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Robust endogenous growth☆

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ABSTRACT

This paper studies a generalization of the Schumpeterian models with endogenous market structure that allows the overall production structure to be more than linear in the growth-driving factor and yet generates endogenous growth, defined as steady-state, constant, exponential growth of income per capita. This version of modern growth theory, therefore, is robust in the sense that its key result obtains for a thick set of parameter values instead of, as often claimed, for a set of measure zero. The paper, moreover, pays close attention to transitional dynamics, showing not only the existence but also the global stability of the endogenous-growth steady state.

1. Introduction

Setting its birth at the publication of Romer (1986), modern endogenous growth economics is now in its 30s and has thus reached full maturity. By all measures of scholarly accomplishment it is a success. The field is vibrant and expanding, empirical and policy relevant. The Schumpeterian branch of the theory, in particular, has generated many insights that have been successfully applied to a wide range of topics.[1]

This paper studies a generalization of Schumpeterian models with endogenous market structure. This particular class of models features two dimensions of technology that play interdependent but distinct roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure. This framework provides a natural way to exploit insights from Industrial Organization to shed light on important aspects of innovation-driven growth. A large literature uses it to study applied issues ranging from the general role of imperfect competition in the growth process, to taxation (with special focus on corporate taxation), corporate governance, natural resource scarcity, the interaction between demography and technology and so on.[2] The generalization studied here yields some novel and deeper insights on

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[1] For reviews, Aghion et al. (2014); Aghion and Howitt (1998, 2005).

[2] To my knowledge, these models were originally developed simultaneously and independently by Peretto (1996) and Smulders (1994) in their PhD dissertations. Follow-up publications that develop the approach are Peretto (1996, 1998a, 1998b, 1999), Smulders and de (1995) and Peretto and Smulders (2002). Also relevant to the initial development of this literature are the contributions of Dinopoulos and Thompson (1998) and Howitt (1999). This class of models has received substantial empirical support, especially as an explanation of long-term historical data. For examples, see Ha and Howitt (2007), Laincz and Peretto (2006), Greasley et al. (2013), Madsen (2008), Madsen (2010), Madsen et al. (2010a), Madsen et al. (2010b), Madsen and Ang (2011), Madsen and Ang (2013), Madsen and Timol (2011), Ulku (2007), Zachariadis (2003). The interested reader can find a sample of applications and extensive references to this literature on my webpage: http://public.econ.duke.edu/~peretto/.

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the relation between the two dimensions of technology. For example, it shows that in steady state vertical innovation creates the opportunity for product proliferation. In previous versions of the theory steady-state product proliferation requires growth of some endowment (e.g., population growth).

One feature of the framework that the literature has singled out for special attention is that the endogenous market structure sterilizes the market size effect and thereby its two key manifestations: (1) the strong scale effect, with the associated need to impose constant endowments to rule out explosive behavior due to external forcings (like, e.g., population growth), and (2) the need to impose a knife-edge condition that rules out explosive behavior due to "excessive" increasing returns. The second property might strike one as a mere technicality. On reflection, however, it is one of the most powerful implications of the generalization pursued in this paper.

As is well known, endogenous growth theory has lived all its life under the shadow of the so called "linearity" critique. The critique consists of two claims. (1) To produce endogenous growth — defined as steady-state, constant, exponential growth of income per capita — the early models posit a production structure that is linear in the factor that the economy accumulates. (2) The theory, therefore, is fragile because if production is less than linear growth dies out and income per capita converges to a constant value, while if production is more than linear growth accelerates and income per capita explodes to infinity in finite time. The seeming requirement of *exact* linearity is why the critique is also known as the knife-edge or razor-edge critique.[3]

The generalization studied in this paper allows the overall production structure to be more than linear in the growth-driving factor, thus violating the knife-edge assumption that is allegedly necessary to the theory, and yet obtains constant exponential growth. In other words, Schumpeterian theory *does not* require linearity of the production structure to generate endogenous growth and, therefore, is robust in the sense that its key result obtains for a thick set of parameter values instead of a set of measure zero. Consequently, researchers can use it with the confidence that the empirical predictions and policy prescriptions that it produces are not fragile because at its core the theory has a robust mechanism that captures critical features of the real-world economy.

The paper works with a tractable version of the theory that allows one to see transparently the forces at play. Section 3 sets up the general framework and notation and Section 4 discusses the main result. Section 5 further explores the mechanism in a variant of the baseline model that yields richer dynamics. Derivations and proofs are in the appendix, which also contains a further extension that allows for a general difficulty of innovation index.

Section 6 dissects the model's mechanism to provide the precise mapping from core assumptions on the primitives to the key property of the state-space representation. Conventional wisdom holds that endogenous growth models postulate *as a primitive* at least one differential equation of the form $\dot{X} = \_X,$ where X is the model-specific variable tasked with growing exponentially in steady state and the blank is a model-specific constant. The section shows that, contrary to this wisdom, none of the model's primitives take such form. Rather, they all are non-linear differential equations that for the sake of this argument one can write in the form $\dot{X} = \_X,$ with the caveat that the blank is not a constant but an equilibrium object that is a function of, *among other things, X* itself. The key property of the model is that such equilibrium object becomes constant (i.e., time-invariant and independent of X) in steady state, *and only in steady state*, so that the differential equation delivers the mathematical description of a constant exponential growth process. But such property is not imposed on any of the model's primitives — in fact, it is not *assumed* anywhere in the paper. It *emerges* as the system converges to the steady state. It is worth stressing this point: the linearity of the core accumulation equation is a steady-state property of the system representing economic decisions. It is not a property of the model's primitives. What makes its emergence possible is that the model interprets X as product quality and fills in the blank with an equilibrium object, i.e., a firm-level investment rate, that depends on X and a very specific other thing that in steady state offsets the growth of X. This very specific other thing is the mass of firms/products, which the model interprets as endogenous market structure.

Of course this is not the first paper discussing core issues raised in the debate on the robustness of endogenous growth theory. Notable contributions are Daalgard and Kreiner (2003), Eicher and Turnovsy (1999) and Cozzi (2017a,b). What distinguishes this paper's attempt is the focus on Schumpeterian theory and the micro-foundations of decentralized market equilibria.

Before jumping into the technical part of the exposition, it is useful to preview the intuition underlying the paper's central result. The next section reduces the paper's argument to its essentials and isolates the key mechanism driving these models. The exercise should help one appreciate the general idea underlying the approach and see why it delivers the results it does.

2. The insight

The model is of the lab-equipment, or one-sector, class where the final good Y can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. This is a desirable feature that allows one to discuss the model's production structure in terms of only one technology, namely, that used to produce the final good. In symmetric equilibrium, that technology has the following

[3] To my knowledge, the clearest articulation of the critique is Jones (2005).

reduced-form representation

$$Y = (\text{constant}) \cdot\ N^{\sigma} Z^{\kappa}\ \cdot\ L,$$

where Y is output, L is labor, and N and Z are, respectively, the variety and the average quality of intermediate goods.

The theory assigns different roles to product variety and product quality. The former provides the proliferation mechanism that sterilizes the market size effect, and thereby the (strong) scale effect, the latter is the economy's engine of growth. The linearity critique says that one must impose $\kappa = 1$ to ensure exact linearity of Y in Z.[4] The paper shows that, instead, this economy converges to a steady state that exhibits constant, exponential growth of income per capita for

$$\kappa \in [1, \kappa^{\max}), \quad \text{with } \kappa^{\max} > 1.$$

The paper's main result therefore is that constant endogenous growth does not require exact linearity of the production structure in the growth-driving factor.

To see why, note that the elasticity of output with respect to labor, 1, is the lower bound below which social returns to quality κ cannot fall without making steady-state constant endogenous growth impossible. Given that one must rule out values $\kappa < 1$, then, the literature's practice has been to impose $\kappa = 1$. It turns out, however, that $\kappa > 1$ is possible because increasing returns to quality in excess of the lower bound 1 are absorbed by the endogenous mass of products.

The second step in the argument is thus to understand the role of product variety. The model's key feature is that there are two activities pushing the technology frontier: (1) creation of new products and the firms that produce and sell such products (horizontal innovation); (2) quality improvement through in-house R&D carried out by firms once in existence (vertical innovation). Accordingly, N is the mass of products/firms competing in the marketplace, a broader concept than just product variety. The model's conditions characterizing agents' behavior say that these returns are functions of the quality-adjusted gross cash flow of the firm:

$$x_i \equiv \frac{X_i(P_i - 1)}{Z_i} = \frac{\text{gross cash flow}}{\text{quality}},$$

where P_i is the price the firm charges, X_i is the quantity it sells, 1 is the marginal cost of production in units of final output, and Z_i is the quality level achieved by the firm. In symmetric equilibrium, this expression becomes

$$x = (\text{constant}) \cdot \underbrace{\frac{Y}{Z}}_{\text{quality-adjusted market size}} \cdot \underbrace{\frac{1}{N}}_{\text{market share}} = (\text{constant}) \cdot \frac{Z^{\kappa-1} L}{N^{1-\sigma}}.$$

If we think of the model's steady state as a situation with unchanging incentives, in this setup this means looking for solutions with constant rate of return to innovation and, therefore, for solutions with constant quality-adjusted gross cash flow.

The first equality in the expression above says that quality-adjusted gross cash flow, x, is constant if the mass of firms, N, grows at the same rate as the quality-adjusted size of the market, Y/Z. That is, *if entry sterilizes the quality-adjusted market size effect*. This *market share effect* is the core of the endogenous market structure mechanism that holds independently of the model's general equilibrium structure because it is an industry-level mechanism.

The second equality shows how the mechanism plays out in general equilibrium, that is, once we consider the aggregate determinants of quality-adjusted market size. In this setup, these determinants are the supply of labor, L, and the aggregation process that yields the social returns to variety, N, and average quality, Z, which differ from the private returns. The restriction $\sigma < 1$ ensures that the market share effect in the intermediate goods market — N at the denominator of the ratio Y/NZ — dominates over the love-of-variety effect in final production, and thereby that the mass of firms/products ends up at the denominator of the expression for the the quality-adjusted gross cash flow of the firm. This expression then captures the model's main property: firm-level decisions depend on the quality-adjusted gross cash flow, which is proportional to the firm's sales and thus increasing in anything that shifts the firm's demand curve to the right. The offsetting force to such rightward shifts is entry, which, via the dominant market share effect, shifts the firm's demand curve to the left.

The expression for the quality-adjusted gross cash flow, then, shows that a steady state with constant x is feasible for $\kappa > 1$ because the market fragmentation mechanism "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models.

To see this, consider the traditional view of the world that ignores endogenous market structure. That is, assume N exogenous and constant, e.g., set $N = 1$. Then, to rule out explosive behavior Y must be exactly linear in Z so to make x independent of Z, i.e., one needs to impose the knife-edge condition $\kappa = 1$. This is not enough, however. As is well known, to rule out explosive behavior one also needs to assume L constant because of the strong scale effect.

[4] One way to see this, popular in the literature (see e.g., Jones, 2005), is to postulate that average quality grows according to $\dot{Z} = I$ and to imagine that aggregate investment in quality growth, NI, is a constant fraction s_I of output, so that investment per firm is a $I = s_I Y/N$. The resulting relation $\dot{Z} = s_I Y/N$ is then used to argue that one must assume $\kappa = 1$ to obtain non-explosive growth. It should be clear that such an argument relies crucially on holding N constant.

Such reasoning does not apply to models with two dimensions of technology that play interdependent, *but distinct*, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that sterilizes the market size effect and thereby its two key manifestations: (1) the strong scale effect, with the associated need to impose constant endowments to rule out explosive behavior due to external forcings (like, e.g., population growth), and (2) the need to impose a knife-edge condition that rules out explosive behavior due to internal "excessive" increasing returns.

3. The model: general setup

The economy is closed. To keep things simple, there is no physical capital. All variables are functions of (continuous) time but to simplify the notation the time argument is omitted unless necessary to avoid confusion. The reader familiar with this literature will recognize in the following a one-sector version of the model developed in Peretto (1998a).

3.1. Households

A representative household consisting of a continuum of identical members supplies labor and trades assets in competitive markets. The mass of household members, i.e., population size, is $L(t) = L_0 e^{\lambda t},\ L_0 \equiv 1$, where $\lambda \gtreqless 0$ so that the model allows for zero or even negative population growth. Each household member is endowed with one unit of time and there is no labor-leisure choice. Consequently, $L(t)$ is the household's total endowment of labor, which equals its supply of labor. The household has preferences

$$U(0) = \int_0^\infty e^{-(\rho-\lambda)t} \frac{(C(t)/L(t))^\eta - 1}{\eta - 1} dt, \quad \rho > \max\{0, \lambda\},\ \eta > 0 \tag{1}$$

where 0 is the point in time when the household makes decisions, ρ is the individual discount rate, η is the inverse of the intertemporal elasticity of substitution and $C(t)$ is aggregate consumption. The household's flow budget constraint is

$$\dot{A} = rA + wL - C, \tag{2}$$

where A is assets holding and r is the rate of return on assets. The intertemporal consumption plan that maximizes (1) subject to (2) then consists of the Euler equation

$$r = \rho + \eta\left(\frac{\dot{C}}{C} - \lambda\right), \tag{3}$$

the budget constraint (2) and the usual boundary conditions.

3.2. Final producers

A competitive representative firm produces a final good Y that can be consumed, used to produce intermediate goods, invested in the improvement of the quality of existing intermediate goods, or invested in the creation of new intermediate goods. The final good is the numeraire so its price is $P_Y \equiv 1$. The production technology is

$$Y = \int_0^N X_i^\theta \left(Z_i^\alpha Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}\right)^{1-\theta} di, \quad 0 < \theta, \alpha < 1 \tag{4}$$

where N is the mass of non-durable intermediate goods, X_i is the quantity of good i and L is labor. Given the inelastic labor supply of the household and the one-sector structure of the model, labor market clearing yields that employment in the final sector is equal to population size. Quality is the good's ability to raise the productivity of labor: the contribution of good i depends on its own quality, Z_i, and on average quality $Z = \int_0^N \left(Z_j/N\right) dj$. The parameters σ and κ measure social returns to variety and quality (see below for the formal argument). There are no restrictions on σ and κ for now, stressing that the model requires no a priori linearity assumption with respect to N or Z. The first-order conditions for the profit maximization problem of the final producer yield that each intermediate producer faces the demand curve

$$X_i = \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} Z_i^\alpha Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}, \tag{5}$$

where P_i is the price of good i. Let w denote the wage. The first-order conditions then yield that the final producer pays total compensation

$$\int_0^N P_i X_i di = \theta Y \quad \text{and} \quad wL = (1-\theta)Y \tag{6}$$

to intermediate goods and labor suppliers, respectively.

3.3. Intermediate producers

The typical intermediate firm operates a technology that requires one unit of final output per unit of intermediate good and a fixed operating cost ϕZ_i, also in units of final output.[5] Using the demand schedule (5) yields the firm's gross profit (i.e., profit before R&D) as

$$\Pi_i = (P_i - 1)\left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} Z_i^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}} - \phi Z_i. \tag{7}$$

Because $\alpha < 1$ the firm's gross profit is concave in Z_i. The firm can increase quality according to the technology

$$\dot{Z}_i = I_i, \tag{8}$$

where I_i is firm-level quality-improving R&D investment in units of final output. The firm chooses the time path of its product's price, $P_i(t)$, and its R&D, $I_i(t)$, to maximize

$$V_i(0) = \int_0^{\infty} e^{-\int_0^t r(s)ds}[\Pi_i(t) - I_i(t)]dt \tag{9}$$

subject to (8) and (7), where r is the interest rate and 0 is the arbitrary point in time when the firm makes decisions. The firm takes average quality, Z, in (7) as given. The characterization of the firm's decision yields a symmetric equilibrium where

$$r = \alpha \frac{(P-1)X}{Z} - \phi \tag{10}$$

is the return to quality innovation (see the Appendix for the derivation).

3.4. Entry

To start up activity an entrant must sink βZ units of final output. It is useful to write this assumption in terms of a production function for new products/firms, i.e.,

$$\dot{N} = \left(\frac{1}{\beta Z}\right) \cdot E,$$

where E is aggregate variety-expanding investment in units of the final good. Alternatively, one can refer to E as aggregate investment in entrepreneurship. Because of the sunk setup cost, the new firm cannot supply an existing good in Bertrand competition with the incumbent monopolist but introduces a new good that expands product variety. New firms finance entry by issuing equity and enter at the average quality level (this simplifying assumption preserves symmetry of equilibrium at all times). Entry is positive if the value of the firm is equal to its setup cost, i.e., if the free-entry condition $V_i = \beta Z$ holds. Taking logs and time derivatives of the free-entry condition and of the evaluation (9), imposing symmetry, and using the expression for the gross profit (7) yields the return to variety innovation

$$r = \left[\frac{(P-1)X}{Z} - \phi - \frac{I}{Z}\right]\frac{1}{\beta} + \frac{\dot{Z}}{Z}. \tag{11}$$

4. Robust endogenous growth

This section discusses the paper's main result. It studies the economy's allocation of final output Y to consumption, production of intermediates, and, when profitable, vertical and horizontal innovation. It then derives the reduced-form representation of the resulting equilibrium dynamics. Finally, it shows that for $\kappa \in [1, \kappa^{\max})$, with $\kappa^{\max} > 1$, the process converges to a steady state that exhibits endogenous constant exponential growth.

4.1. Structure of the equilibrium

Intermediate producers set $P = 1/\theta$ and in symmetric equilibrium receive $N \cdot PX = \theta Y$ from the final producer. Consequently, $NX = \theta^2 Y$. Imposing symmetry in the production function (4) and using this result to eliminate X yields:

$$Y = \theta^{\frac{2\theta}{1-\theta}} \cdot N^{\sigma} Z^{\kappa} \cdot L. \tag{12}$$

This reduced-form production function is not linear in N, not linear in Z and not linear in the combination of the two, $N^{\sigma} Z^{\kappa}$, in the precise sense that $\sigma + \kappa$ is not restricted to be 1.

[5] Modeling the fixed cost as $\phi Z_i^{\varphi} Z^{1-\varphi}$, $0 \le \varphi \le 1$, complicates the analysis without changing the results. Setting $\phi = 0$ reduces this model to the exact one-sector version of Peretto (1998b).

The definition of gross profit (7) and Eqs. (10)–(11) show that the returns to vertical and horizontal innovation depend on the gross cash flow of the firm $X(P-1)$ — i.e., revenues minus variable production costs — since this is the appropriate measure of profitability for firms that spread fixed costs, including the cost of developing innovations, over their volume of sales. On closer inspection, moreover, one can see that both returns are functions of the quality-adjusted gross cash flow of the firm, $X(P-1)/Z$. It is thus useful to define

$$x \equiv \frac{X(P-1)}{Z} = \frac{\text{gross cash flow}}{\text{quality}}$$

and use it as the model's stationary state variable.

Recall now that $P = 1/\theta$ and $NX = \theta^2 Y$. Eq. (12) then yields

$$x = (1-\theta)\frac{\theta Y}{NZ} = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}}\frac{Z^{\kappa-1}L}{N^{1-\sigma}}. \tag{13}$$

The restriction $\sigma < 1$ ensures that the *market share effect* in the intermediate goods market — N at the denominator of the ratio $\theta Y/NZ$ — dominates over the love-of-variety effect in final production so that the mass of firms ends up at the denominator of the expression for the the quality-adjusted gross cash flow of the firm.[6] Substituting (13) in (10) and (11) the returns to innovation become:

$$r = \alpha x - \phi; \tag{14}$$

$$r = \left(x - \phi - \frac{I}{Z}\right)\frac{1}{\beta} + \frac{\dot{Z}}{Z}. \tag{15}$$

These expressions capture the model's main property: firm-level decisions depend on the quality-adjusted gross cash flow, which is proportional to the firm's sales and thus increasing in labor use in the downstream final sector since production of final goods drives the demand for intermediate goods. It should be clear, thus, that from the viewpoint of the managers of incumbent firms and of the entrepreneurs that set up new firms the critical market size variable is quality-adjusted total expenditure on intermediates, $\theta Y/Z$. This structure yields a transparent characterization of how the state of the economy drives incentives for quality and variety innovation and thus it allows one to study the conditions under which a steady state with positive, non-explosive growth exists and whether such steady state is stable.

To see this more precisely, let

$$z \equiv \frac{\dot{Z}}{Z} \quad \text{and} \quad n \equiv \frac{\dot{N}}{N}$$

be, respectively, the rates of vertical and horizontal innovation. Also let

$$c \equiv \frac{C}{Y}$$

be the consumption-output ratio. In equilibrium, the two rates of innovation and the associated rate of return can be written as three functions, $z(x)$, $n(x, c)$ and $r(x)$, that have properties, characterized in detail below, dictated by the no-arbitrage condition that variety and quality innovation deliver the same rate of return and by the additional general-equilibrium conditions of the economy. In particular, the quality-adjusted gross cash flow, x, obeys the differential equation

$$\frac{\dot{x}}{x} = \underbrace{\lambda + (\kappa-1)z(x)}_{\text{market growth}} - \underbrace{(1-\sigma)n(x,c)}_{\text{market fragmentation}}. \tag{16}$$

To appreciate the mechanism at work here, it is useful to stress that thus far we have been dealing with a variable per firm, not per capita. The difference in denominators is crucial. At any point in time, the growth rate of output per capita is:

$$\frac{\dot{Y}}{Y} - \lambda = \kappa z(x) + \sigma n(x,c). \tag{17}$$

Comparing (16) to (17) one sees that while the quality-adjusted gross cash flow growth falls with the rate of entry, n, output per capita growth rises with the rate of entry. This difference is due to the $\sigma < 1$ restriction that makes the market share effect dominate the love-of-variety effect in the determination of the firm's quality-adjusted gross cash flow. In other words, product variety expansion is good for output growth, and thus for output per capita growth, whereas, on net, it is bad for

[6] For $\sigma \geq 1$ entry is self-sustaining and the model degenerates to Romer-style growth. How to interpret this outcome? The general model with both vertical and horizontal innovation nests variety-driven endogenous growth as a special case. The key is that for $\sigma > 1$ variety is so productive that each new good expands the market by more than it is needed to support itself profitably and thereby fuels further entry. This property is captured by the fact that the mass of firms/products, N, ends up at the numerator of the firm's profitability, x, signaling that each firm/product is actually making all the others *more* profitable. In this case, the model reverts back to the standard two-sector interpretation of the theory that assigns no special role to endogenous market structure. In fact, under $\sigma \geq 1$ the incentives to entry become so strong that vertical innovation does not take place — this is the sense in which the model degenerates to the standard variety-expansion story.

the growth of the firm's quality-adjusted gross cash flow. Quality growth, in contrast, is good for both output per capita growth and for per firm profit growth.

The general equilibrium system must also account for the dynamics of the consumption-output ratio, c, so it must satisfy the equation

$$\frac{\dot{c}}{c} = \frac{r(x) - \rho}{\eta} - \kappa z(x) - \sigma n(x, c). \tag{18}$$

The interpretation of this expression is the usual one: equilibrium of the financial market obtains when the rate of return to investment generated by firms is equal to the rate of return on saving demanded by the household.

4.2. The problem: growth on a knife edge or on a wide highway?

Before proceeding further it is useful to state more precisely the paper's research question. If we think of the model's steady state as a situation with unchanging incentives, in this setup this means looking for solutions with constant rate of return to innovation and, therefore, for solutions with constant quality-adjusted gross cash flow, i.e., constant x; see equation (14). The economics of the model then says that the quality-adjusted gross cash flow is constant if the mass of firms, N, grows at the same rate as the quality-adjusted size of the market for intermediate goods, $\theta Y/Z$. That is, *if entry sterilizes the quality-adjusted market size effect*. It follows that the mass of firms absorbs any upward pressure on market size that makes it grow "in excess" of average quality, Z.

It is useful to recall, before studying the mechanism in detail, that thus far the analysis has imposed only one restriction on κ and σ, namely, $\sigma < 1$ which says that the market share effect dominates over the love-of-variety effect so that in equilibrium the firm's the quality-adjusted gross cash flow x_i is decreasing in the mass of firms/products, N. This restriction plus the existence of fixed operating costs implies that *by construction* product variety is not the engine of growth of this economy. The discussion, therefore, concerns the social returns to quality. With this framework in mind, one can ask the following two questions about the degree of social returns to quality, κ.

- What is the lower bound on κ that ensures non-decreasing growth?
- What is the upper bound on κ that ensures non-explosive growth?

The first issue is well understood; the second is at the heart of the linearity critique.

The lower bound on κ follows from the straightforward observation that *the return to the accumulation of the growth-driving factor must be non-decreasing in the factor*. In the context of this paper this means that $r = \alpha x - \phi$ must be non-decreasing in Z, that is x must be non-decreasing in Z. The upper bound on κ follows from similar reasoning: *the return to the accumulation of the growth-driving factor must be non-increasing in the factor*. This means that $r = \alpha x - \phi$ must be non-increasing in Z, that is, x must be non-increasing in Z.

The key to the paper's argument, therefore, is Eq. (13), which describes how the quality-adjusted gross cash flow responds to variety, average quality and the other determinants of market size, namely, employment, L. Looking at the equation, two things stand out. First, x is non-decreasing in Z for $\kappa \geq 1$. Hence, $\kappa = 1$ is the lower bound on κ that gives the non-decreasing property because it makes x *independent* of Z. Imposing $\kappa = 1$ to ensure that constant growth is feasible in steady state, the literature has typically stopped at showing that in general equilibrium the endogeneity of the mass of intermediate goods sterilizes the effect of employment, so that x is constant in steady state. On reflection, however, it turns out that $\kappa = 1$ is *not necessary* since the same market fragmentation mechanism that stabilizes x under $\kappa = 1$ can stabilize it under $\kappa > 1$. If the mass of firms can absorb changes in employment, L, then it can also absorb changes in $Z^{\kappa-1}$.

4.3. The main result

A key step in the characterization of the model's equilibrium is the construction of the two functions mentioned above, $z(x)$ and $n(x, c)$, that describe the rates of vertical and horizontal innovation. The properties of these functions are dictated by the no-arbitrage condition that variety and quality innovation deliver the same rate of return. Specifically, the functions are built by comparing the rates of return to vertical and horizontal innovation, and checking whether they can be equal or one of the two activities is return-dominated and shuts down. Moreover, the two rates of return must be compared to the reservation rate of return demanded by households. The resulting no-arbitrage conditions identify regions of the state space $x \in (\phi, \infty)$ where investment in variety innovation and/or investment in quality innovation are zero. Depending on the location and shape of these regions, there exist two classes of transition paths: one where the economy activates variety innovation first and one where it activates quality innovation first. Either way, the economy converges to the region of state space where both variety and quality innovation take place. In that region it is possible to have the steady state with constant, exponential growth.

4.3.1. Existence

The no-arbitrage condition between quality and variety innovation, i.e., equalization of the returns in Eqs. (14)–(15), yields

$$z(x) = \begin{cases} 0 & \phi \le x \le x_Z \equiv \dfrac{\beta - 1}{\beta\alpha - 1}\phi \\ \dfrac{\beta\alpha - 1}{\beta - 1}x - \phi & x > x_Z \end{cases}. \tag{NA}$$

Given its construction, we refer to this equation as the *no arbitrage locus*. It's worth noting that in this specification of the model the investment rate of incumbents, $I/Z = z$, is determined solely by the equalization of returns to quality and variety innovation and depends on the rest of the general-equilibrium conditions of the model only indirectly, that is, through quality-adjusted firm size, x. The next section discusses a specification that relaxes this property and allows $I/Z = z$ to depend directly on the whole general equilibrium structure.

The rate of return to innovation arising from the no-arbitrage condition is

$$r(x) = \begin{cases} \dfrac{1}{\beta}(x - \phi) & \phi \le x \le x_Z \\ \alpha x - \phi & x > x_Z \end{cases}. \tag{19}$$

The remainder of the model's general equilibrium conditions yield

$$n(x, c) = \begin{cases} \dfrac{1}{\beta}\left[\dfrac{x\left(1 - \theta^2 - c\right)}{\theta(1 - \theta)} - \phi\right] & \phi \le x \le x_Z \\ \dfrac{1}{\beta}\left[\dfrac{x\left(1 - \theta^2 - c\right)}{\theta(1 - \theta)} - \phi - z(x)\right] & x > x_Z \end{cases}. \tag{20}$$

The steady-state system is:

$$\dot{x} = 0: \quad n(x, c) = \frac{\lambda}{1 - \sigma} + \frac{\kappa - 1}{1 - \sigma}z(x); \tag{21}$$

$$\dot{c} = 0: \quad n(x, c) = \frac{r(x) - \rho}{\sigma\eta} - \frac{\kappa}{\sigma}z(x). \tag{22}$$

Combining these two equations we obtain

$$z(x) = \begin{cases} 0 & \phi \le x \le \dfrac{1}{\alpha}\left(\phi + \rho + \eta\dfrac{\sigma\lambda}{1 - \sigma}\right) \\ \dfrac{1 - \sigma}{\kappa - \sigma}\left(\dfrac{\alpha x - \phi - \rho}{\eta} - \dfrac{\sigma\lambda}{1 - \sigma}\right) & x > \dfrac{1}{\alpha}\left(\phi + \rho + \eta\dfrac{\sigma\lambda}{1 - \sigma}\right) \end{cases}. \tag{GE}$$

Given its construction we refer to this equation as the *general equilibrium locus*.

The steady state is the intersection in (x, z) space of (NA)-(GE). By construction, the economy is at all times on the (NA) locus while the (GE) locus has the property that to its left x grows and to its right it shrinks. The formal derivation of this property is in the next subsection but already at this stage of the analysis the intuition is clear. Points to the right of the locus are combinations of x and z where quality-adjusted firm size is too large, given what incumbents spend on quality innovation, and there is room left for entrants to compete away the associated profits; accordingly, the rate of entry is high and causes quality-adjusted firm size to fall. Points to the left of the locus, in contrast, are combinations of x and z where quality-adjusted firm size is too small, given what incumbents spend on quality innovation, so that the rate of entry is low causing quality-adjusted firm size to rise. With the aid of Fig. 1 we can then establish the following result.

Proposition 1. *The system (NA)-(GE) has a unique solution* (x^*, z^*), *with* $x^* > x_Z$ *and* $z^* > 0$, *that is stable under the model's dynamics if and only if line (GE) intersects line (NA) from below, that is, if and only if the following slope and intercept conditions hold:*

$$\textit{slope} \qquad \frac{1 - \sigma}{\kappa - \sigma}\frac{\alpha}{\eta} > \frac{\beta\alpha - 1}{\beta - 1}; \tag{23}$$

$$\textit{intercept} \qquad \rho + \frac{\lambda\eta\sigma}{1 - \sigma} > \phi\frac{1 - \alpha}{\beta\alpha - 1}. \tag{24}$$

The slope condition (23) can be rearranged as

$$\kappa < \kappa^{\max} \equiv 1 + (1 - \sigma)\left(\frac{1}{\eta}\frac{\beta - 1}{\beta - 1/\alpha} - 1\right). \tag{25}$$

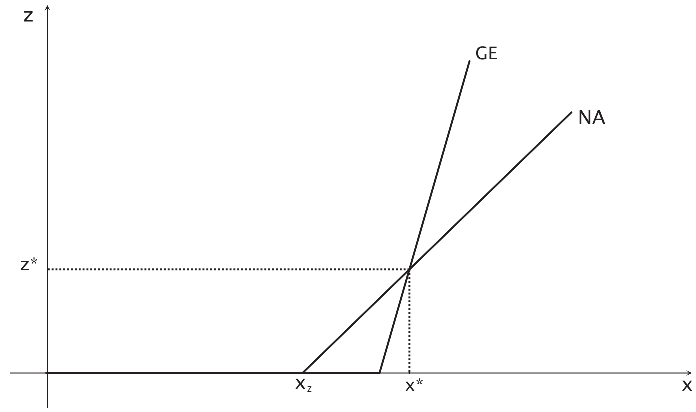

Fig. 1. Existence of the steady state with constant exponential growth when entry cost is βZ.

The intuition for this result follows from its construction. It is useful to proceed in two steps. First consider Eqs. (21) and (22) and think of the steady state as the intersection in (x, n) space of two straight lines. According to (NA) $z(\phi) = 0$. Thus, line (22) cuts the horizontal axis at a positive value of x while line (21) has a positive intercept. Therefore, an intersection exists if and only if the former is steeper than the latter, i.e., if and only if:

$$\frac{r'(x)}{\sigma\eta} - \frac{\kappa}{\sigma} z'(x) > \frac{\kappa - 1}{1 - \sigma} z'(x). \tag{26}$$

Eqs. (NA) and (19) say that this condition is surely satisfied for $x \le x_Z$ since $z'(x) = 0$ and $r'(x) = 1/\beta$. For $x > x_Z$, instead, $z'(x) = \frac{\beta\alpha - 1}{\beta - 1}$ and $r'(x) = \alpha$; manipulating terms then yields the slope condition (23). The next step is to check that x^* does in fact yield $z^* > 0$. Substituting x^* in (NA) yields this requirement as the intercept condition (24).

This procedure highlights that the model: (i) always has a steady state; (ii) nests as a corner solution, occurring when either one of conditions (23) and (24) fails, the semi-endogenous growth case where $\bar{z} = 0$ and $\bar{n} > 0$ if and only if population growth is positive; (iii) allows for a steady state with endogenous exponential growth, occurring when both conditions (23) and (24) hold, that is the solution of a linear equation in quality-adjusted firm size, x. Accordingly, we can summarize the steady state as follows:

$$(\bar{x}, \bar{n}, \bar{z}) = \left(\frac{\phi + \rho}{\alpha}, \frac{\lambda}{1 - \sigma}, 0 \right) \quad \text{if either of conditions (23)-(24) fails;}$$

$$(x^*, n^*, z^*) = \left(x^*, \frac{\lambda + (\kappa - 1) z^*}{1 - \sigma}, z^* \right) \quad \text{if both conditions (23)-(24) hold,}$$

where, solving jointly (NA)-(GE):

$$x^* = \frac{\left(\frac{1}{\eta} - \frac{\kappa - \sigma}{1 - \sigma}\right)\phi + \frac{\sigma\lambda}{1 - \sigma} + \frac{\rho}{\eta}}{\frac{\alpha}{\eta} - \frac{\beta\alpha - 1}{\beta - 1}\frac{\kappa - \sigma}{1 - \sigma}}; \quad z^* = \frac{\frac{\sigma\lambda}{1 - \sigma} + \frac{\rho}{\eta} - \frac{1 - \alpha}{\beta\alpha - 1}\frac{\phi}{\eta}}{\frac{\beta - 1}{\beta\alpha - 1}\frac{\alpha}{\eta} - \frac{\kappa - \sigma}{1 - \sigma}}.$$

Eq. (17) yields the associated growth rate of output per capita:

$$\left(\frac{\dot{Y}}{Y}\right)^* - \lambda = \kappa z^* + \sigma n^*$$

$$= \frac{\kappa - \sigma}{1 - \sigma} \frac{\frac{\sigma\lambda}{1 - \sigma} + \frac{\rho}{\eta} - \frac{1 - \alpha}{\beta\alpha - 1}\frac{\phi}{\eta}}{\frac{\beta - 1}{\beta\alpha - 1}\frac{\alpha}{\eta} - \frac{\kappa - \sigma}{1 - \sigma}} + \frac{\sigma\lambda}{1 - \sigma}.$$

This solution makes clear that in steady state the model's engine of growth is product quality. Product variety plays a supporting role, crucial for the stability of the system, soaking up the growth of the size of the market to stabilize quality-adjusted profitability and thereby the economy's interest rate.

4.3.2. Dynamics and stability

We now turn to the dynamics of the model. The results are best understood qualitatively; see the appendix for the algebraic details. First, recall that we are characterizing the dynamical system (16)–(18), where the key ingredients are the two functions $z(x)$ and $n(x, c)$ discussed above. The analysis is standard except that it must pay attention to the potential corner solutions due the non-negativity constraints that apply to investment in entry and to investment in quality. These corner solutions identify regions of state space where z and/or n are zero.

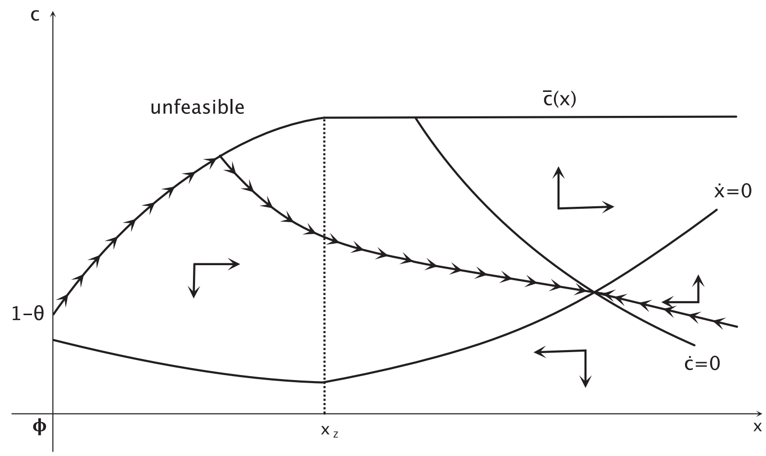

Fig. 2. Example of transitional dynamics.

Fig. 2 provides an example of the type of transitional dynamics that the model produces. The pattern that emerges consists of three cases identified by the progression upward of the $\dot{c}=0$ locus relative to the other loci.

1. The $\dot{c}=0$ locus intersects the $\dot{x}=0$ locus for $x<x_Z$. Two paths are possible: (i) the saddle path going to the steady state $(\bar{x},\bar{c})$ starts at $x=\phi$ below the point $c=1-\theta$; (ii) the saddle path going to $(\bar{x},\bar{c})$ intersects the boundary of the unfeasible region at some value $\phi<x<x_Z$. Path (i) features entry all the time; path (ii) features no entry initially, until the economy turns it on when market size is sufficiently large.
2. The $\dot{c}=0$ locus is higher and intersects the $\dot{x}=0$ locus for $x>x_Z$. The same two paths as above are possible with the difference that in both cases the economy eventually enters the region $x>x_Z$, activates quality growth and converges to the steady state (x^*, c^*).
3. The $\dot{c}=0$ locus is so high that it intersects the boundary of the unfeasible region for $x>x_Z$. Two paths are possible: (i) the saddle path going to (x^*, c^*) intersects the boundary of the unfeasible region for some value $x<x_Z$; (ii) the saddle path going to (x^*, c^*) intersects the boundary of the unfeasible region for some value $x>x_Z$. Path (i) has the same features as those above; path (ii) features no entry initially, a phase of no entry and quality growth, until the economy turns on entry and converges to (x^*, c^*).

Because these dynamics are rather cumbersome, it is best to focus this discussion on the intuition behind the core existence result. The next section works with a specification of the model that yields more transparent dynamics and thus sheds more light on the operating of the key mechanism out of steady state.

4.4. Intuition

The intuition behind the result that non-explosive endogenous growth does not require exact linearity of the production structure is as follows. The elasticity of output with respect to labor, 1, is the lower bound below which social returns to quality κ cannot fall without making steady-state constant endogenous growth impossible. But why is $\kappa>1$ possible? The answer is that $\kappa>1$ yields that quality growth fuels market growth "in excess" of each firm's quality growth but that such "excess" market growth is absorbed by entry. In other words, if social returns to quality are larger than what is needed to support constant exponential growth, the "excess" growth in market size is absorbed by the endogenous mass of firms.

More specifically, Proposition 1 says that $\kappa>1$ is feasible because there is a thick region of parameters' space where the market fragmentation mechanism "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models. Such reasoning does not apply to models with two dimensions of technology that play interdependent, but distinct, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that *sterilizes the market size effect* and thereby its manifestations — the (strong) scale effect and the need to impose a knife-edge condition that rules out explosive behavior.

4.5. An important special case: zero entry cost

Conditions (23) and (24) allow $\beta=0$. This property is important. It says that the core steady-state results delivered by this class of models *do not depend* on the specifics of the entry technology. This fact has been known for a long time but is often neglected. What it means is that the entry technology does not determine the existence or non-existence of the steady state but determines only (i) how the economy converges to the steady state and (ii) some features of the steady state like, for example, the relationship between growth and endowments that arise from the allocation of resources to vertical or horizontal innovation.

To see this point most vividly, note that under $\beta = 0$ condition (24) is satisfied for *any* values of the other parameters so that the steady state, if it exists, surely features $z^* > 0$. In other words, the model accommodates zero entry costs naturally: Proposition 1 applies as is and one simply sets $\beta = 0$ in all the relevant expressions. The only thing that changes is the dynamics. Under zero entry costs the mass of firms, N, becomes a jumping variable and, consequently, quality-adjusted firm size, x, also becomes a jumping variable. The model then "degenerates" to the straightforward saddle-point dynamics studied in many of the papers cited in the introduction that laid the foundations of the approach.

4.6. Additional remarks

It might surprise the reader that the condition determining whether the model delivers steady-state endogenous growth does not depend on κ. On reflection, the property stems from the fact that in this specification the no arbitrage locus (NA) depends only on the parameters driving the *private* returns to innovation. The specification studied in the next section relaxes this feature and produces an extended analog to condition (24) that includes the parameters driving the social returns to innovation, κ and σ.

If we set $\kappa = 1$ we obtain the conditions that the literature so far has imposed in models of this class. This fact suggests that $\kappa > 1$ is a natural extension of such models. In the language of Proposition 1, we can say that under the shadow of the linearity critique the literature has been imposing a restriction like $z'(x) < \alpha/\eta$ without much reflection on what it means. The proposition says that we can allow $\kappa > 1$ and simply choose parameters to satisfy the condition

$$z'(x) = \frac{\beta\alpha - 1}{\beta - 1} < \frac{\alpha}{\eta}\frac{1-\sigma}{\kappa-\sigma} < \frac{\alpha}{\eta}.$$

This requirement is not hard to meet since it boils down to compensate the larger value of κ with changes in any of β, α, σ or η.

The case $\eta = 1$ (log utility) yields

$$\kappa^{\max} \equiv 1 + (1-\sigma)\frac{1-\alpha}{\beta-1} > 1.$$

It follows that the theory's mechanism fails only when preferences feature high η. More precisely, η needs to be high enough to yield $\frac{1}{\eta}\frac{\beta-1}{\beta-1/\alpha} - 1 \leq 0$. But the property that sufficiently high η kills endogenous growth holds also in the traditional case $\kappa = 1$; it is not specific to the robustness issue discussed in this paper. To see this, it is enough to recall that condition (24) does not contain κ and that it is the one dictating whether the economy converges to the steady state (x^*, c^*) or to the steady state $(\bar{x}, \bar{c})$. Thus, regardless of κ, one can always choose η high enough to enforce the steady state with zero quality growth.

An interesting implication of the mechanism explored in this paper is that steady-state entry is possible even when population growth is zero or even *negative*, i.e., $\lambda \leq 0$.[7] To see this, note that in steady state

$$n^* = \frac{1}{1-\sigma} \cdot [\lambda + (\kappa - 1)z^*],$$

which says that entry is positive as long as quality growth compensates for the potentially negative λ. In fact, this result says that quality growth *causes* variety expansion in that it creates the profit opportunity for entrants. In the case $\lambda = 0$, we have

$$n^* = \frac{\kappa - 1}{1-\sigma} \cdot z^*,$$

that is, variety expansion is proportional to quality growth.

It is also useful to stress that in this structure there is no add-up-to-one restriction on the parameters regulating returns to scale, as in standard two-sector models, because this is not a two-sector model. As stated, the key to the insight is that in the determination of the return to innovation quality and variety end up at the opposite ends of the ratio $\theta Y/NZ$ that drives the dynamics of the quality-adjusted gross cash flow, x. Why is this difference so important? Because in a standard two-sector model the quantity of one factor, say human capital, *raises* the marginal product of the other factor, say physical capital. Therefore, in such a view of the world the factors can only reinforce each other in the determination of the returns to investment so that to have non-decreasing, non-explosive growth the overall production structure must have the well-known property that the degrees of social returns in the two sectors add up to one. In the Schumpeterian view of the world, in contrast, the mass of firms/products is not just another factor of production but the measure of endogenous market structure that *lowers* the quality-adjusted gross cash flow of the firm through the dominant market share effect. Consequently, more product variety, while it raises final output, *lowers* the return to vertical (in-house) innovation.

It is worth to emphasize two other aspects of the class of models that this paper generalizes.[8] First, the models feature important growth effects of policy instruments (e.g., tax rates), both in steady state and along the transition to the steady

[7] To avoid possible confusion, let me stress that this observation applies to the region $x > x_Z$. I have already argued that to get there population growth might be necessary.

[8] I thank an anonymous referee for suggesting this.

state. Second, the models have quantitative relevance both in the sense that calibrations typically deliver reasonable parameter values and in the sense that the calibrated models produce quantitatively important effects of changes in fundamentals and in policy instruments. The first property has been established in several prior applications. Limiting attention to published works that use the very model generalized in this paper, see Peretto (2007, 2011) on the role of taxation and the tax cuts of 2003 in the US. The second paper, in particular, contains a calibration that illustrates well the second property. Moreover, in recent work (Ferraro et al., 2017) develop a DSGE version of the model in Peretto (2007), extended in Peretto (2015) and further generalized here, and calibrate it to NIPA data according to the state of the art in business-cycle macro. Iacopetta and Peretto (2018) develop a version of the model in Peretto (2015) that allows for an important role of corporate governance and the internal organization of the firm. They then calibrate it to the Maddison data to study whether the theory augmented with corporate governance replicates well the historical evolution of the world both in terms of the paths of development of individual countries and of the income gaps across countries. Overall, therefore, there is ample evidence that models of this class have both policy and quantitative relevance.

5. Alternative specification of entry costs

This section uses an alternative specification of the entry cost. Namely, an entrant must sink βX units of final output. In terms of the production function for new products/firms, we now work with

$$\dot{N} = \left(\frac{1}{\beta X}\right) \cdot E,$$

where E is variety-expanding investment in units of the final good. Proceeding as in the previous case, the free-entry condition is $V = \beta X$ and the associated rate of return to entry is

$$r = \left[\frac{(P-1)X}{Z} - \phi - \frac{I}{Z}\right]\frac{Z}{\beta X} + \frac{\dot{X}}{X}.$$

The main differences with respect to the expression used in the previous exercise are that this return is non-linear in X and, because in equilibrium $NX = \theta^2 Y$, its capital gain/loss component makes it dependent on the growth rate of output $\dot{Y}/Y$ and on the entry rate $n = \dot{N}/N$.

More specifically, this modification of the entry technology adds the following features to the analysis: (i) the entry cost is scaled by average firm size, X, and thus, in equilibrium, by market size θY and by the firm's market share $1/N$; (ii) as a consequence, the wealth-output ratio is constant, i.e., $A/Y = NV/Y = \beta\theta^2$; (iii) moreover, the innovation rates z and n are determined at any point in time by the full general-equilibrium structure of the model.

Overall, this specification allows for a feedback of the interest rate on the innovation rate of incumbents, z, and thereby for such innovation rate to depend directly on preference parameters that regulate the household reservation rate of return on assets. Through this channel $z = I/Z$ depends on the parameters driving social return to quality and variety, κ and σ. The most interesting feature of this extension, however, is that under some additional conditions, $\kappa^{\max} \to \infty$.

5.1. Existence

The steady state loci $\dot{c} = 0$ and $\dot{x} = 0$ have the same structure as in the previous case and thus yield again equation (GE). The key difference is that instead of (NA) the equation representing quality growth as a function of quality-adjusted firm size is

$$z(x) = \frac{(\alpha x - \phi - \rho)[1-(1-\eta)\sigma] - \eta\sigma\left(\frac{x-\phi}{\pi x} - \rho + \lambda\right)}{\eta(\kappa - \sigma/\pi x)}. \qquad NA'$$

The steady state is the intersection in (x, z) space of (NA')-(GE).

Proposition 2. *The steady state is the solution of the equation*

$$\frac{1-\sigma}{\kappa-\sigma}\left(\frac{\alpha x - \phi - \rho}{\eta} - \frac{\sigma\lambda}{1-\sigma}\right) = \frac{(\alpha x - \phi - \rho)[1-(1-\eta)\sigma] - \eta\sigma\left(\frac{x-\phi}{\pi x} - \rho + \lambda\right)}{\eta(\kappa - \sigma/\pi x)},$$

which takes the form

$$a_1 x^2 + a_2 x + a_3 = 0,$$

with coefficients that are functions of the deep parameters of the model. The equation has two real roots, x^* and $\tilde{x}$, with $x^* < \tilde{x}$ for $1 \le \kappa < \kappa^{\max}$, where $\kappa^{\max} > 1$ is a finite value that depends on the other parameters. If, in addition, $(1+\phi\pi)^2 > 4\alpha\pi\phi$ there exists a thick region of parameter space where $\kappa^{\max} \to \infty$.

Proof. See the Appendix. □

Fig. 3 illustrates the mechanism. It differs from that obtained in the previous section only in that the no-arbitrage locus is non-linear. As said, its curvature follows from the fact that the return to entry depends on the growth rate of the economy.

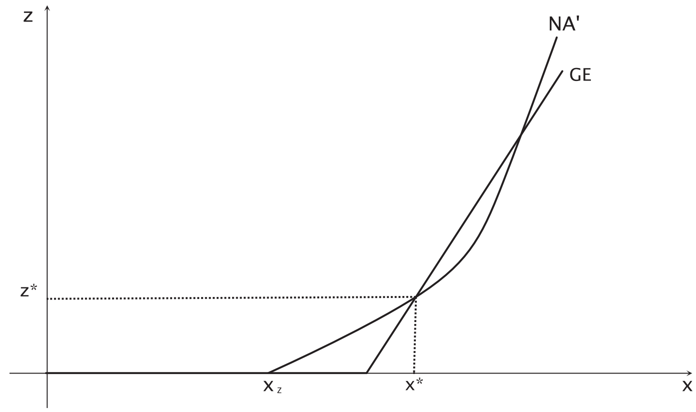

Fig. 3. Existence of the steady state with constant exponential growth when entry cost is βX.

Consequently, the no-arbitrage condition determining investment in quality now includes the reservation rate of return on saving of the household.

The existence proof exploits the fact that the equation $\Psi(x) = 0$ yields the quadratic form $a_1 x^2 + a_2 x + a_3 = 0$, with coefficients that are functions of the parameters of the model, especially κ (see the appendix for the expressions). The condition for existence of the stable steady state with constant exponential growth is then that two real roots exist in the interval $x \in (x_Z, +\infty)$. To check when this is the case, the proof looks for values of $\kappa \geq 1$ such that $\Delta(\kappa) \equiv (a_2(\kappa))^2 - 4a_1(\kappa)a_3(\kappa) > 0$. A straightforward calculation yields

$$\Delta(\kappa) = b_1(\kappa - 1)^2 + b_2(\kappa - 1) + b_3,$$

where the coefficients b_1, b_2 and b_3 are functions of the other parameters (see the appendix for the expressions). In particular, $b_3 > 0$ so that $\Delta(1) > 0$. The sign of b_1 is thus important. For $b_1 > 0$, that is, for $(1 + \phi\pi)^2 > 4\alpha\pi\phi$ we can obtain a condition such that $\Delta(\kappa)$ is always positive because the quadratic equation in $\kappa - 1$ does not have real solutions. Specifically, we can chose parameters that yield

$$b_2^2 - 4b_1 b_3 < 0.$$

For $b_1 < 0$, that is, for $(1 + \phi\pi)^2 < 4\alpha\pi\phi$, instead, we have $\Delta(\kappa) > 0$ for

$$1 \leq \kappa < 1 + \frac{b_2 + \sqrt{b_2^2 - 4b_1 b_3}}{-2b_1} \equiv \kappa^{\max}.$$

To summarize: (i) there always exists a finite value $\kappa^{\max} > 1$ such that for $1 \leq \kappa < \kappa^{\max}$ there exists a steady state with constant endogenous growth; (ii) if, in addition, $(1 + \phi\pi)^2 > 4\alpha\pi\phi$, then there exists a region of parameter space where $\kappa^{\max} \to \infty$ and the steady state with constant endogenous growth exists for all $\kappa \geq 1$.

5.2. Dynamics and stability

To study the dynamics of this specification it is convenient to restrict attention to the case $\eta = 1$ (log-utility). The analysis of the general case $\eta \neq 1$ has the same features as that carried out in the previous section and thus adds no novel insight. On the other hand, under log-utility this specification has the nice property that the ratio $C/Y \equiv c$ is constant at all times and therefore one can reduce the general-equilibrium dynamical system to only one dimension, x. A manifestation of this property is that the two functions describing the rates of vertical and horizontal innovation have the form, $z(x)$ and $n(x)$, that is, both rates of innovation depend only on quality-adjusted firm size, x. Moreover, the threshold for activation of horizontal innovation is also just a value of x, denoted x_N, instead of a locus in (x, c) space. The resulting structure yields a transparent characterization of the dynamics. The ordering of the thresholds determines the sequence of activation of the two types of innovation. Most importantly, as in the previous case, the economy converges to the right-most region of state space, $x > \max\{x_N, x_Z\}$, where both variety and quality innovation take place.

We begin with the case in which the economy activates variety innovation first. The expressions for the functions $z(x)$ and $n(x)$ are as follows.

Proposition 3. *In the case $x_N < x_Z$ the rates of variety and quality innovation are:*

$$z(x) = \begin{cases} 0 & \phi \leq x \leq x_N \\ 0 & x_N < x \leq x_Z \\ \dfrac{\alpha x - \phi - \sigma \frac{x - \phi}{\pi x} - [\rho(1 - \sigma) + \sigma\lambda]}{\kappa - \frac{\sigma}{\pi x}} & x_z < x < \infty \end{cases};$$

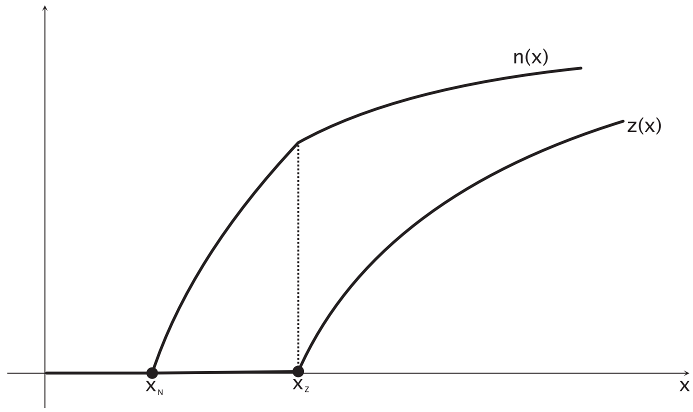

Fig. 4. Innovation rates as functions of the quality-adjusted gross cash flow in the case $x_N < x_Z$.

$$n(x) = \begin{cases} 0 & \phi \le x \le x_N \\ \dfrac{x-\phi}{\pi x} - \rho + \lambda & x_N < x \le x_Z \\ \dfrac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho(1-\sigma)+\sigma\lambda]}{\kappa\pi x - \sigma} - \rho + \lambda & x_Z < x < \infty \end{cases},$$

where:

$$x_N \equiv \frac{\phi}{1-\pi(\rho-\lambda)};$$

$$x_Z \equiv \arg\text{solve}\left\{\alpha x - \phi - \sigma\frac{x-\phi}{\pi x} = \rho(1-\sigma)+\sigma\lambda\right\}$$
$$= \frac{\phi + \rho(1-\sigma) + \sigma\lambda + \frac{\sigma}{\pi} + \sqrt{\left[\phi+\rho(1-\sigma)+\sigma\lambda+\frac{\sigma}{\pi}\right]^2 - 4\alpha\frac{\sigma\phi}{\pi}}}{2\alpha}.$$

Proof. See the Appendix. □

Fig. 4 illustrates the properties of the two functions. $z(x)$ is initially zero, becomes positive at x_Z, leaving that point with positive derivative, can be either concave or convex, and becomes asymptotically linear. $n(x)$ is initially zero, becomes positive at x_N, leaving that point with positive derivative, is always concave and converges from below to a finite upper bound.

When the economy activates quality innovation first things are a bit different. The expressions for the two functions are as follows.

Proposition 4. *In the case $x_Z < x_N$ the rates of variety and quality innovation are:*

$$z(x) = \begin{cases} 0 & \phi \le x \le x_Z \\ \tilde{z}(x) & x_Z < x \le x_N \\ \dfrac{\alpha x - \phi - \sigma\frac{x-\phi}{\pi x} - [\rho(1-\sigma)+\sigma\lambda]}{\kappa - \frac{\sigma}{\pi x}} & x_Z < x < \infty \end{cases};$$

$$n(x) = \begin{cases} 0 & \phi \le x \le x_Z \\ 0 & x_Z < x \le x_N \\ \dfrac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho(1-\sigma)+\sigma\lambda]}{\kappa\pi x - \sigma} - \rho + \lambda & x_N < x < \infty \end{cases},$$

where:

$$\tilde{z}(x) = x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] - \phi$$

and $\tilde{c}(x)$ is the solution of the partial differential equation

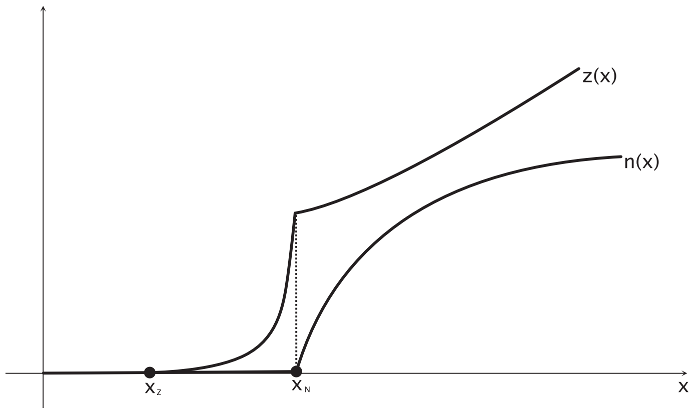

Fig. 5. Innovation rates as functions of the quality-adjusted gross cash flow in the case $x_Z < x_N$.

$$\frac{dc}{dx} = \frac{\kappa c}{x} \frac{\frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right) - \frac{\rho}{\kappa} - (1-\alpha)x}{\lambda + (\kappa - 1)\left[x - \phi - \frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right)\right]}.$$

The thresholds are:

$$x_N \equiv \arg\text{solve}\left\{\frac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho(1-\sigma) + \sigma\lambda]}{\kappa\pi x - \sigma} = \rho - \lambda\right\};$$

$$x_Z \equiv \arg\text{solve}\left\{x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] = \phi\right\}.$$

The function $z(x)$ has zero derivative at $x = x_Z$, is increasing and has positive derivative at x_N.

Proof. See the Appendix. □

Fig. 5 illustrates the properties of the two functions. $z(x)$ becomes positive at x_Z leaving that point with zero derivative, is convex up to x_N and then has the same properties as in the previous case. $n(x)$ becomes positive at x_N and has the same properties as in the previous case, i.e., is increasing, concave and bounded above.

The following proposition states the main result.

Proposition 5. *The roots x^* and $\tilde{x}$ studied in Proposition 2 identify two steady states with constant, exponential, endogenous growth, one stable, x^*, and the other unstable, $\tilde{x}$. For initial condition $x(0) \in (\phi, \tilde{x})$, the steady state x^* is the attractor of the economy's dynamics if*

$$x_Z < \frac{\phi}{1 - \left(\rho + \frac{\sigma\lambda}{1-\sigma}\right)\pi}, \tag{27}$$

where x_Z is the activation threshold for quality innovation in Proposition 3. For initial condition $x(0) > \tilde{x}$ the economy's dynamics is explosive.

Proof. See the Appendix. □

The result is best understood qualitatively; see the proof of the proposition for the algebraic details. First, note that we are characterizing the dynamical system

$$\frac{\dot{x}}{x} = \Psi(x) \equiv \lambda + (\kappa - 1)z(x) - (1-\sigma)n(x),$$

where the key ingredients are the two functions $z(x)$ and $n(x)$ discussed above. Figs. 6 and 7 illustrate the dynamics. Consider the case $x_N < x_Z$ in which the economy activates variety innovation first.

For $x \le x_N < x_Z$, we have $z(x) = n(x) = 0$ because both returns are too low. Accordingly, the growth rate of the quality-adjusted gross cash flow is $\dot{x}/x = \lambda > 0$ and the economy crosses the threshold for entry in finite time. For $x_N < x < x_Z$, $z(x) = 0$ and the growth rate of the quality-adjusted gross cash flow is

$$\frac{\dot{x}}{x} = \lambda - (1-\sigma)n(x).$$

This is a converging process. The system, therefore, crosses the threshold for quality innovation in finite time only if $\dot{x}/x = \lambda - (1-\sigma)n(x_Z) > 0$, that is, if the quality-adjusted gross cash flow is still growing at $x = x_Z$. Rearranging terms yields condition (27). The intuition is that product proliferation is not so extreme that the quality-adjusted gross cash flow stops

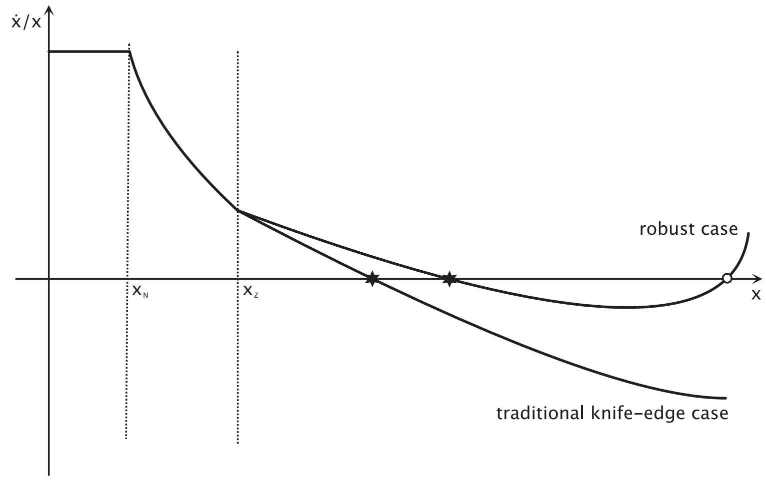

Fig. 6. Dynamics when $x_N < x_Z$ in the traditional case $\kappa = 1$ and in the robust case $\kappa > 1$. The stars denote the stable steady states; the hollow circle the unstable one.

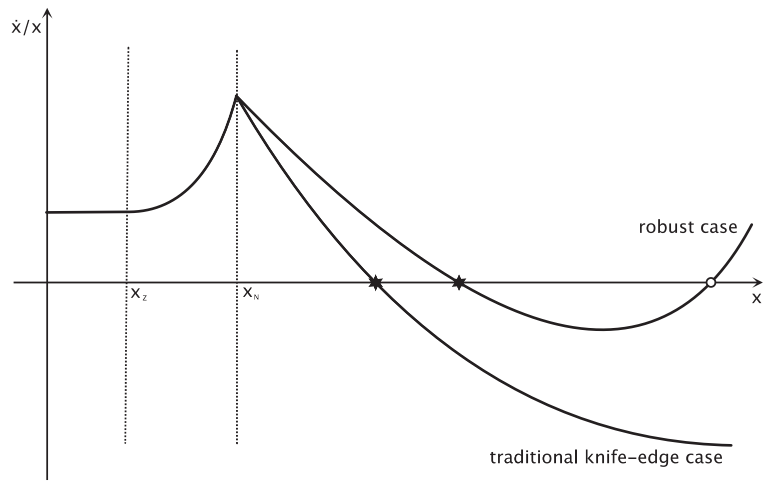

Fig. 7. Dynamics when $x_Z < x_N$ in the traditional case $\kappa = 1$ and in the robust case $\kappa > 1$. The stars denote the stable steady states; the hollow circle the unstable one.

growing before the activation of vertical innovation. In other words, the economy avoids premature market saturation. The case $x_N > x_Z$, in which the economy activates quality innovation first, is rather different.

Instead of a deceleration of the rate of growth of the quality-adjusted gross cash flow, at $x = x_Z$ we have an acceleration and therefore we don't need any restriction to ensure that the economy crosses the threshold x_N in finite time. Only for $x > x_N$ the quality-adjusted gross cash flow growth starts slowing down and converging to zero.

For comparison with the existing literature, it is instructive to look at

$$\begin{aligned}\lim_{x\to\infty} \Psi(x) &= \lim_{x\to\infty} [\lambda + (\kappa - 1)z(x) - (1-\sigma)n(x)] \\ &= \lambda + \lim_{x\to\infty} (\kappa - 1)z(x) - \lim_{x\to\infty} (1-\sigma)n(x).\end{aligned}$$

The proof of the proposition shows that as $x \to \infty$, $n(x) \to \frac{\kappa-\alpha-\pi(\rho-\lambda)}{\pi}$ while $z(x) \to \alpha x$. Hence, the restriction for the return to innovation to be non-decreasing in Z, $\kappa \geq 1$, implies that the quality-adjusted gross cash flow explodes to infinity unless one imposes the knife-edge condition $\kappa = 1$ because doing so kills the term $z(x)$. This then allows one to ensure that a solution $\Psi(x) = 0$ exists by simply assuming

$$\lambda - \lim_{x\to\infty} (1-\sigma)n(x) = \lambda - \frac{\kappa - \alpha - \pi(\rho - \lambda)}{\pi} < 0,$$

which yields $\lim_{x\to\infty} \Psi(x) < 0$. The literature has thus far considered only this case: linearity of production in the growth-driving factor, Z in this case, to ensure that constant steady-state growth is both (1) technologically feasible and (2) non-explosive, and restrictions on the other parameters to ensure that, once agents' behavior is taken into account, the steady state with constant growth exists as a market equilibrium and is the attractor of the associated dynamical system. Feasibility in this context means that the economy exhibits non-decreasing returns to Z.

This approach, however, exploits a *sufficient* condition, which, as such, can be relaxed. So, what happens if $\kappa > 1$? Propositions 2 and 5 states that there are two sets of conditions that yield the following outcome:

- the economy converges to the steady state with both quality and variety innovation;
- such steady state exhibits constant, exponential growth of income per capita for $\kappa \in [1, \kappa^{\max})$, where $\kappa^{\max}$ can be infinity.

The proof shows that the equation $\Psi(x) = 0$ yields the quadratic form $a_1x^2 + a_2x + a_3 = 0$, with coefficients that are functions of the primitive parameters of the model, especially κ. This quadratic form is exactly that used in Proposition 2 with the restriction $\eta = 1$. The condition for existence of the stable steady state with constant exponential growth is then already established there.

6. Interpretation: where are the linearities?

It is often claimed that endogenous growth models are linear in the sense that they must postulate some differential equation of the form $\dot{X} = \_X$. The primitives used in deriving this paper's results do not exhibit such property, i.e., they are not linear in this sense. One can thus ask: How is it possible to obtain endogenous growth without *assuming* the necessary functional form?

6.1. Answering the question: first-principles

The italicized word "assuming" in the question above is at the heart of the linearity critique and its relevance to this paper's contribution. In short, the answer to the question is that the model developed in this paper *does not assume* the required form but it *produces it endogenously*.

One reading of the linearity critique is that we need not worry about what the blank is and where it comes from, but we can focus exclusively on the fact that, whatever it is, it must be independent of calendar time t and, crucially, independent of X itself.[9] Henceforth, the word "constant" refers to a model-specific filler of the blank with this property, i.e., time-invariant and independent of the accumulated endogenous variable. On this reading, the linearity critique reduces to the self-evident assertion that endogenous growth models exhibit the mathematical property that the variable X grows at a constant exponential rate in steady state. To mean anything, therefore, the critique must refer to the mechanism that in the context of a specific model delivers the property that the blank is constant (as defined above). It is only in this very specific sense that one can say that $\dot{X} = \_X$ represents a core element of the model. If, for example, the blank stands for the economy's intensity of investment in the growth of X, then the critique's point is that such intensity must be independent of X and of calendar time t. To determine whether this is the case requires working out how the economy determines such intensity. If this process of unpacking the blank reveals that the model indeed *assumes* that $\dot{X}$ is proportional to X, with coefficient of proportionality independent of calendar time t, by postulating such property as a primitive, then the critique applies. If not, the critique reduces to the tautological claim that models of exponential growth processes work with the mathematical representation of exponential growth processes.

In what follows we unpack this paper's model to obtain the precise mapping from the primitives to the state-space representation of the equilibrium. We restrict attention to average product quality, Z, since it is the dimension in which the model exhibits endogenous growth, and note that agents invest according to a function $I(\cdot)$ that *over time* converges to the form

$$I = Z \cdot (\text{constant})$$

so that in steady state, and *only in steady state*,

$$\dot{Z} = I = Z \cdot (\text{constant}).$$

The goal of this section is to develop this argument in detail. The main step is the characterization of the properties of the function $I(\cdot)$, namely, (1) the identification of what, exactly, are the arguments inside it and (2) the determination of its shape. The crucial claim is going to be that $I(\cdot)$ is, among other things, a function of Z itself so that the model's core equation violates the core assertion of the linearity critique. The exception is the steady state where the function takes the form mathematically necessary to describe a constant growth rate.

Before developing the argument, it is useful to note that with the focus of the discussion on what properties delivers endogenous growth, one might think that the variety dimension of technology, N, is not subject to the critique discussed above because product variety is not tasked with supporting endogenous growth — in fact, it is tasked with "taming" explosive growth. Nevertheless, the mechanism that yields the properties of the $\dot{Z}$ equation applies to the $\dot{N}$ equation as well. The reason is that N exhibits the semi-endogenous growth property and thus *must* have the steady-state mathematical representation $\dot{N} = (\text{constant}) \cdot N$ for the same reason why average quality *must* have the steady-state mathematical representation $\dot{Z} = (\text{constant}) \cdot Z$.

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[9] The critique conflates the two implicit claims on the blank — independent of X and independent of calendar time t — because it is an attempt at summarizing a very special class of models, namely, those whose entire economic structure can be reduced to the accumulation of a single stock. It is self-evident from the equation that to be independent of calendar time t the blank must be independent of the variable X since X is tasked with growing exponentially over time.

6.2. Answering the question: details

It is useful to work with the second version of the model because, as argued, it allows for richer interactions in the determination of the investment rates z and n. We start from the model's key primitives written in the following terms:

$$\dot{N} = \frac{1}{\beta X} \cdot E; \tag{28}$$

$$\dot{Z}_i = I_i. \tag{29}$$

In this notation, E is aggregate investment in entry and I_i is investment in quality by firm i. To ease the exposition, henceforth I focus on the symmetric equilibrium and drop the subscript i in the firm-level equation. Now recall that this is a one-sector model, that is, the terms E and I are interpreted as composite inputs produced with the same technology as that for the final good Y. In other words, the model is understood as taking a shortcut with respect to the much more cumbersome version that one could write, where on the right-hand side of Eqs. (28) and (29) one specifies functions of the quantities of intermediates and labor allocated to, respectively, horizontal and vertical innovation. Such functions would have the same mathematical form as the right-hand side of Eq. (4). Therefore, the right-hand side of the *primitives* (28) and (29) is non-linear in N and non-linear in Z.

It is now useful to rewrite Eqs. (28) and (29) as:

$$\dot{N} = N \cdot \frac{E}{\beta N X};$$

$$\dot{Z} = Z \cdot \frac{I}{Z}.$$

On the right-hand side we now have two measures of investment intensity, $E/\beta NX$ and I/Z. Recall that in solving the model we defined

$$n \equiv \frac{\dot{N}}{N} = \frac{E}{\beta N X} \quad \text{and} \quad z \equiv \frac{\dot{Z}}{Z} = \frac{I}{Z}$$

and then obtained from the equilibrium conditions expressions that give these two growth rates — or, equivalently, the investment intensity in variety and quality, $E/\beta NX$ and I/Z — as functions of the state variable x; see Propositions 1 and 2. For notational convenience we called such functions $n(x)$ and $z(x)$, where, from its definition and the model's equilibrium conditions,

$$x = \frac{(P-1)X}{Z} = \frac{P-1}{P}\frac{PX}{Z} = \frac{P-1}{P}\frac{\theta Y}{NZ} = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}, \quad \sigma < 1 \le \kappa. \tag{30}$$

Recall that x is the quality-adjusted gross cash flow of the firm and that using it as my state variable is suggested by the equations characterizing behavior, since they say that the returns to vertical and horizontal innovation are functions of the quality-adjusted gross profit earned by the typical firm in symmetric equilibrium. For the purposes of this discussion, it is insightful to summarize this step of the argument as saying that we can write x as a function $x(N, Z, L)$, which, clearly, is non-linear in Z, non-linear in N and non-linear in the combination of the two in the precise sense that the exponents $\kappa - 1$ and $1 - \sigma$ do not add up to one.

With this notation in hand, we can write:

$$\dot{N} = N \cdot n(x), \quad x = x(N, Z, L); \tag{31}$$

$$\dot{Z} = Z \cdot z(x), \quad x = x(N, Z, L). \tag{32}$$

At the most basic level, therefore, addressing the "missing linearities question" reduces to pointing out that the terms $n(x)$ and $z(x)$ in these expressions are: (1) the blanks that the linearity critique talks about; (2) functions of the state vector (N, Z, L); (3) constant in steady state. It is more interesting, however, to develop the argument in a form that brings to the forefront the role played by agents' behavior and the model's equilibrium.

First, note that comparing expressions (31) and (32) to the primitives (28) and (29) highlights that the model works with:

$$E = \beta N X \cdot n(x) \Leftrightarrow n(x) = \frac{E}{\beta N X};$$

$$I = Z \cdot z(x) \Leftrightarrow z(x) = \frac{I}{Z}.$$

In other words, the equilibrium objects $n(x)$ and $z(x)$ represent the investment decisions expressed in terms familiar to most readers. Next, note that the terms $E/\beta NX$ and I/Z are measures of investment intensity that can be mapped into, respectively, the share of output allocated to entry and the firm-level R&D/sales ratio. Specifically, using the equilibrium results $P = 1/\theta$ and $NX = \theta^2 Y$ we can define $s_N \equiv E/Y$ and $s_Z \equiv I/PX$ and write:

$$\dot{N} = N \cdot \frac{1}{\beta\theta^2} \cdot s_N;$$

$$\dot{Z} = Z \cdot \frac{X}{\theta Z} \cdot s_Z.$$

We now recall the definition $x \equiv \frac{(P-1)X}{Z}$ and write:

$$\dot{N} = N \cdot \frac{1}{\beta\theta^2} \cdot s_N;$$

$$\dot{Z} = Z \cdot \frac{x}{1-\theta} \cdot s_Z.$$

The mapping between the analysis in the previous sections and this representation is the following:

$$s_N(x) \equiv \beta\theta^2 \cdot n(x);$$

$$s_Z(x) \equiv \frac{1-\theta}{x} \cdot z(x).$$

Accordingly, the new terms s_N and s_Z are not arbitrary objects but are specific functions of the state variable x constructed through manipulation of the model's equilibrium conditions.[10] Determining which representation is better is purely a matter of expositional convenience and idiosyncratic taste. What we did here is simply decompose the core objects constructed earlier, the functions $n(x)$ and $z(x)$, in terms that translate into the representation that many readers might be more accustomed to. Neither the economics nor the math have changed.

6.3. Answering the question: the punchline

Now refer back to Eqs. (28) and (29) with their right-hand sides interpreted according to the observations just made, i.e., as measures of investment intensity. Simply looking at the graphical representation of the functions $n(x)$ and $z(x)$ makes the point that we are dealing with non-linear functions. Hence, we only need to highlight that (1) x converges to a constant value x^* and (2) x is a function of the model's state vector (N, Z, L). Since this is the punchline of this section, it is useful to reproduce here the key string of relations used above,

$$x = \frac{(P-1)X}{Z} = \frac{P-1}{P}\frac{PX}{Z} = \frac{P-1}{P}\frac{\theta Y}{NZ} = (1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}, \quad \sigma < 1 \leq \kappa.$$

The full system that we are working with is:

$$\dot{N} = N \cdot \frac{1}{\beta\theta^2} \cdot s_N(x) = N \cdot \frac{1}{\beta\theta^2} \cdot s_N\left((1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{Z} = Z \cdot \frac{x}{1-\theta} \cdot s_Z(x) = Z \cdot \left(\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right) \cdot s_Z\left((1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{L} = L \cdot \lambda.$$

Section 4 of the paper simply uses the more compact notation:

$$\dot{N} = N \cdot n(x) = N \cdot n\left((1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{Z} = Z \cdot z(x) = Z \cdot z\left((1-\theta)\theta^{\frac{1+\theta}{1-\theta}} \cdot \frac{Z^{\kappa-1}L}{N^{1-\sigma}}\right);$$

$$\dot{L} = L \cdot \lambda.$$

The two representations are not just equivalent but two ways to say the same thing.

The structure of the model's equilibrium system is thus as follows. The growth rates of variety, $\dot{N}/N$, and average quality, $\dot{Z}/Z$, are given by two functions $n(x)$ and $z(x)$ that are non-linear in x, which is itself non-linear in Z, non-linear in N and

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[10] The interested reader might want to notice that $s_Z = I/PX = NI/NPX = NI/\theta Y$ so that the argument can be cast in terms of aggregate spending on quality R&D as a fraction of final output.

non-linear in the combination of the two in the sense tat $\sigma+\kappa\neq 1$. It follows that the $\dot{N}$ equation above is non-linear in N, non-linear in Z and non-linear in the combination of the two. Similarly, the $\dot{Z}$ equation above is non-linear in N, non-linear in Z and non-linear in the combination of the two. The only linear differential equation is the third. For the purposes of this argument, however, the law of motion of population is an uninteresting object since it is only a forcing process and we are trying to understand the endogenous forces at work in the model. We can thus set $\lambda=0$ and strip the system down to only the endogenous state variables N and Z. This leaves us with a pair of autonomous non-linear differential equations with a full Jacobian matrix.[11] More precisely, not only the differential equations in the model's equilibrium system system do not have the form $\dot{N}=\_N$ and/or $\dot{Z}=\_Z$ with the blanks independent of N and Z, but we are also dealing with a system that does not allow us to think about one equation in isolation from the other. The model is about the interdependence of Z and N. Another way to say this is that one cannot abstract from the cross-equations restrictions that make the equilibrium system a system. The exception, as already said, is the steady state.

6.4. Answering the question: interpretation

We can now reconsider the question motivating this exercise appropriately refined, namely: How is it possible to obtain endogenous growth without *assuming* the necessary functional form $\dot{Z}=\_Z$ with the blank independent of Z and of calendar time t? The answer is remarkably simple. No assuming is needed. Along the dynamics, as $x\to x^*$ *both* equations *become* of the class mathematically consistent with constant exponential growth, namely, $\dot{N}=N\cdot(\text{constant})$ and $\dot{Z}=Z\cdot(\text{constant})$. But such property is not imposed on any of the model's primitives — in fact, it is not assumed anywhere in the paper.[12] It *emerges* as the system converges to the steady state. Is is worth stressing this point: the linearity of the accumulation equations for N and Z is a steady-state property of the system representing economic decisions and the interaction among the two dimensions of technology. It is not a property of the primitives per se. As such, it cannot be intuited, or guessed at, by simply inspecting one of the primitives in isolation from the others and without reference to the model's mechanism.

To explore further this point, one could say that each equation in the system (28) and (29) consists of two parts. The first is the technology (i.e., the primitive) part, the second is the behavior part. For example, in the case of quality, the technology part is $\dot{Z}=I$ and the behavior part is $I=Z\cdot z(x)$, where $z(x)$ is simply compact notation for a function obtained from the equilibrium conditions. In words, this says that agents invest in quality according to a function of the state vector (N, Z, L) that can be decomposed in the product of two terms, Z and $z(x)$, with x given by (30). The decomposition is not arbitrary, but stems from the property that agents care about rates of returns and that rates of return are functions of the quality-adjusted gross cash flow x and of the rates of innovation z and n. Similarly, one can write $\dot{N}=(1/\beta X)\cdot E$ as the technology (i.e., primitive) part and $(1/\beta X)\cdot E=N\cdot n(x)$ as the behavior part, where one can think in terms of a function $E(N, Z, L)$ that because of the properties of the model's equilibrium takes the specific form $E=\beta\theta^2 Y\cdot n(x)$ with $Y=\theta^{2\theta}N^{\sigma}Z^{\kappa}L$.

To conclude this discussion, we turn to the extension of the linearity critique due to Growiec (2007). Consider a vector of variables X whose growth rate, denoted $\hat{X}$, is governed by the equation $\hat{X}=F(X)$. A balanced growth path exists if $d\hat{X}/dt=DF(X)\cdot\dot{X}=0$, that is, if either $\dot{X}=0$ or the matrix DF is singular. Since we are interested in $\hat{X}>0$, the necessary condition for steady-state growth is singularity of DF, which requires either at least one of the differential equations in the system to be linear or a subset of the differential equations to be in linear combination. As this is a theorem, Growiec (2007) argues, such a condition is imposed either explicitly or implicitly in all growth models. Is this the case also for the model developed in this paper? No. The condition is not imposed. What happens is that the theorem's condition is trivially satisfied by the fact that *all* differential equations in the system (28) and (29) become linear. Moreover, in steady state the Jacobian matrix of the system is diagonal and thus the theorem (and any such heavy-duty mathematical tool) becomes redundant since it is designed to apply to systems with cross-equations restrictions and in this model — one could argue — the steady-state representation of the dynamics is no longer a system in the sense of interdependent equations.

More importantly, however, the theorem applies to the reduced-form representation of the economy's dynamics and therefore — exactly as the one-dimensional $\dot{X}=\_X$ articulation of the linearity critique — has nothing to say about the model's primitives, which is what this paper is about. The claim of the paper is not that the differential equation for the equilibrium evolution of Z in steady state is not of the form $\dot{Z}=Z\cdot(\text{constant})$ — such a claim would be a mathematical absurdity! The claim of the paper is that the form $\dot{Z}=Z\cdot(\text{constant})$ need not be assumed as a property of the primitive $\dot{Z}=I$ representing how agents (i.e., firms) combine factors of production to obtain an increase in the quality of the product that they sell. Rather, it emerges as agents follow an equilibrium investment rule that takes the form $I=Z\cdot z(x)$, where $z(x)$ is compact notation for a function obtained from the equilibrium conditions and the variable x, which is a summary statistic for how the state vector (N, Z, L) drives the quality-adjusted gross cash flow of the firm, is constant in steady state.

In this light, replacing $z(x)$ with a blank reduces the critique to the tautological claim that a model of an exponential growth process must be based on the mathematical representation of an exponential growth process. What such a claim adds to our understanding of the model and of its contribution to the field is unclear. The economics of the model is about how it obtains such exponential representation, i.e., about what the blank is and where it comes from.

[11] Let me emphasize that the system becomes autonomous only *after* we kill population growth. The argument I am making is general and deals with time dependence as well, as I stressed in my definition of the term "constant".

[12] The only linearity assumption, based on the traditional replication argument, is constant returns to scale with respect to the rival factors of production X_i and L in the production function for the final good Y.

7. Conclusion

This paper has studied a generalization of Schumpeterian models with endogenous market structure that allows the overall production structure to be more than linear in the growth-driving factor. Despite the seeming explosiveness of the setup, the proposed generalization generates endogenous growth, defined as steady-state, constant, exponential growth of income per capita. The implication is that the Schumpeterian theory of endogenous innovation *does not* require a linear production structure to generate endogenous growth. This version of modern growth theory, therefore, is robust in the sense that its key result obtains for a thick set of parameter values instead of, as often claimed, for a set of measure zero.

The mechanism delivering this property is the market fragmentation process, originally studied by Peretto (1994) and Smulders (1994), that "tames" potentially explosive growth due to increasing returns that look excessive in terms of the traditional theory. To date, such increasing returns have been deemed impossible by a priori reasoning that extrapolates the properties of one-dimensional models. Such reasoning does not apply to models with two dimensions of technology that play interdependent, but distinct, roles: the vertical dimension provides the engine of growth, the horizontal dimension provides the endogenous market structure that sterilizes the market size effect and thereby its two key manifestations: (1) the strong scale effect, with the associated need to impose constant endowments to rule out explosive behavior due to external forcings, and (2) the need to impose a knife-edge condition that rules out explosive behavior due to internal "excessive" increasing returns.

In addition to this result, the proposed generalization yields some novel and deeper insights on the relation between the two dimensions of technology. For example, it shows that in steady state vertical innovation creates the opportunity for product proliferation because it expands the size of the market for all suppliers (old and new) of high-tech goods. In previous versions of the theory steady-state product proliferation requires growth of some endowment (e.g., population growth).

Appendix A

To facilitate the reader, all the equations from the text needed for the proofs are replicated in this document with self-contained numbering.

A1. Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the innovation, q_i, is equal to its cost,

$$1 = q_i \Leftrightarrow I_i > 0. \tag{A.1}$$

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$r = \frac{\partial \Pi_i}{\partial Z_i}\frac{1}{q_i} + \frac{\dot{q}_i}{q_i}. \tag{A.2}$$

To calculate the marginal profit, observe that the firm's problem is separable in the price and investment decisions. Facing the isoelastic demand

$$X_i = \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} Z_i^{\alpha} Z^{\kappa-\alpha}\frac{L}{N^{1-\sigma}}, \tag{A.3}$$

and a marginal cost of production equal to one, the firm sets $P_i = 1/\theta$. Substituting this result into the firm's cash flow,

$$\Pi_i = \left(\frac{1}{\theta} - 1\right)\theta^{\frac{2}{1-\theta}} Z_i^{\alpha} Z^{\kappa-\alpha}\frac{L}{N^{1-\sigma}} - \phi Z_i, \tag{A.4}$$

differentiating with respect to Z_i, substituting into (A.2) and imposing symmetry yields

$$r = \frac{\alpha}{Z_i}\cdot \underbrace{\left(\frac{1}{\theta} - 1\right)\theta^{\frac{2}{1-\theta}} Z_i^{\alpha} Z^{\kappa-\alpha}\frac{L}{N^{1-\sigma}}}_{(P_i-1)X_i} - \phi. \tag{A.5}$$

A2. Proof of Proposition 3

The proof is in several steps.

A2.1. Step 1: consumption/saving decision

Recall that consumption, production of intermediates, quality and variety innovation, and resource extraction are all in units of the final good so that the resource allocation problem of this economy is the allocation across its alternative uses of the quantity Y produced according to the technology

$$Y = \theta^{\frac{2\theta}{1-\theta}} \cdot N^{\sigma} Z^{\kappa} \cdot L. \tag{A.6}$$

The consumption flow that results from such allocation is:

$$\frac{C}{Y} \equiv c = \begin{cases} (1-\theta)\left[\theta\left(1 - \dfrac{\phi + z}{x}\right) + 1\right] & n = 0 \quad z \geq 0 \\ (1-\theta)[\theta(\rho - \lambda)\pi + 1] & n > 0 \quad z \geq 0 \end{cases}. \tag{A.7}$$

This equation is obtained as follows.

When $n > 0$ assets market equilibrium requires

$$A = NV = \beta\theta^2 Y, \tag{A.8}$$

which says that the wealth ratio A/Y is constant. This result and the saving schedule

$$r = \rho - \lambda + \dot{C}/C \tag{A.9}$$

allow me to rewrite the household budget

$$\dot{A} = rA + wL - C \tag{A.10}$$

as the following unstable differential equation in $c \equiv C/Y$:

$$0 = \rho - \lambda + \frac{\dot{c}}{c} + \frac{1 - \theta - c}{\beta\theta^2}.$$

Accordingly, to satisfy the transversality condition c jumps to the constant value $(\rho - \lambda)\beta\theta^2 + 1 - \theta$. Using the definition of π yields the bottom line of (A.7).

When $n = 0$ assets market equilibrium still requires $A = NV$ but it is no longer true that $V = \beta X$ since by definition the free-entry condition does not hold. This means that the wealth ratio A/Y is not constant. However, the relation

$$r = \frac{\Pi_i - I_i}{V_i} + \frac{\dot{V}_i}{V_i} \tag{A.11}$$

holds, since it is the arbitrage condition on equity holding that characterizes the value of an existing firm regardless of how it came into existence in the first place. Imposing symmetry and inserting (A.4), (A.11), and (A.8) into the household budget (A.10) yields

$$0 = N[(1/\theta - 1)X - \phi Z - I] + (1 - \theta)Y - C.$$

The definition of x, the R&D technology

$$\dot{Z}_i = I_i, \tag{A.12}$$

and the fact that $NX = \theta^2 Y$, allow me to rewrite this expression as the top line of (A.7).

With this result in hand, I now construct the functions $z(x)$ and $n(x)$.

A2.2. Step 2: innovation rates as functions of the state variable

Here I prove Propositions 1 and 2. I begin with the case $x_N < x_Z$ in Proposition 1 and then deal with the case $x_N > x_Z$ in Proposition 2.

Proposition 1. *The ratio c is constant when there is entry, i.e., when $n > 0$, and in such case the return to saving (A.9) becomes $r = \rho - \lambda + \dot{Y}/Y$. Therefore, we can use the expression for the return to entry,*

$$r = \frac{\Pi - I}{\beta X} + \frac{\dot{X}}{X}, \tag{A.13}$$

(A.4), (A.12) and the definition

$$x \equiv (1 - \theta)\frac{\theta Y}{NZ} = (1 - \theta)\theta^{\frac{1+\theta}{1-\theta}} \frac{Z^{\kappa - 1} L}{N^{1-\sigma}} \tag{A.14}$$

to obtain

$$n = \frac{x - \phi - z}{\pi x} - \rho + \lambda, \quad z \geq 0, \tag{A.15}$$

which holds for positive values of the right-hand side. The Euler Eq. (A.9) and the reduced-form production function (A.6) yield:

$$\begin{aligned} r &= \rho - \lambda + \dot{Y}/Y \\ &= \rho + \kappa z + \sigma n. \end{aligned}$$

Combining this expression with the return to quality

$$r = \alpha x - \phi \tag{A.16}$$

yields

$$\alpha x - \phi = \rho + \kappa z + \sigma n.$$

Combining this expression with the rate of entry in (A.15) and solving for z yields

$$z = \frac{\alpha x - \phi - \sigma \frac{x-\phi}{\pi x} - [\rho - \sigma(\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x}}.$$

Substituting this result back into (A.15) yields

$$\begin{aligned} n &= \frac{x - \phi - z}{\pi x} - \rho + \lambda \\ &= \frac{x-\phi}{\pi x} - \frac{z}{\pi x} - \rho + \lambda \\ &= \frac{(\kappa - \alpha)x - (\kappa - 1)\phi + [\rho - \sigma(\rho - \lambda)]}{\kappa \pi x - \sigma} - \rho + \lambda. \end{aligned}$$

Consider now the thresholds. Suppose $x_N < x_Z$. Then $n(x) > 0$ for

$$\frac{x-\phi}{\pi x} - \rho + \lambda > 0,$$

since $z = 0$, which yields

$$x > x_N \equiv \frac{\phi}{1 - \pi(\rho - \lambda)}.$$

On the other hand, $z(x) > 0$ for

$$\alpha x - \phi - \sigma \frac{x-\phi}{\pi x} - [\rho - \sigma(\rho - \lambda)] > 0,$$

because entry is already active, which yields

$$x > x_Z \equiv \arg \text{solve} \left\{ \alpha x - \phi - \sigma \frac{x-\phi}{\pi x} = \rho - \sigma(\rho - \lambda) \right\}.$$

The inequality

$$z(x_N) = \frac{\alpha x_N - \phi - \sigma \frac{x_N - \phi}{\pi x_N} - [\rho - \sigma(\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x_N}} < 0$$

identifies the region of parameter space such that $x_N < x_Z$.

Combining all of these results, I can write:

$$z(x) = \begin{cases} 0 & \phi \le x \le x_N \\ 0 & x_N < x \le x_Z \\ \dfrac{\alpha x - \phi - \sigma \frac{x-\phi}{\pi x} - [\rho - \sigma(\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x}} & x_z < x < \infty \end{cases};$$

$$n(x) = \begin{cases} 0 & \phi \le x \le x_N \\ \dfrac{x-\phi}{\pi x} - \rho + \lambda & x_N < x \le x_Z \\ \dfrac{(\kappa - \alpha)x - (\kappa - 1)\phi + [\rho - \sigma(\rho - \lambda)]}{\kappa \pi x - \sigma} - \rho + \lambda & x_z < x < \infty \end{cases},$$

where:

$$x_N \equiv \frac{\phi}{1-\pi(\rho-\lambda)};$$

$$x_Z \equiv \arg\text{solve}\left\{\alpha\kappa x-\phi-\sigma\frac{x-\phi}{\pi x}=\rho-\sigma(\rho-\lambda)\right\}.$$

Proposition 2. *As before, over the range $\phi \le x \le x_Z$ the function $c(x)$ is given by (A.7) evaluated at $z=0$. To characterize it over the range $x_Z < x \le x_N$, set the rate of return to vertical innovation equal to the reservation rate of return of savers to obtain:*

$$\rho-\lambda+\frac{\dot{c}}{c}+\kappa z+\lambda=\alpha x-\phi.$$

Solving the household budget constraint for z, yields

$$z=x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right).$$

Combining these two expressions yields

$$\frac{\dot{c}}{c}=(\kappa-\alpha)x-(\kappa-1)\phi+\kappa\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)-(\rho-\lambda)-\lambda.$$

The $\dot{c}\ge 0$ locus is thus

$$c\ge(1-\theta)\left[1+\theta\frac{\rho-(\kappa-\alpha)x+(\kappa-1)\phi}{\kappa x}\right].$$

In this region, the law of motion of x is

$$\begin{aligned}\frac{\dot{x}}{x}&=\lambda+(\kappa-1)z\\&=\lambda+(\kappa-1)\left[x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)\right].\end{aligned}$$

Recall, however, that $z\ge 0$ so that $\dot{x}/x$ is strictly positive. There is then a unique equilibrium trajectory: the economy jumps on the saddle path in (x, c) space that converges to (x^*, c^*) with smooth pasting. Writing

$$\frac{\dot{c}}{\dot{x}}=\frac{dc}{dx}=\frac{c}{x}\frac{(\kappa-\alpha)x-(\kappa-1)\phi-\rho+\kappa\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)}{\lambda+(\kappa-1)\left[x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)\right]}$$

yields a partial differential equation that doesn't have a closed-form solution. However, we can show that the function $\tilde{c}(x)$ that solves it has the same derivative from the left and the right at $x=x_Z$ and approaches the value c^* with zero derivative at $x=x_N$:

$$\begin{aligned}\frac{dc\left(x_Z^-\right)}{dx}&=\frac{dc\left(x_Z^+\right)}{dx};\\\frac{dc(x_N)}{dx}&=0.\end{aligned}$$

In other words, it is increasing, concave and has no kinks. The associated expression for z is

$$\tilde{z}(x)=x\left[1-\frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta}-1\right)\right]-\phi.$$

Once again, we can show that $\tilde{z}(x)$ starts out at $x=x_Z$ with zero derivative and approaches the line that holds for $x>x_N$ with positive derivative:

$$\begin{aligned}\frac{dz(x_Z)}{dx}&=1-\frac{1}{\theta}\left(\frac{c(x_Z)}{1-\theta}-1\right)-\frac{x_Z}{\theta}\frac{dc(x_Z)/dx}{1-\theta}=0;\\\frac{dz(x_N)}{dx}&=1-\frac{1}{\theta}\left(\frac{c(x_N)}{1-\theta}-1\right)-\frac{x_N}{\theta}\frac{dc(x_N)/dx}{1-\theta}>0.\end{aligned}$$

The function $\tilde{z}(x)$ exhibits a kink at $x=x_N$ because when entry begins quality innovation attracts only a fraction of the economy's saving flow, which is now a constant fraction of final output.

Combining all of these results, I can write:

$$z(x) = \begin{cases} 0 & \phi \le x \le x_Z \\ \tilde{z}(x) & x_Z < x \le x_N \\ \dfrac{\alpha x - \phi - \sigma \frac{x-\phi}{\pi x} - [\rho - \sigma(\rho - \lambda)]}{\kappa - \frac{\sigma}{\pi x}} & x_z < x < \infty \end{cases};$$

$$n(x) = \begin{cases} 0 & \phi \le x \le x_Z \\ 0 & x_Z < x \le x_N \\ \frac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa \pi x - \sigma} - \rho + \lambda & x_N < x < \infty \end{cases},$$

where:

$$\tilde{z}(x) = x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] - \phi$$

and $\tilde{c}(x)$ is the solution of the partial differential equation

$$\frac{dc}{dx} = \frac{\kappa c}{x} \frac{\frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right) - \frac{\rho}{\kappa} - (1-\alpha)x}{\lambda + (\kappa - 1)\left[x - \phi - \frac{x}{\theta}\left(\frac{c}{1-\theta} - 1\right)\right]}.$$

The thresholds are:

$$x_N \equiv \arg\text{solve}\left\{\frac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa\pi x - \sigma} = \rho - \lambda\right\};$$

$$x_Z \equiv \arg\text{solve}\left\{x\left[1 - \frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta} - 1\right)\right] = \phi\right\}.$$

The function $z(x)$ has zero derivative at $x = x_Z$, is increasing and has positive derivative at x_N.

According to these results, the only difference between the two cases is the middle region. With the functions $z(x)$ and $n(x)$ in hand, I can now prove the main result.

A2.3. Step 3: existence

After some algebra, the equation $\Psi(x) = 0$ yields

$$\Psi(x) = a_1 x^2 + a_2 x + a_3 = 0,$$

where:

$$a_1(\kappa) \equiv \alpha\pi(\kappa - 1) > 0;$$

$$a_2(\kappa) \equiv -(\kappa-1)(1+\phi\pi) - (1-\sigma)(1-\alpha) + [(1-\sigma)\rho + \lambda\sigma]\pi;$$

$$a_3(\kappa) \equiv (\kappa-1)\phi - [(1-\sigma)\rho + \lambda\sigma].$$

I am thus looking for values of κ such that $\Delta(\kappa) \equiv (a_2(\kappa))^2 - 4a_1(\kappa)a_3(\kappa) > 0$ and the quadratic equation has two solutions in the region $x > \max\{x_N, x_Z\}$. To obtain this result, it is sufficient to assume

$$a_3(\kappa) \le 0 \Rightarrow \kappa \le 1 + \frac{(1-\sigma)\rho + \lambda\sigma}{\phi}.$$

But I can go further. Brute force calculation yields

$$\begin{aligned} \Delta(\kappa) &= ((\kappa-1)(1+\phi\pi) + (1-\sigma)(1-\alpha) - [(1-\sigma)\rho + \lambda\sigma]\pi)^2 - 4\alpha\pi(\kappa-1)[(\kappa-1)\phi - [(1-\sigma)\rho + \lambda\sigma]] \\ &= \left[(1+\phi\pi)^2 - 4\alpha\pi\phi\right](\kappa-1)^2 + 2(\kappa-1)\{(1+\phi\pi)(1-\sigma)(1-\alpha) - (1+\phi\pi - 2\alpha)[(1-\sigma)\rho + \lambda\sigma]\pi\} \\ &\quad + [(1-\sigma)(1-\alpha) - [(1-\sigma)\rho + \lambda\sigma]\pi]^2. \end{aligned}$$

I thus have a quadratic equation in κ,

$$\Delta(\kappa) = b_1(\kappa-1)^2 + b_2(\kappa-1) + b_3,$$

where:

$$b_1 = (1+\phi\pi)^2 - 4\alpha\pi\phi;$$

$$b_2 = 2\{(1+\phi\pi)(1-\sigma)(1-\alpha) - (1+\phi\pi - 2\alpha)[(1-\sigma)\rho + \lambda\sigma]\pi\};$$

$$b_3 = [(1-\sigma)(1-\alpha) - [(1-\sigma)\rho + \lambda\sigma]\pi]^2 > 0.$$

Since $b_3 > 0$, $\Delta(1) > 0$. To obtain a condition such that $\Delta(\kappa)$ is always positive, I look for parameters such that this quadratic equation has negative delta, that is, for parameters that satisfy

$$b_2^2 - 4b_1 b_3 < 0.$$

This inequality can hold only if $b_1 > 0$, that is, only if $(1+\phi\pi)^2 > 4\alpha\pi\phi$. If this condition fails so that $b_1 < 0$, we have that $\Delta(\kappa) > 0$ for

$$1 \le \kappa < 1 + \frac{b_2 + \sqrt{b_2^2 - 4b_1 b_3}}{-2b_1} \equiv \kappa^{\max}.$$

To summarize: there always exists a finite value $\kappa^{\max} > 1$ such that for $1 \le \kappa < \kappa^{\max}$ there exists a steady state with constant endogenous growth. If in addition $(1+\phi\pi)^2 > 4\alpha\pi\phi$, then there exists a region of parameter space where $\kappa^{\max} \to \infty$ and the steady state with constant endogenous growth exists for all $\kappa \ge 1$.

A2.4. Step 4: Stability

Figs. 3 and 4 in the text illustrate the dynamics. Consider first the case $x_N < x_Z$, in which the economy activates first variety innovation. For $x \le x_N < x_Z$ the growth rate of profitability is $\dot{x}/x = \lambda$ and the economy crosses the threshold for entry in finite time. For $x_N < x < x_Z$ the growth rate is

$$\frac{\dot{x}}{x} = \lambda - (1-\sigma)\left(\frac{x-\phi}{\pi x} - \rho + \lambda\right).$$

This expression identifies a steady-state value

$$x_N^* \equiv \arg\text{solve}\left\{(1-\sigma)\frac{x-\phi}{\pi x} = \rho - \sigma(\rho - \lambda)\right\}.$$

Now recall that

$$x_Z \equiv \arg\text{solve}\left\{\alpha x - \phi - \sigma\frac{x-\phi}{\pi x} = \rho - \sigma(\rho-\lambda)\right\}$$

and note that a *sufficient* condition for $x_Z < x_N^*$ is that $x_N^* \to \infty$, that is, that

$$(1-\sigma)\frac{x-\phi}{\pi x} \le \rho - \sigma(\rho-\lambda) \quad \forall x \in (\phi, \infty).$$

Since the left-hand side is an increasing, concave, bounded above function, the condition we seek is

$$(1-\sigma)\frac{1}{\pi} \le \rho - \sigma(\rho-\lambda) \Rightarrow \frac{1-\sigma}{\rho - \sigma(\rho-\lambda)} \le \pi.$$

Interestingly, this condition does not depend on κ, since we are looking for parameter combinations that boost incentives to variety growth when quality growth is still zero. The intuition for this condition is that it prevents premature market saturation.

The case where $x_N < x_Z$ features an acceleration of the rate of growth of profitability at $x = x_Z$ so that the economy crosses the threshold x_N in finite time. I conclude, therefore, that the condition stated in the proposition is sufficient for convergence to the steady state x^* for any initial condition $x(0) \in (\phi, \bar{x})$.

For the comparison with the existing literature, it is instructive to look at

$$\lim_{x\to\infty}\Psi(x) = \lim_{x\to\infty}[\lambda + (\kappa-1)z(x) - (1-\sigma)n(x)],$$

where, using the expressions above:

$$\lim_{x\to\infty} z(x) = \lim_{x\to\infty}\frac{\alpha x - \phi - \sigma\frac{x-\phi}{\pi x} - [\rho - \sigma(\rho-\lambda)]}{\kappa - \frac{\sigma}{\pi x}} = \lim_{x\to\infty}\alpha x;$$

$$\lim_{x\to\infty} n(x) = \lim_{x\to\infty}\frac{(\kappa-\alpha)x - (\kappa-1)\phi + [\rho - \sigma(\rho-\lambda)]}{\kappa\pi x - \sigma} - \rho + \lambda = \lim_{x\to\infty}\frac{\kappa-\alpha}{\pi} - \rho + \lambda.$$

This concludes the long proof of Proposition 3.

A3. A CIES economy

I now sketch the results for a generic CIES economy with welfare function

$$U(0)=\int_0^{\infty} e^{-(\rho-\lambda)t}\frac{1}{1-\eta}\left[\left(\frac{C(t)}{L(t)}\right)^{1-\eta}-1\right]dt, \quad \rho>\lambda\geq 0,\ \eta>0.$$

The associated Euler equation is

$$r=\rho+\eta\left(\frac{\dot{C}}{C}-\lambda\right).$$

I shall focus this discussion on the steady-state properties of this generalization for two reasons. First, it is sufficient to make the point that the robust endogenous growth studied in the paper is in fact a general property and that the assumption of log-utility in the main text just a simplification. Second, the analysis of the transitional dynamics in the general case is much more cumbersome because the model looses the nice feature that the ratio $C/Y\equiv c$ is constant at all times and therefore one needs to study the dynamical system in two dimensions, c and x. The main difficulty is that the thresholds for activation of vertical innovation become non-linear locus in (x, c) space. The phase diagram is doable but much more cumbersome to present. The key advantage of the log-utility specification, therefore, is that it yields a transparent characterization of the dynamics.

To work out the steady state of the CIES economy, I start by using the reduced-form production function (A.6) to rewrite the Euler equation as

$$r=\rho+\eta\left(\frac{\dot{Y}}{Y}-\lambda\right)+\eta\left(\frac{\dot{C}}{C}-\frac{\dot{Y}}{Y}\right)=\rho+\eta(\kappa z+\sigma n)+\eta\frac{\dot{c}}{c}. \tag{A.17}$$

In steady state c is constant and I can proceed as in the proof of Proposition 3. Combining (A.17) with the return to quality (A.16) yields

$$\alpha x-\phi=\rho+\eta(\kappa z+\sigma n). \tag{A.18}$$

The return to entry (A.13), the definition of x in (A.14) and the Euler equation (A.17) yield

$$n=\frac{x-\phi-z}{\pi x}-\rho+\lambda+(1-\eta)(\kappa z+\sigma n), \quad z\geq 0.$$

Solving this expression for n yields

$$n=\frac{\frac{x-\phi}{\pi x}-\rho+\lambda}{1-(1-\eta)\sigma}+\frac{\kappa(1-\eta)-1/\pi x}{1-(1-\eta)\sigma}z.$$

Substituting this result in (A.18) and solving for z yields

$$z(x)=\frac{(\alpha x-\phi-\rho)[1-(1-\eta)\sigma]-\eta\sigma\left(\frac{x-\phi}{\pi x}-\rho+\lambda\right)}{\eta(\kappa-\sigma/\pi x)}.$$

Substituting this solution back in the expression for n yields

$$\begin{aligned}n(x)&=\frac{\frac{x-\phi}{\pi x}-\rho+\lambda}{1-(1-\eta)\sigma}+\frac{\kappa(1-\eta)-1/\pi x}{1-(1-\eta)\sigma}z(x)\\&=\left(\frac{x-\phi}{\pi x}-\rho+\lambda\right)\frac{\kappa}{\kappa-\sigma/\pi x}+\frac{\kappa(1-\eta)-1/\pi x}{\eta(\kappa-\sigma/\pi x)}(\alpha x-\phi-\rho).\end{aligned}$$

The definition of x in (A.14) and the reduced-form production function (A.6) yield

$$\frac{\dot{x}}{x}=\lambda+(\kappa-1)z(x)-(1-\sigma)n(x).$$

The equation

$$0=\lambda+(\kappa-1)z(x)-(1-\sigma)n(x),$$

yields again a quadratic form $0=a_1x^2+a_2x+a_3$, where:

$$a_1(\kappa)\equiv[\kappa\eta-1+(1-\eta)\sigma]\alpha\pi;$$

$$a_2(\kappa)\equiv[[\kappa\eta-1+(1-\eta)\sigma]\phi+\rho(1-\sigma)+\eta\sigma\lambda]\pi+(1-\sigma)\alpha-\eta(\kappa-\sigma);$$

$$a_3(\kappa)\equiv(\eta\kappa-\eta\sigma-1+\sigma)\phi-[\eta\lambda\sigma+(1-\sigma)\rho].$$

The problem thus has the same structure as that studied for the log-utility case and delivers a similar result. (The reader can check that setting $\eta = 1$ yields exactly the equation studied above.)

If $x_N < x_Z$ the threshold for $n(x) > 0$ is the same as in the log-utility case since the function $n(x)$ reduces to

$$n(x) = \frac{\frac{x-\phi}{\pi x} - \rho + \lambda}{1-(1-\eta)\sigma}$$

and thus yields

$$x > x_N \equiv \frac{\phi}{1-\pi(\rho-\lambda)}.$$

Then, $z(x) > 0$ for

$$(\alpha x - \phi - \rho)[1-(1-\eta)\sigma] - \eta\sigma\left(\frac{x-\phi}{\pi x} - \rho + \lambda\right) > 0,$$

which yields

$$x > x_Z \equiv \arg\text{solve}\left\{1-(1-\eta)\sigma = \frac{\eta\sigma\left(\frac{x-\phi}{\pi x} - \rho + \lambda\right)}{\alpha x - \phi - \gamma\rho}\right\}.$$

The assumption

$$z(x_N) = \frac{[\alpha(x_N - \phi) - \rho][1-(1-\eta)\sigma] - \eta\sigma\left(\frac{x_N-\phi}{\pi x_N} - \rho + \lambda\right)}{\eta(\kappa - \sigma/\pi x_N)} < 0$$

ensures that $x_N < x_Z$.

A4. Derivation of the returns to innovation in the extended model of Section 6

The typical firm's Hamiltonian is:

$$CVH_i = \left(\frac{1}{\theta} - 1\right)\theta^{\frac{2}{1-\theta}} Z_i^\alpha Z^{\kappa-\alpha}\frac{L}{N^{1-\sigma}} - \phi Z_i D(Z_i; Z, N) - I_i + q_i I_i \frac{1}{D(Z_i; Z, N)}.$$

This yields:

$$r = \frac{\partial \Pi_i}{\partial Z_i}\frac{1}{q_i} - \frac{\partial D(Z_i; Z, N)}{\partial Z_i}\frac{Z_i}{D(Z_i; Z, N)}\frac{I_i}{Z_i D(Z_i; Z, N)} + \frac{\dot{q}_i}{q_i}, \quad q_i = D(Z_i; Z, N),$$

where

$$\frac{\partial \Pi_i}{\partial Z_i} = \frac{\partial\left(Z_i^\alpha Z^{\kappa-\alpha}\right)/\partial Z_i}{Z_i^\alpha Z^{\kappa-\alpha}} \cdot \underbrace{\left(\frac{1}{\theta} - 1\right)\theta^{\frac{2}{1-\theta}} Z_i^\alpha Z^{\kappa-\alpha}\frac{L}{N^{1-\sigma}}}_{(P_i-1)X_i} - \phi\left(D(Z_i; Z, N) + Z_i\frac{\partial D(Z_i; Z, N)}{\partial Z_i}\right)$$

$$= \alpha\frac{(P_i-1)X_i}{Z_i} - \phi\left(D(Z_i; Z, N) + Z_i\frac{\partial D(Z_i; Z, N)}{\partial Z_i}\right).$$

With the functional form

$$D(Z_i; Z, N) = Z_i^{\delta_1} Z^{\delta_2} N^{\delta_3}$$

I have:

$$r = \alpha\kappa\frac{(P_i-1)X_i}{D(Z_i; Z, N)} - \delta_1 z_i - \phi(1+\delta_1) + \frac{\dot{q}_i}{q_i}, \quad q_i = D(Z_i; Z, N);$$

$$r = \frac{(P_i-1)X_i - \phi Z_i D(Z_i; Z, N) - I_i}{V_i} + \frac{\dot{V}_i}{V_i}, \quad V_i = \frac{\beta Y}{N}.$$

The price-dividend ratio in the return to entry can be written:

$$\frac{(P_i-1)X_i - \phi Z_i D(Z_i; Z, N) - I_i}{V_i} = \frac{\frac{(P_i-1)X_i}{Z_i D(Z_i;Z,N)} - \frac{F_i+I_i}{Z_i D(Z_i;Z,N)}}{\frac{\beta Y}{N}} Z_i D(Z_i; Z, N)$$

$$= \frac{\frac{(P_i-1)X_i}{Z_i D(Z_i;Z,N)} - \phi - z_i}{\frac{\beta Y}{N}} Z_i D(Z_i; Z, N).$$

Imposing symmetry and recalling that $NPX = \theta Y$ yields:

$$r = \alpha \frac{(P-1)X}{ZD(Z;Z,N)} - \delta_1 z - \phi(1+\delta_1) + \frac{\dot{q}}{q}, \quad q = D(Z;Z,N) = Z^{\delta_1+\delta_2} N^{\delta_3};$$

$$r = \left[1 - \frac{\phi + z}{\frac{(P-1)X}{ZD(Z;Z,N)}} \right] \frac{\theta(P-1)}{\beta P} + \frac{\dot{V}}{V}, \quad V = \frac{\beta Y}{N}.$$

These show that we have the same mechanism as the basic model with the only difference that we now define

$$x \equiv \frac{(P-1)X}{ZD(Z;Z,N)} = \frac{(P-1)X}{Z^{\delta_1+\delta_2} N^{\delta_3}}.$$

Concavity of the revenue function holds for

$$\frac{\partial^2}{\partial Z_i^2} \left[\left(\frac{1}{\theta} - 1 \right) \theta^{\frac{2}{1-\theta}} \frac{LZ^{\kappa-\alpha}}{N^{1-\sigma}} \cdot Z_i^{\alpha} - \phi Z^{\delta_2} N^{\delta_3} \cdot Z_i^{1+\delta_1} \right] < 0,$$

that is, for

$$\left(\frac{1}{\theta} - 1 \right) \theta^{\frac{2}{1-\theta}} \frac{LZ^{\kappa-\alpha}}{N^{1-\sigma}} \cdot \alpha(\alpha-1) Z_i^{\alpha-2} - \phi Z^{\delta_2} N^{\delta_3} \cdot (1+\delta_1)\delta_1 Z_i^{\delta_1-1} < 0.$$

We thus get the sufficient condition:

$$\alpha \leq 1.$$

Quasi-convexity of the innovation plus management cost component holds for $\delta_1 \geq 0$. Nothing else is needed.

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