper does the opposite: it opens the black box of technological change and keeps the model tractable by abstracting from reproduction and education decisions.[18]

Second, the core logic of the Malthusian regime is that population growth is the channel through which technological change yields larger output while living standards stagnate. The idea is that population size fully absorbs improvements in the economy's production capacity. Crucially, therefore, the Malthusian regime says that the economy develops and eventually activates the quantity–quality trade-off for children only *because* there is technological change, either exogenous or associated with population size, that drives the effective supply of land.[19] But then, precisely because it models

(*footnote continued*)
Benthamite case discussed in the text ($\mu = 1$). With this specification, the expression for steady-state firm size becomes

$$x^* = \frac{(1-\alpha)\phi - \left(\rho - \mu\lambda + \frac{\gamma\lambda}{1-\sigma}\right)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \beta\left(\rho - \mu\lambda + \frac{\gamma\lambda}{1-\sigma}\right)}$$

while

$$g^* = \alpha\left(\left(\frac{1}{\theta} - 1\right)x^* - \phi\right) - \rho + \mu\lambda - \lambda,$$

[18] Recent notable attempts to integrate endogenous innovation mechanisms in UGT are Desmet and Parente (2012) and Strulik et al. (2013). Although they ask different research questions, both papers propose full-fledged UGT models that, because of their ambition and large number of key ingredients, are quite complex. Desmet and Parente (2012) ask a question and deploy a mechanism that are sufficiently close to what this paper does to deserve some discussion. They argue that a central mechanism at the heart of the Industrial Revolution was the rise in market competition, visible in falling mark-ups. They credit Peretto (1998, 1999a,b) for being the first to integrate such mechanisms in endogenous growth theory, and extend the scope of the analysis by adding the key ingredients of UGT (e.g., fertility choice) and an agricultural sector whose productivity grows exogenously in the early history of the model. Given the complexity of the model, they need to resort to numerical simulations. This paper, in contrast, shuts down those additional UGT channels and concentrates on the industry-level Schumpeterian response to market expansion and provides an analytically transparent characterization of the forces at play. Moreover, because it shuts down endogenous mark-ups as well, it emphasizes different drivers of the incentives to innovation.

technological change as a black box, UGT leaves unexplained the key driver of the dynamics. To see why this is important, note that if, applying the Malthusian logic, the economy studied in this paper sets population growth at zero to stabilize output per capita, it cannot take off since it fails to cross the threshold of market size for positive technological progress. This observation highlights the danger of the black-box assumption in UGT and, perhaps more interestingly, suggests that initially shrinking output per capita due to a growing population is the price that society pays to create the aggregate market size needed to support profit-driven investment in innovation. This feature seems to turn the logic of Malthusian equilibria on its head and is worth exploring since it refines our understanding of the dynamic interactions among land scarcity, technology and demography.

A third difference between UGT and this paper, therefore, is that this paper posits that population growth drives aggregate market growth and eventually takes the economy across the threshold of quality-adjusted firm size — which is really a threshold of profitability — where investment in new technology yields a sufficiently high rate of return. In this perspective, the paper articulates a vision of the take-off process in line with that proposed by Simon (2000), who argued that (exogenous) population growth triggered the " great breakthrough" and the consequent acceleration of world growth. The paper's characterization, which yields an analytical solution for the growth path, sheds new light on long-standing questions concerning the features of the process. Why is this difference important? Aside from it being a modeling simplification analogous to UGT's black boxing of technological change — and thus something to be explicitly acknowledged — this assumption focuses directly on the role of population growth as the trigger of the economy's phase transition independent of the underlying forces driving it. Furthermore, it fully acknowledges that the existence of a corner solution where technological change is zero suggests the potential for a chicken-and-egg problem: which comes first, technological change or population growth? UGT takes the view that land-augmenting technological change comes first and drives (i.e., causes) population growth; this paper simplifies things by going to the other extreme: population expansion triggers the onset of Schumpeterian innovation.

There are other, less prominent, differences between this paper and UGT that are worth highlighting. UGT considers only land as the factor that induces diminishing returns to labor and, typically, has no market for land. The second feature implies that UGT lacks a scarcity price signal and consequently has limited applicability to broader issues arising from the interactions of technology, demography and natural resources. Likewise, the first feature leaves out sources of scarcity that play an important role in the debate on the future growth potential of modern economies. This paper, in contrast, develops a framework that has a scarcity price signal and that extends seamlessly to the case of exhaustible and renewable natural resources. The ambition is to apply the ideas developed here to a broader class of questions.

Another difference is that UGT typically has no consumption/saving decision. This model does, and thus applies a framework that assigns an important role to the financial market in channeling resources from savers to the agents with the investment projects in need of funding. In this perspective, this paper embeds the question of the long-run acceleration of the economy at the time of the Industrial Revolution in a more traditional macroeconomic framework.

None of these observations are criticisms of UGT. On the contrary, they are meant to highlight the *complementarity* between UGT and the Schumpeterian approach proposed in this paper. UGT has accomplished much in advancing our understanding of issues that for a long time have resisted our best analytical efforts. Much remains to be done, however, and the Schumpeterian approach to endogenous innovation allows us to make further progress in the areas that UGT had to simplify to keep the models tractable.

A.2. Explicit comparisons of the paper's production structure with Aghion–Howitt (1998)

I adapted the production function in Eq. (4) from Aghion–Howitt (1998). Specifically, in p. 107 they introduce the following structure:

$$Y_t = C_t + I_t + N_t = L_t^{1-\alpha} Q_t^{\alpha-1} \int_0^{Q_t} A_{it} x_{it}^{\alpha}\, di,$$

where in their notation Y is the aggregate output of the final good, I is the aggregate investment in capital accumulation, N is the aggregate vertical/quality R&D, L is the aggregate employment, Q is the mass of intermediate goods, A_i is the quality of good i and x_i is the quantity of good i. Moreover, intermediate goods are produced with capital only, according to $x_i = k_i / A_i$ and subject to the market clearing condition $\int_0^Q k_i\, di = K$, where K is the aggregate capital stock. Later in the book (p. 408, footnote 6) they acknowledge that this structure implies zero social returns to variety and, to take the model to the data, modify it to

$$Y_t = C_t + I_t + N_t = \left(\frac{Q_t}{L_t}\right)^{\beta} \cdot L_t^{1-\alpha} Q_t^{\alpha-1} \int_0^{Q_t} A_{it} x_{it}^{\alpha}\, di, \quad 0 < \beta < 1.$$

[19] UGT models this process in reduced-form by positing that the rate of land-augmenting technological change is an increasing function of population size that yields a strictly positive rate of technological change for all values of population size. If, instead, it admits a threshold level of population below which technological change is zero, then escaping the Malthusian trap is not a necessary outcome of the model. In fact, such a variant of the theory would yield the same prediction as the one developed here. Namely, the economy needs an initial period of population growth *not* driven by technological change to cross the population size threshold and activate the engine of growth.

The mapping between what I do and their structure is thus as follows. Their C, L and x_i are my C, L and X_i. Their Q is my N. And since I call firm-specific investment in quality I, their N is my aggregate quality-R&D flow $\int_0^N I_i\, di$. The first key difference of substance, then, is that I do not want the extra state variable K so I make my intermediates non-durable (and obviously not capital intensive since I eliminate capital). Hence, their I is my aggregate flow of intermediate production $\int_0^N X_i\, di$. The second key difference is that I add land and interpret quality as Hicks-neutral augmentation of the composite $L^\gamma \Omega^{1-\gamma}$. Note that quality as Hicks-neutral augmentation of L is exactly what their structure yields. So my adaptation amounts to simply changing the exponent on quality and making the interpretation explicit. The third difference is that to support a symmetric equilibrium I introduce diminishing returns to own quality compensated by a spillover through average quality.

A.3. Derivation of the returns to quality and variety innovation

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the quality innovation, q_i, is equal to its cost:

$$1 = q_i \Leftrightarrow I_i > 0. \tag{34}$$

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$r = \frac{\partial \Pi_i}{\partial Z_i}\frac{1}{q_i} + \frac{\dot{q}_i}{q_i}. \tag{35}$$

To calculate the marginal profit, observe that the firm's problem is separable in the price and investment decisions. Facing the isoelastic demand (5) and a marginal cost of production equal to one, the firm sets $P_i = 1/\theta$. Substituting this result into (8), differentiating with respect to Z_i, substituting into (35) and imposing symmetry yield (10).

To obtain the return to variety innovation, observe first that the value of the new firm is given by (9) because the post-entry profit flow that accrues to the entrant in is given by (8). Entrepreneurs undertake R&D up to the point where the value of the variety innovation, V_i, is equal to its cost:

$$\beta X = V_i \Leftrightarrow \dot{N} > 0. \tag{36}$$

Taking logs and time derivatives of the value of the firm yields the rate of return to entry as

$$r = \frac{\Pi_i - I_i}{V_i} + \frac{\dot{V}_i}{V_i}. \tag{37}$$

Using the free-entry condition (36) and imposing symmetry yield (11).

A.4. Proof of Proposition 1

When $n > 0$ assets market equilibrium requires

$$A = NV = \beta\theta^2 Y, \tag{38}$$

which says that the wealth ratio A/Y is constant. This result and the saving schedule (3) allow me to rewrite the household budget (2) as the following unstable differential equation in C/Y:

$$0 = \rho - \lambda + \left(\frac{\dot{C}}{C} - \frac{\dot{Y}}{Y}\right) + \frac{1-\theta-(C/Y)}{\beta\theta^2},$$

which says that to satisfy the transversality condition C/Y jumps to the constant value $(\rho-\lambda)\beta\theta^2 + 1 - \theta$. Using the definition of π yields the bottom line of (17).

When $n=0$ assets market equilibrium still requires $A=NV$ but it is no longer true that $V=\beta X$ since by definition the free-entry condition does not hold. This means that the wealth ratio A/Y is not constant. However, (37) holds since it is the arbitrage condition on equity holding that characterizes the value of an existing firm regardless of how it came into existence in the first place. Imposing symmetry and inserting (8), (37) and (38) in the household budget (2) yields

$$0 = N\left[\left(\frac{1}{\theta} - 1\right)X - \phi Z - I\right] + (1-\theta)Y - C.$$

The definition of x, the R&D technology (7), and the fact that $NX = \theta^2 Y$ allow me to rewrite this expression as the top line of (17).□

A.5. Proof of Proposition 2

The key to the proof is that it looks for a Nash equilibrium where both R&D activities yield a rate of a return that matches the reservation rate of return on saving of the household. Consider (18) and (19). In (z, n) space these are two negatively sloped straight lines – the quality innovation line and the variety innovation line – with intersections with the axes that depend positively on x. Solving the top lines of (18) and (19) for n and z yields (20) and (21). This solution is a stable Nash equilibrium if the variety innovation line (18) is flatter than the quality innovation line (19), that is, if $\beta x > \sigma$, which is surely true under $\beta\phi/(1/\theta-1) > \sigma$ since this model requires $x \geq \phi/(1/\theta-1)$.□

A.6. Proof of Proposition 3

The definition of quality-adjusted firm size $x \equiv X/Z$ and the reduced-form production function (12) yield $\dot{x}/x = \gamma\lambda-(1-\sigma)n$. Setting $\dot{x}=0$ yields (23). Inserting (23) in (20) yields (22). Inserting (22) in (21) yields (24). The values x^* and z^* are positive if

$$1-\alpha > \frac{1}{\phi}\left(\rho-\lambda+\frac{\gamma\lambda}{1-\sigma}\right);$$

$$1-\alpha > \frac{\beta}{\frac{1}{\theta}-1}\left(\rho-\lambda+\frac{\gamma\lambda}{1-\sigma}\right);$$

$$\frac{\alpha\left(\phi\dfrac{\beta}{\dfrac{1}{\theta}-1}-1\right)}{1-\alpha-\dfrac{\beta}{\dfrac{1}{\theta}-1}\left(\rho-\lambda+\dfrac{\gamma\lambda}{1-\sigma}\right)} > 1.$$

Observing that the third inequality can hold only if $\beta\phi/(1/\theta-1) > 1$, which implies

$$1-\frac{1}{\phi}\left(\rho-\lambda+\frac{\gamma\lambda}{1-\sigma}\right) > 1-\frac{\beta}{\dfrac{1}{\theta}-1}\left(\rho-\lambda+\frac{\gamma\lambda}{1-\sigma}\right),$$

yields the two conditions in the text of the proposition. Finally, noticing that C/Y is constant, one can use the Euler equation (3) to write

$$y^* = r^*-\rho$$

$$y^* = \alpha\left(\left(\frac{1}{\theta}-1\right)x^*-\phi\right)-\rho > 0$$

iff (25) holds.□

A.7. Proof of Proposition 4

The proof is a generalization of that of Proposition 2: it looks for a Nash equilibrium where either at least one of the two R&D activities yields a rate of a return that matches the reservation rate of return on saving of the household or there is no TFP growth. As seen in Proposition 2, the Nash equilibrium features both vertical and horizontal R&D if they yield equal rates of return. Now consider (18) and (19) in (z,n) space and suppose that initially x is so small that both lines lie entirely below the origin. This configuration arises when both vertical and horizontal R&D yield a rate of return lower than what the household demands to postpone consumption and therefore the economy is in an equilibrium with zero TFP growth. As x grows two sequences of events are possible:

- The variety innovation line (18) reaches the origin and enters the positive quadrant before the quality innovation line (19). In this case, the economy crosses the quality-adjusted firm size threshold that activates horizontal innovation while agents anticipate zero vertical innovation. That is, agents anticipate $z=0$ and therefore (18) says that $n>0$ if

$$x > x_N \equiv \frac{\phi}{\dfrac{1}{\theta}-1-(\rho-\lambda)\beta}.$$

  As x keeps growing, the quality innovation line (19) enters the positive quadrant and, *if* it catches up and overtakes the variety innovation line (18), the Nash equilibrium with both vertical and horizontal R&D takes hold. Specifically, given that along this path agents anticipate $n>0$ at the switch point, (21) says that $z>0$ if

$$\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)\left(\alpha-\frac{\sigma}{\beta x}\right) > (1-\sigma)(\rho-\lambda)+\gamma\lambda.$$

  The left-hand side starts out at zero for $x=\phi/(1/\theta-1)$ and is monotonically increasing in x. The inequality thus identifies a unique value

$$x_Z = \text{arg solve}\left\{\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)\left(\alpha-\frac{\sigma}{\beta x}\right)=(1-\sigma)(\rho-\lambda)+\gamma\lambda\right\}>x_N.$$

- The quality innovation line (18) reaches the origin and enters the positive quadrant before the variety innovation line (20). In this case, the economy crosses the quality-adjusted firm size threshold that activates vertical innovation while agents anticipate zero variety innovation, that is, agents anticipate $n=0$ and $z>0$ if $\alpha((1/\theta-1)x-\phi)>\rho-\lambda+\gamma\lambda+(c-y)$, where the value of $(c-y)$ comes from log-differentiating (17) with respect to time. It is useful to think of $z=0$ if $\alpha((1/\theta-1)x-\phi)\le\rho-\lambda+\gamma\lambda+(c-y)$ so that I can compute $(c-y)$ from (17) under $z=0$. Recalling that I am working in the region where $\dot{x}/x=\gamma\lambda$, I then have that $z=0$ for

$$\alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)\le\rho-\lambda+\frac{\theta^2\frac{\phi}{x}}{1-\theta^2\left(1+\frac{\phi}{x}\right)}\gamma\lambda,$$

which yields the unique value

$$x_Z = \text{arg solve}\left\{\alpha\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)=\rho-\lambda+\frac{\theta^2\frac{\phi}{x}}{1-\theta^2\left(1+\frac{\phi}{x}\right)}\gamma\lambda\right\}.$$

As x keeps growing, the variety innovation line (18) enters the positive quadrant and eventually catches up and overtakes the variety innovation line (19), at which point the Nash equilibrium with both vertical and horizontal R&D takes hold. Specifically, given that along this path agents anticipate $z>0$ at the switch point, (20) yields that $n>0$ if

$$x>x_N\equiv\frac{(1-\alpha)\phi-\rho+\lambda-\gamma\lambda}{(1-\alpha)\left(\frac{1}{\theta}-1\right)-(\rho-\lambda)\beta}>x_Z.$$

To identify the condition on the fundamentals that yields which one of the two scenarios arises it is then sufficient to check for what values of the parameters (18) and (19) go through the origin for the same value of x. According to (19) $z=0$ for $n=0$ if

$$\alpha\left(\frac{1}{\theta}-1-\frac{\phi}{x}\right)x=\rho-\lambda+\frac{\theta^2\frac{\phi}{x}}{1-\theta^2\left(1+\frac{\phi}{x}\right)}\gamma\lambda$$

while according to (18) $n=0$ for $z=0$ if

$$\frac{1}{\theta}-1-\frac{\phi}{x}=(\rho-\lambda)\beta.$$

These two equations hold for the same x when

$$\frac{\alpha(\rho-\lambda)\beta\phi}{\frac{1}{\theta}-1-(\rho-\lambda)\beta}=\rho-\lambda+\frac{\theta^2\left[\frac{1}{\theta}-1-(\rho-\lambda)\beta\right]}{1-\theta^2\left(\frac{1}{\theta}-(\rho-\lambda)\beta\right)}\gamma\lambda,$$

which yields

$$\alpha=\overline{\alpha}\equiv\frac{\frac{1}{\theta}-1-(\rho-\lambda)\beta}{(\rho-\lambda)\beta\phi}\left[\rho-\lambda+\frac{\theta^2\left[\frac{1}{\theta}-1-(\rho-\lambda)\beta\right]}{1-\theta^2\left(\frac{1}{\theta}-(\rho-\lambda)\beta\right)}\gamma\lambda\right].$$

Now note that the quality-adjusted firm size threshold at which the variety innovation line (18) goes through the origin is independent of α while the quality-adjusted firm size threshold at which the quality innovation line (19) goes through the origin is decreasing in α. It then follows that for $\alpha<\overline{\alpha}$ the first scenario occurs, while the second occurs for $\alpha>\overline{\alpha}$. Substituting in the expression for $\overline{\alpha}$ derived above yields the inequalities in the text of the proposition.□

A.8. Proof of Proposition 5

The definition of quality-adjusted firm size $x \equiv X/Z$ and the reduced-form production function (12) yield the differential equation

$$\frac{\dot{x}}{x} = y - n - z = \gamma\lambda - (1-\sigma)n.$$

The behavior of entrants in (18) and (20) then yields the law of motion of quality-adjusted firm size that holds in each case.

- In the variety-first case, I have

$$\dot{x} = \begin{cases} \gamma\lambda x, & x \le x_N \\ \overline{\nu}(\overline{x}^* - x), & x_N < x \le x_Z \\ \nu(x^* - x), & x > x_Z \end{cases},$$

where

$$\overline{\nu} \equiv \frac{1-\sigma}{\beta}\left[\frac{1}{\theta} - 1 - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)\right];$$

$$\overline{x}^* \equiv \frac{\phi}{\frac{1}{\theta} - 1 - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)};$$

$$\nu \equiv \frac{1-\sigma}{\beta - \sigma/x}\left[(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \beta\left(\rho - \lambda + \frac{\gamma\lambda}{1-\sigma}\right)\right].$$

Without loss of generality, I approximate $\sigma/x \cong 0$ for $x > \max\{x_N, x_Z\}$ so that the coefficient ν becomes constant and therefore the law of motion of x is piecewise linear. Fig. 7 shows the phase diagram for x with the approximation. Integrating the first line between time 0 and time t yields

$$x(t) = x_0 e^{\gamma\lambda t}.$$

Since x grows exponentially, there exists a value T_N such that

$$x(T_N) = x_0 e^{\gamma\lambda T_N} = x_N,$$

which yields (27). Integrating the second line between time T_N and time t yields

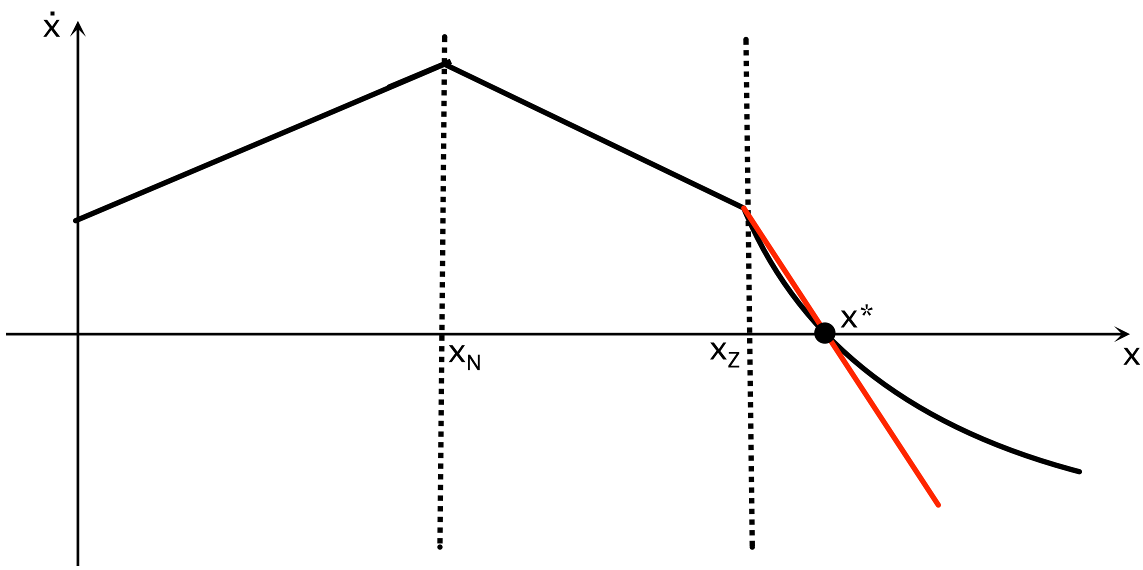

Fig. 7. The phase diagram for firm size with the approximation (in red) that yields a linear differential equation in the last phase. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

$$x(t) = x_N e^{\overline{\nu}(T_N - t)} + \overline{x}^* \left(1 - e^{\overline{\nu}(T_N - t)}\right).$$

For $\overline{x}^* > x_Z$ there exists a finite value T_Z such that

$$x(T_Z) = x_N e^{\overline{\nu}(T_N - T_Z)} + \overline{x}^* \left(1 - e^{\overline{\nu}(T_N - T_Z)}\right) = x_Z,$$

which yields (28). Thereafter the economy follows the third line and converges to the value x^* given in (22). Integrating between time T_Z and time t yields

$$x(t) = x_Z e^{\nu(T_Z - t)} + x^* \left(1 - e^{\nu(T_Z - t)}\right).$$

- In the quality-first case, instead, I have

$$\dot{x} = \begin{cases} \gamma\lambda x, & x \le x_Z \\ \gamma\lambda x, & x_Z < x \le x_N \\ \nu(x^* - x), & x > x_N. \end{cases}$$

Integrating the first line of between time 0 and time t yields $x(t) = x_0 e^{\gamma\lambda t}$. Accordingly, there exists a value T_Z such that

$$x(T_Z) = x_0 e^{\gamma\lambda T_Z} = x_Z,$$

which yields (30). After crossing this threshold quality-adjusted firm size keeps growing exponentially and therefore *must* cross the entry threshold x_N in finite time. Integrating between time T_Z and time t yields

$$x(t) = x_Z e^{\gamma\lambda(t - T_Z)}.$$

There thus exists a value T_N such that

$$x(T_N) = x_Z e^{\gamma\lambda(T_N - T_Z)} = x_N,$$

which yields (31). Thereafter the economy follows the third line of (29), which is identical to the third line of (26), and converges to x^*. Integrating between time T_N and time t yields

$$x(t) = x_N e^{\nu(T_N - t)} + x^* \left(1 - e^{\nu(T_N - t)}\right). \square$$

A.9. The policy functions for the quality-first case

Let $b \equiv C/Y$. Consider $b(x)$ first. Recall that for $x > x_N$ one has $b(x) = b^* \equiv (\rho - \lambda)\beta\theta^2 + 1 - \theta$. At issue, then, is only what happens in the region where entrants are not active. Consider first the case of variety innovation first. This is straightforward since firms turn on quality growth only after free entry already applies and thus $b(x)$ is given by (17) evaluated at $z = 0$ over the entire range $\phi/(1/\theta - 1) \le x \le x_N$. The case of quality innovation first is more interesting, since it requires taking into account the dynamic feedbacks through $z > 0$. As before, over the range $\phi/(1/\theta - 1) \le x \le x_Z$ the function $b(x)$ is given by (17) evaluated at $z = 0$. To characterize it over the range $x_Z < x \le x_N$, substitute $z = \alpha\left((1/\theta - 1)x - \phi\right) - \rho + \lambda - \gamma\lambda - (c - y)$ into (17) and rearrange terms to get

$$\frac{\dot{b}}{b} = \frac{\left(\frac{1}{\theta} - 1\right)x}{\theta}\left(\frac{b}{1-\theta} - 1\right) - \rho + \lambda - \gamma\lambda - (1-\alpha)\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right).$$

This yields the $\dot{b} \ge 0$ locus:

$$b \ge (1-\theta)\left[1 + \theta\frac{\rho + \lambda + \gamma\lambda + (1-\alpha)\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right)}{\left(\frac{1}{\theta} - 1\right)x}\right].$$

The dynamics then imply that the unique equilibrium trajectory is for the economy to jump on the saddle path in (x, b) space that converges to (x^*, b^*). Writing

$$\frac{\dot{b}}{\dot{x}} = \frac{db}{dx} = \frac{b\left[\frac{\left(\frac{1}{\theta} - 1\right)x}{\theta}\left(\frac{b}{1-\theta} - 1\right) - \rho + \lambda - \gamma\lambda - (1-\alpha)\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right)\right]}{\gamma\lambda\left(\frac{1}{\theta} - 1\right)x}$$

characterizes the saddle path more sharply. Although this partial differential equation does not have a closed-form solution, it is straightforward to show that the function $b(x)$ has the same derivative from the left and the right at $x=x_Z$ and approaches the value b^* with zero derivative at $x=x_N$:

$$\frac{db(x_Z)}{dx}=\frac{b(x_Z)\left[\frac{\left(\frac{1}{\theta}-1\right)x_Z}{\theta}\left(\frac{b(x_Z)}{1-\theta}-1\right)-\rho+\lambda-\gamma\lambda-(1-\alpha)\left(\left(\frac{1}{\theta}-1\right)x_Z-\phi\right)\right]}{\gamma\lambda\left(\frac{1}{\theta}-1\right)x_Z}$$
$$=\frac{\theta(1-\theta)\phi}{\left(\frac{1}{\theta}-1\right)^2 x_Z^2};$$

$$\frac{db(x_N)}{dx}=\frac{b(x_N)\left[\frac{\left(\frac{1}{\theta}-1\right)x_N}{\theta}\left(\frac{b(x_N)}{1-\theta}-1\right)-\rho+\lambda-\gamma\lambda-(1-\alpha)\left(\left(\frac{1}{\theta}-1\right)x_N-\phi\right)\right]}{\gamma\lambda\left(\frac{1}{\theta}-1\right)x_N}=0.$$

In other words, it is increasing, concave and has no kinks. Solving (17) for z yields

$$z(x)=\left(\frac{1}{\theta}-1\right)x-\frac{\left(\frac{1}{\theta}-1\right)x}{\theta}\left(\frac{b(x)}{1-\theta}-1\right)-\phi.$$

Once again, it is straightforward to show that $z(x)$ starts out at $x=x_Z$ with zero derivative and approaches the line $z(x)=\alpha((1/\theta-1)x-\phi)-\rho+\lambda-\gamma\lambda$, which holds for $x>x_N$, with positive derivative:

$$\frac{dz(x_Z)}{dx}=1-\frac{1}{\theta}\left(\frac{b(x_Z)}{1-\theta}-1\right)-\frac{\left(\frac{1}{\theta}-1\right)x_Z}{\theta}\frac{db(x_Z)/dx}{1-\theta}$$
$$=\frac{\phi}{\left(\frac{1}{\theta}-1\right)x_Z}-\frac{\left(\frac{1}{\theta}-1\right)x_Z}{\theta}\frac{1}{1-\theta}\frac{\theta(1-\theta)\phi}{\left(\frac{1}{\theta}-1\right)^2 x_Z^2}=0;$$

$$\frac{dz(x_N)}{dx}=1-\frac{1}{\theta}\left(\frac{b(x_N)}{1-\theta}-1\right)-\frac{\left(\frac{1}{\theta}-1\right)x}{\theta}\frac{db(x_N)/dx}{1-\theta}=1-(\rho-\lambda)\frac{\beta}{\frac{1}{\theta}-1}>\alpha.$$

The function $z(x)$ exhibits a kink at $x=x_N$ because when entry begins quality innovation attracts only a fraction of the economy's saving flow, which is now a constant fraction of final output.

References

Allen, R.C., 2009. The British Industrial Revolution in Global Perspective. Cambridge University Press, Cambridge, UK.
Aghion, P., Howitt, P., 1998. Endogenous Growth Theory. MIT University Press, Cambridge, MA.

Ang, James B., Madsen, Jakob B., 2011. Can Second-Generation Endogenous Growth Models Explain the Productivity Trends and Knowledge Production in the Asian Miracle Economies? The Review of Economics and Statistics 93 (4), 1360–1373. MIT Press.
Barro, R., Sala-i-Martin, X., 2004. In: Economic GrowthMIT Press, Cambridge MA.
Clark, G., 2015. The industrial revolution. In: Aghion, P., Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 2A. , Elsevier, Amsterdam, pp. 217–262.
Crafts, N., O'Rourke, K.H., 2015. Twentieth century growth. In: Aghion, P., Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 2A. , Elsevier, Amsterdam, pp. 263–346.
Desmet, K., Parente, S.L., 2012. The evolution of markets and the revolution of industry: a unified theory of growth. J. Econ. Growth 17, 205–234.
Easterlin, R.A., 1996. Growth Triumphant: The Twenty-first Century in Perspective. The University of Michigan Press, Ann Arbor.
Galor, O., 2005. From stagnation to growth: unified growth theory. In: Aghion, P., Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 1A. , Elsevier, Amsterdam, pp. 171–293.
Galor, O., 2011. Unified Growth Theory. Princeton University Press, Princeton, NJ.
Galor, O., Weil, D.N., 2000. Population, technology and growth: from the Malthusian regime to the demographic transition. Am. Econ. Rev. 110, 806–828.
Ha, J., Howitt, P., 2007. Accounting for trends in productivity and R&D: a Schumpeterian critique of semi-endogenous growth theory. J. Money Credit Bank. 33, 733–774.
Herrendorf, B., Rogerson, R., Valentinyi, A., 2015. Growth and structural transformation. In: Aghion, P., Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 2A. , Elsevier, Amsterdam, pp. 855–942.
Jones, C.I., 2005. Growth and ideas. In: Aghion, P., Durlauf, S.N. (Eds.), Handbook of Economic Growth, vol. 1A. , Elsevier, Amsterdam, pp. 1063–1111.
Jones, E.L., 1988. Growth Recurring: Economic Change in World History. University of Michigan Press, Ann Arbor.
Kelly, M., 1997. The dynamics of Smithian growth. Q. J. Econ. 112, 939–964.