# Robust Endogenous Growth: Appendix 

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## A Derivations and proofs

To facilitate the reader, all the equations from the text needed for the proofs are replicated in this document with self-contained numbering.

## A. 1 Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R\&D up to the point where the shadow value of the innovation, $q_{i}$, is equal to its cost,

$$
\begin{equation*}
1=q_{i} \Leftrightarrow I_{i}>0 . \tag{A.1}
\end{equation*}
$$

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$
\begin{equation*}
r=\frac{\partial \Pi_{i}}{\partial Z_{i}} \frac{1}{q_{i}}+\frac{\dot{q}_{i}}{q_{i}} . \tag{A.2}
\end{equation*}
$$

To calculate the marginal profit, observe that the firm's problem is separable in the price and investment decisions. Facing the isoelastic demand

$$
\begin{equation*}
X_{i}=\left(\frac{\theta}{P_{i}}\right)^{\frac{1}{1-\theta}} Z_{i}^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}} \tag{A.3}
\end{equation*}
$$

and a marginal cost of production equal to one, the firm sets $P_{i}=1 / \theta$. Substituting this result into the firm's cash flow,

$$
\begin{equation*}
\Pi_{i}=\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} Z_{i}^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}-\phi Z_{i} \tag{A.4}
\end{equation*}
$$

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differentiating with respect to $Z_{i}$, substituting into (A.2) and imposing symmetry yields

$$
\begin{equation*}
r=\frac{\alpha}{Z_{i}} \cdot \underbrace{\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} Z_{i}^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}}_{\left(P_{i}-1\right) X_{i}}-\phi . \tag{A.5}
\end{equation*}
$$

## A. 2 Solution of the baseline model in Section 3

Recall that this economy allocates across its alternative uses - consumption, production of intermediates, quality and variety innovation - final output produced with the technology

$$
\begin{equation*}
Y=\theta^{\frac{2 \theta}{1-\theta}} \cdot N^{\sigma} Z^{\kappa} \cdot L \tag{A.6}
\end{equation*}
$$

The main challenge in studying the dynamics resulting from this allocation is the existence of corner solutions associated to the non-negativity constraints on vertical and horizontal R\&D. It is thus useful to proceed as follows.

Consider first a situation where $n>0$ and the free-entry condition holds. Asset market equilibrium yields $A=N V=N \cdot \beta Z$ so that $\dot{A} / A=n+z$. This result, the household budget,

$$
\begin{equation*}
\dot{A}=r A+w L-C, \tag{A.7}
\end{equation*}
$$

the equilibrium labor income, $w L=(1-\theta) Y$, and the definitions

$$
\begin{equation*}
x \equiv(1-\theta) \frac{\theta Y}{N Z}=(1-\theta) \theta^{\frac{1+\theta}{1-\theta}} \frac{Z^{\kappa-1} L}{N^{1-\sigma}} \tag{A.8}
\end{equation*}
$$

and $c \equiv C / Y$, yield

$$
\begin{equation*}
\frac{\dot{A}}{A}=r+\frac{w L}{A}-\frac{C}{A} \Rightarrow n+z=r+\frac{1}{\beta} \frac{x(1-\theta-c)}{\theta(1-\theta)} . \tag{A.9}
\end{equation*}
$$

This is just the economy's resource constraint written in simple-to-use terms. Equalization of the returns to quality and variety innovation yields

$$
z(x)=\left\{\begin{array}{cc}
0 & \phi \leq x \leq x_{Z} \equiv \frac{\beta-1}{\beta \alpha-1} \phi  \tag{NA}\\
\frac{\beta \alpha-1}{\beta-1} x-\phi & x>x_{Z}
\end{array}\right.
$$

The associated expression for the return to innovation is

$$
r(x)=\left\{\begin{array}{cc}
\frac{1}{\beta}(x-\phi) & \phi \leq x \leq x_{Z}  \tag{A.10}\\
\alpha x-\phi & x>x_{Z}
\end{array} .\right.
$$

The functions $z(x)$ and $r(x)$ take into account the non-negativity constraint on quality $\mathrm{R} \& \mathrm{D}$ and they have been derived using the free-entry condition $V=\beta Z$, that is, under the assumption $n>0$. Solving the resource constraint (A.9) for $n$ yields

$$
n(x, c)=\left\{\begin{array}{cc}
\frac{1}{\beta}\left[\left(1+\frac{1}{\theta}\right) x-\phi-\frac{c}{\theta(1-\theta)} x\right] & \phi \leq x \leq x_{Z}  \tag{A.11}\\
{\left[\frac{1-\alpha}{\beta-1}+\frac{1}{\beta \theta}-\frac{c}{\beta \theta(1-\theta)}\right] x} & x>x_{Z} .
\end{array} .\right.
$$

This equation identifies the locus where $n>0$, that is,

$$
n(x, c)>0 \Longleftrightarrow c<\bar{c}(x) \equiv\left\{\begin{array}{cc}
\theta(1-\theta)\left(\frac{1}{\theta}+1-\frac{\phi}{x}\right) & \phi \leq x \leq x_{Z} \\
\theta(1-\theta)\left(\frac{1}{\theta}+\beta \frac{1-\alpha}{\beta-1}\right) & x>x_{Z}
\end{array}\right.
$$

For values of $c \geq \bar{c}(x)$, we have $n=0$.
Recall now the Euler equation

$$
r=\rho+\eta\left(\frac{\dot{C}}{C}-\lambda\right) .
$$

The reduced-form production function (A.6) yields

$$
\begin{equation*}
r=\rho+\eta\left(\frac{\dot{Y}}{Y}-\lambda\right)+\eta\left(\frac{\dot{C}}{C}-\frac{\dot{Y}}{Y}\right)=\rho+\eta(\kappa z+\sigma n)+\eta \frac{\dot{c}}{c} . \tag{A.12}
\end{equation*}
$$

The definition of $x$ in (A.8) and the reduced-form production function (A.6) yield

$$
\begin{equation*}
\frac{\dot{x}}{x}=\lambda+(\kappa-1) z-(1-\sigma) n . \tag{A.13}
\end{equation*}
$$

Using the functions derived above, one obtains the dynamical system in $(x, c)$ space that holds for $c<\bar{c}(x)$. The steady-state loci are:

$$
\begin{gathered}
\dot{c}=0 \quad r(x)=\rho+\eta[\kappa z(x)+\sigma n(x, c)] ; \\
\dot{x}=0 \quad 0=\lambda+(\kappa-1) z(x)-(1-\sigma) n(x, c) .
\end{gathered}
$$

The expressions just derived provide the ingredients used in the construction of Proposition 1.
To complete the characterization of dynamics, consider now what happens for $c \geq \bar{c}(x)$. First note that by construction values $c>\bar{c}(x)$ violate the economy's resource constraint and are thus unfeasible. Therefore, the no-entry region is actually $c=\bar{c}(x)$, i.e., the boundary of the unfeasible region. In the no-entry region assets market equilibrium still requires $A=N V$ but it is no longer true that $V=\beta Z$ since by definition the free-entry condition does not hold. However, the relation

$$
\begin{equation*}
r=\frac{\Pi_{i}-I_{i}}{V_{i}}+\frac{\dot{V}_{i}}{V_{i}} \tag{A.14}
\end{equation*}
$$

holds, since it is the arbitrage condition on equity holding that characterizes the value of an existing firm regardless of how it came into existence in the first place. Imposing symmetry and substituting (A.4), $w L=(1-\theta) Y$, and (A.14) in the household budget (A.7) yields

$$
\frac{\dot{V}}{V}=\frac{\Pi-I}{V}+\frac{\dot{V}}{V}+\frac{w L}{N V}-\frac{C}{N V} \Rightarrow 0=N[(1 / \theta-1) X-\phi Z-I]+(1-\theta) Y-C .
$$

This is the resource constraint when investment in entry is zero. One can rewrite this expression as

$$
z(x, c)=\left\{\begin{array}{cc}
0 & \phi \leq x \leq x_{Z} \\
{\left[1-\left(\frac{c}{1-\theta}-1\right) \frac{1}{\theta}\right] x-\phi} & x>x_{Z}
\end{array} .\right.
$$

This is the equivalent of the no-arbitrage locus in equation (NA) that holds in the no-entry region. Next, substituting the return to quality R\&D $r=\alpha x-\phi$ in the Euler equation yields

$$
\alpha x-\phi=\rho+\eta \kappa z(x, c)+\eta \frac{\dot{c}}{c} .
$$

Similarly, rewriting equation (A.13) yields

$$
\frac{\dot{x}}{x}=\lambda+(\kappa-1) z(x, c) .
$$

This equation says that in the no-entry region the growth rate of $x$ is strictly positive so that the economy will eventually leave it.

The dynamics of the system are as follows. Depending on the relative positions of the stationary loci derived above, the process can feature either an activation sequence whereby the economy turns on variety innovation first and quality innovation later, or an activation sequence whereby the economy turns on quality innovation first and variety innovation later. The paper provides a taxonomic summary of the possible cases.

## A. 3 Solution of the alternative specification in Section 5

For expositional purposes, the paper presents the steady-state result in Proposition 2 separately from the dynamics summarized in Proposition 3-5. In particular, Proposition 2 applies to the CIES model with $\eta \neq 1$ while Propositions apply to the simpler case $\eta=1$ (log utility).

## A.3.1 The CIES economy

As mentioned in the text, the analysis of the transitional dynamics in this case is much more cumbersome because the model looses the nice feature that the ratio $C / Y \equiv c$ is constant at all times and therefore one needs to study the dynamical system in two dimensions, $c$ and $x$. The baseline model of Section 2 shares this property but the fact that under no arbitrage it yields equations (NA) and (A.10) simplifies things. The main difficulty in the specification of Section 3 is that the thresholds for activation of vertical innovation become non-linear loci in $(x, c)$ space. The phase diagram is doable but much more cumbersome to present. The key advantage of the log-utility specification, therefore, is that it yields a transparent characterization of the dynamics.

To obtain the main claim of Proposition 2, consider a situation where both $z$ and $n$ are positive. In steady state $c$ is constant and one can combine the Euler equation (A.12) with the return to quality (A.22) to obtain

$$
\begin{equation*}
\alpha x-\phi=\rho+\eta(\kappa z+\sigma n) . \tag{A.15}
\end{equation*}
$$

The return to entry (A.20), the definition of $x$ in (A.8) and the Euler equation (A.12) yield

$$
n=\frac{x-\phi-z}{\pi x}-\rho+\lambda+(1-\eta)(\kappa z+\sigma n) .
$$

Solving this expression for $n$ yields

$$
n=\frac{\frac{x-\phi}{\pi x}-\rho+\lambda}{1-(1-\eta) \sigma}+\frac{\kappa(1-\eta)-1 / \pi x}{1-(1-\eta) \sigma} z .
$$

Substituting this result in (A.15) and solving for $z$ yields

$$
z(x)=\frac{(\alpha x-\phi-\rho)[1-(1-\eta) \sigma]-\eta \sigma\left(\frac{x-\phi}{\pi x}-\rho+\lambda\right)}{\eta(\kappa-\sigma / \pi x)} .
$$

Substituting this solution back in the expression for $n$ yields

$$
\begin{aligned}
n(x) & =\frac{\frac{x-\phi}{\pi x}-\rho+\lambda}{1-(1-\eta) \sigma}+\frac{\kappa(1-\eta)-1 / \pi x}{1-(1-\eta) \sigma} z(x) \\
& =\left(\frac{x-\phi}{\pi x}-\rho+\lambda\right) \frac{\kappa}{\kappa-\sigma / \pi x}+\frac{\kappa(1-\eta)-1 / \pi x}{\eta(\kappa-\sigma / \pi x)}(\alpha x-\phi-\rho)
\end{aligned}
$$

The main simplification in obtaining these two functions $z(x)$ and $n(x)$ is that the system is in steady state. To obtain the out-of-steady-state dynamics requires a lot of algebraic work that is not particularly illuminating.

## A.3.2 Proof of Proposition 2

The steady state is the solution of the equation

$$
\frac{\dot{x}}{x}=\lambda+(\kappa-1) z(x)-(1-\sigma) n(x)=0 .
$$

Using the two functions $z(x)$ and $n(x)$ just derived, some algebra reduces the steady-state condition to the quadratic form

$$
a_{1} x^{2}+a_{2} x+a_{3}=0,
$$

$$
\begin{aligned}
& \text { where: } \\
& \qquad \begin{array}{c}
a_{1}(\kappa) \equiv(\kappa-1) \alpha \pi-\alpha \pi \frac{(1-\eta)(1-\sigma)}{\eta} ; \\
a_{2}(\kappa) \equiv-(1+\phi \pi)(\kappa-1)+\left[(1-\sigma) \frac{\alpha}{\eta}-(1-\sigma)-\sigma(\rho-\lambda) \pi+\rho \pi+(1-\sigma)(\phi+\rho) \pi\left(\frac{1}{\eta}-1\right)\right] ; \\
a_{3}(\kappa) \equiv \phi(\kappa-1)-\left[(1-\sigma) \phi\left(\frac{1}{\eta}-1\right)+(1-\sigma) \rho \frac{1}{\eta}+\sigma \lambda\right] .
\end{array}
\end{aligned}
$$

To establish existence of the steady state, thus, one needs to prove that there exist values of $\kappa$ such that $\Delta(\kappa) \equiv\left(a_{2}(\kappa)\right)^{2}-4 a_{1}(\kappa) a_{3}(\kappa)>0$ and the quadratic equation has two solutions in the region $x>\max \left\{x_{N}, x_{Z}\right\}$. Brute force calculation yields

$$
\Delta(\kappa)=b_{1}(\kappa-1)^{2}+b_{2}(\kappa-1)+b_{3},
$$

where:

$$
b_{1}=(1+\phi \pi)^{2}-4 \alpha \pi \phi ;
$$

$$
\begin{aligned}
b_{2}= & 4 \alpha \pi \frac{(1-\eta)(1-\sigma)}{\eta} \phi \\
& +4 \alpha \pi\left[(1-\sigma) \phi\left(\frac{1}{\eta}-1\right)+(1-\sigma) \rho \frac{1}{\eta}+\sigma \lambda\right] \\
- & 2\left[(1-\sigma) \frac{\alpha}{\eta}-(1-\sigma)-\sigma(\rho-\lambda) \pi+\rho \pi+(1-\sigma)(\phi+\rho) \pi\left(\frac{1}{\eta}-1\right)\right](1+\phi \pi) ; \\
b_{3}= & {\left[(1-\sigma) \frac{\alpha}{\eta}-(1-\sigma)-\sigma(\rho-\lambda) \pi+\rho \pi+(1-\sigma)(\phi+\rho) \pi\left(\frac{1}{\eta}-1\right)\right]^{2} } \\
& -4 \alpha \pi \frac{(1-\eta)(1-\sigma)}{\eta}\left[(1-\sigma) \phi\left(\frac{1}{\eta}-1\right)+(1-\sigma) \rho \frac{1}{\eta}+\sigma \lambda\right] .
\end{aligned}
$$

With this expression in hand, one can obtain a condition such that $\Delta(\kappa)$ is always positive so that the $\dot{x}=0$ equation surely has two real solutions.

Note now that $\Delta(1)=b_{3}$. Therefore, one can simply look for parameters such that (i) $b_{3}>0$ and (ii) this quadratic equation in $\kappa$ has negative determinant, that is, for parameters that satisfy $b_{2}^{2}-4 b_{1} b_{3}<0$. Note also that by construction, $\Delta(1)=b_{3}>0$ is the condition for existence of the steady state under $\kappa=1$. It would thus make sense to maintain it to study the model's robustness to $\kappa>1$. It is nevertheless interesting to consider also the case $b_{3}<0$ to fully understand the existence argument.

The key to the argument is that restricting the coefficients of the quadratic form $\Delta(\kappa)$ does not involve $\kappa$ itself. Therefore, it is always possible to choose values of the other parameters that yield $\Delta(\kappa)>0$. Specifically, we have that $\Delta(\kappa)>0$ for

$$
1 \leq \kappa<1+\frac{-b_{2}+\sqrt{b_{2}^{2}-4 b_{1} b_{3}}}{2 b_{1}} \equiv \kappa^{\max }
$$

Inspecting this expression, allows one to obtain the following pattern.
Note first that $b_{1}>0$ for $(1+\phi \pi)^{2}>4 \alpha \pi \phi$. So, under this condition one can set parameters such that $b_{2}^{2}-4 b_{1} b_{3}<0$ and $\Delta(\kappa)>0$ because the quadratic equation does not have solutions in the range $\kappa \geq 1$. If the condition fails so that $b_{1}<0$, then to obtain $\Delta(\kappa)>0$ one must set parameters such that $b_{2}^{2}-4 b_{1} b_{3}>0$ and the quadratic equation has two solutions.

To understand better the above result, think of two cases. 1) For $b_{1}>0$ we are working with a convex parabola and $\Delta(\kappa)>0$ for $(\kappa-1)<r_{1}$ and $(\kappa-1)>r_{2}$ where

$$
r_{1,2}=\frac{-b_{2} \pm \sqrt{b_{2}^{2}-4 b_{1} b_{3}}}{2 b_{1}}
$$

If the roots do not exist, then $\Delta(\kappa)>0$ for all $\kappa \geq 1$.2) For $b_{1}<0$ the parabola is concave and $\Delta(\kappa)>0$ for $r_{1}<(\kappa-1)<r_{2}$.

To summarize: there always exists a finite value $\kappa^{\max }>1$ such that for $1 \leq \kappa<\kappa^{\max }$ there exists a steady state with constant endogenous growth. If in addition $(1+\phi \pi)^{2}>4 \alpha \pi \phi$, then there exists a region of parameter space where $\kappa^{\max } \rightarrow \infty$ and the steady state with constant endogenous growth exists for all $\kappa \geq 1$.

## A.3.3 Proof of Propositions 3-5

As said, the specification with $\eta=1$ yields tractable dynamics. It is useful to proceed in steps.
Step 1: consumption/saving decision The key to the tractable dynamics is that the equilibrium consumption flow is:

$$
\frac{C}{Y} \equiv c=\left\{\begin{array}{ccc}
(1-\theta)\left[\theta\left(1-\frac{\phi+z}{x}\right)+1\right] & n=0 & z \geq 0  \tag{A.16}\\
(1-\theta)[\theta(\rho-\lambda) \pi+1] & n>0 & z \geq 0
\end{array} .\right.
$$

This equation is obtained as follows.
When $n>0$ assets market equilibrium requires

$$
\begin{equation*}
A=N V=\beta \theta^{2} Y \tag{A.17}
\end{equation*}
$$

which says that the wealth ratio $A / Y$ is constant. This result and the saving schedule

$$
\begin{equation*}
r=\rho-\lambda+\dot{C} / C \tag{A.18}
\end{equation*}
$$

allow one to rewrite the household budget as the following unstable differential equation in $c \equiv C / Y$ :

$$
0=\rho-\lambda+\frac{\dot{c}}{c}+\frac{1-\theta-c}{\beta \theta^{2}} .
$$

Accordingly, to satisfy the transversality condition $c$ jumps to the constant value $(\rho-\lambda) \beta \theta^{2}+1-\theta$. Using the definition of $\pi$ yields the bottom line of (A.16).

When $n=0$ assets market equilibrium still requires $A=N V$ but it is no longer true that $V=\beta X$ since by definition the free-entry condition does not hold. This means that the wealth ratio $A / Y$ is not constant. However, one can proceed as in the previous case and obtain again

$$
0=N[(1 / \theta-1) X-\phi Z-I]+(1-\theta) Y-C .
$$

The definition of $x$, the $\mathrm{R} \& \mathrm{D}$ technology

$$
\begin{equation*}
\dot{Z}_{i}=I_{i}, \tag{A.19}
\end{equation*}
$$

and the fact that $N X=\theta^{2} Y$, allow me to rewrite this expression as the top line of (A.16).

Step 2: innovation rates as functions of the state variable We begin with the case $x_{N}<x_{Z}$ and then deal with the case $x_{N}>x_{Z}$.

Proposition 3 The ratio $c$ is constant when there is entry, i.e., when $n>0$, and in such case the return to saving (A.18) becomes $r=\rho-\lambda+\dot{Y} / Y$. Therefore, one can use the expression for the return to entry,

$$
\begin{equation*}
r=\frac{\Pi-I}{\beta X}+\frac{\dot{X}}{X}, \tag{A.20}
\end{equation*}
$$

(A.4), (A.19) and the definition of $x$ to obtain

$$
\begin{equation*}
n=\frac{x-\phi-z}{\pi x}-\rho+\lambda, \quad z \geq 0 \tag{A.21}
\end{equation*}
$$

which holds for positive values of the right-hand side. The Euler equation (A.18) and the reducedform production function (A.6) yield:

$$
\begin{aligned}
r & =\rho-\lambda+\dot{Y} / Y \\
& =\rho+\kappa z+\sigma n
\end{aligned}
$$

Combining this expression with the return to quality

$$
\begin{equation*}
r=\alpha x-\phi \tag{A.22}
\end{equation*}
$$

yields

$$
\alpha x-\phi=\rho+\kappa z+\sigma n .
$$

Combining this expression with the rate of entry in (A.21) and solving for $z$ yields

$$
z=\frac{\alpha x-\phi-\sigma \frac{x-\phi}{\pi x}-[\rho-\sigma(\rho-\lambda)]}{\kappa-\frac{\sigma}{\pi x}} .
$$

Substituting this result back into (A.21) yields

$$
\begin{aligned}
n & =\frac{x-\phi-z}{\pi x}-\rho+\lambda \\
& =\frac{x-\phi}{\pi x}-\frac{z}{\pi x}-\rho+\lambda \\
& =\frac{(\kappa-\alpha) x-(\kappa-1) \phi+[\rho-\sigma(\rho-\lambda)]}{\kappa \pi x-\sigma}-\rho+\lambda .
\end{aligned}
$$

Consider now the thresholds. Suppose $x_{N}<x_{Z}$. Then $n(x)>0$ for

$$
\frac{x-\phi}{\pi x}-\rho+\lambda>0
$$

since $z=0$, which yields

$$
x>x_{N} \equiv \frac{\phi}{1-\pi(\rho-\lambda)} .
$$

On the other hand, $z(x)>0$ for

$$
\alpha x-\phi-\sigma \frac{x-\phi}{\pi x}-[\rho-\sigma(\rho-\lambda)]>0,
$$

because entry is already active, which yields

$$
x>x_{Z} \equiv \arg \text { solve }\left\{\alpha x-\phi-\sigma \frac{x-\phi}{\pi x}=\rho-\sigma(\rho-\lambda)\right\} .
$$

This equation has two roots, one less than $\phi$ and one larger than $\phi$. Only the latter is admissible. Hence,

$$
x_{Z} \equiv \frac{\phi+\rho(1-\sigma)+\sigma \lambda+\frac{\sigma}{\pi}+\sqrt{\left[\phi+\rho(1-\sigma)+\sigma \lambda+\frac{\sigma}{\pi}\right]^{2}-4 \alpha \frac{\sigma \phi}{\pi}}}{2 \alpha} .
$$

The inequality

$$
z\left(x_{N}\right)=\frac{\alpha x_{N}-\phi-\sigma \frac{x_{N}-\phi}{\pi x_{N}}-[\rho-\sigma(\rho-\lambda)]}{\kappa-\frac{\sigma}{\pi x_{N}}}<0
$$

identifies the region of parameter space such that $x_{N}<x_{Z}$.
Combining all of these results, one can write:

$$
\begin{gathered}
z(x)=\left\{\begin{array}{cc}
0 & \phi \leq x \leq x_{N} \\
0 & x_{N}<x \leq x_{Z} ; \\
\frac{\alpha x-\phi-\sigma \frac{x-\phi}{\pi x}-[\rho-\sigma(\rho-\lambda)]}{\kappa-\frac{\sigma}{\pi x}} & x_{z}<x<\infty
\end{array}\right. \\
n(x)=\left\{\begin{array}{cc}
\phi \leq x \leq x_{N} \\
0 & x_{N}<x \leq x_{Z} \\
\frac{x-\phi}{\pi x}-\rho+\lambda
\end{array},\right.
\end{gathered}
$$

where:

$$
\begin{gathered}
x_{N} \equiv \frac{\phi}{1-\pi(\rho-\lambda)} ; \\
x_{Z} \equiv \arg \text { solve }\left\{\alpha \kappa x-\phi-\sigma \frac{x-\phi}{\pi x}=\rho-\sigma(\rho-\lambda)\right\}
\end{gathered}
$$

Proposition 4 As before, over the range $\phi \leq x \leq x_{Z}$ the function $c(x)$ is given by (A.16) evaluated at $z=0$. To characterize it over the range $x_{Z}<x \leq x_{N}$, set the rate of return to vertical innovation equal to the reservation rate of return of savers to obtain:

$$
\rho-\lambda+\frac{\dot{c}}{c}+\kappa z+\lambda=\alpha x-\phi .
$$

Solving the household budget constraint for $z$, yields

$$
z=x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right) .
$$

Combining these two expressions yields

$$
\frac{\dot{c}}{c}=(\kappa-\alpha) x-(\kappa-1) \phi+\kappa \frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)-(\rho-\lambda)-\lambda .
$$

The $\dot{c} \geq 0$ locus is thus

$$
c \geq(1-\theta)\left[1+\theta \frac{\rho-(\kappa-\alpha) x+(\kappa-1) \phi}{\kappa x}\right]
$$

In this region, the law of motion of $x$ is

$$
\begin{aligned}
\frac{\dot{x}}{x} & =\lambda+(\kappa-1) z \\
& =\lambda+(\kappa-1)\left[x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)\right]
\end{aligned}
$$

Recall, however, that $z \geq 0$ so that $\dot{x} / x$ is strictly positive. There is then a unique equilibrium trajectory: the economy jumps on the saddle path in $(x, c)$ space that converges to $\left(x^{*}, c^{*}\right)$ with smooth pasting. Writing

$$
\frac{\dot{c}}{\dot{x}}=\frac{d c}{d x}=\frac{c}{x} \frac{(\kappa-\alpha) x-(\kappa-1) \phi-\rho+\kappa \frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)}{\lambda+(\kappa-1)\left[x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)\right]}
$$

yields a partial differential equation that doesn't have a closed-form solution. However, one can show that the function $\tilde{c}(x)$ that solves it has the same derivative from the left and the right at $x=x_{Z}$ and approaches the value $c^{*}$ with zero derivative at $x=x_{N}$ :

$$
\begin{aligned}
\frac{d c\left(x_{Z}^{-}\right)}{d x} & =\frac{d c\left(x_{Z}^{+}\right)}{d x} \\
\frac{d c\left(x_{N}\right)}{d x} & =0
\end{aligned}
$$

In other words, it is increasing, concave and has no kinks. The associated expression for $z$ is

$$
\tilde{z}(x)=x\left[1-\frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta}-1\right)\right]-\phi
$$

Once again, one can show that $\tilde{z}(x)$ starts out at $x=x_{Z}$ with zero derivative and approaches the line that holds for $x>x_{N}$ with positive derivative:

$$
\begin{aligned}
\frac{d z\left(x_{Z}\right)}{d x} & =1-\frac{1}{\theta}\left(\frac{c\left(x_{Z}\right)}{1-\theta}-1\right)-\frac{x_{Z}}{\theta} \frac{d c\left(x_{Z}\right) / d x}{1-\theta}=0 \\
\frac{d z\left(x_{N}\right)}{d x} & =1-\frac{1}{\theta}\left(\frac{c\left(x_{N}\right)}{1-\theta}-1\right)-\frac{x_{N}}{\theta} \frac{d c\left(x_{N}\right) / d x}{1-\theta}>0
\end{aligned}
$$

The function $\tilde{z}(x)$ exhibits a kink at $x=x_{N}$ because when entry begins quality innovation attracts only a fraction of the economy's saving flow, which is now a constant fraction of final output.

Combining all of these results, one can write:

$$
\begin{gathered}
z(x)=\left\{\begin{array}{cc}
0 & \phi \leq x \leq x_{Z} \\
\tilde{z}(x) & x_{Z}<x \leq x_{N} \\
\frac{\alpha x-\phi-\sigma \frac{x-\phi}{\pi x}-[\rho-\sigma(\rho-\lambda)]}{\kappa-\frac{\sigma}{\pi x}} & x_{z}<x<\infty
\end{array}\right. \\
n(x)=\left\{\begin{array}{cc}
0 & \phi \leq x \leq x_{Z} \\
0 & x_{Z}<x \leq x_{N} \\
0 & x_{N}<x<\infty
\end{array}\right.
\end{gathered}
$$

where:

$$
\tilde{z}(x)=x\left[1-\frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta}-1\right)\right]-\phi
$$

and $\tilde{c}(x)$ is the solution of the partial differential equation

$$
\frac{d c}{d x}=\frac{\kappa c}{x} \frac{\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)-\frac{\rho}{\kappa}-(1-\alpha) x}{\lambda+(\kappa-1)\left[x-\phi-\frac{x}{\theta}\left(\frac{c}{1-\theta}-1\right)\right]} .
$$

The thresholds are:

$$
\begin{gathered}
x_{N} \equiv \arg \text { solve }\left\{\frac{(\kappa-\alpha) x-(\kappa-1) \phi+[\rho-\sigma(\rho-\lambda)]}{\kappa \pi x-\sigma}=\rho-\lambda\right\} \\
x_{Z} \equiv \arg \text { solve }\left\{x\left[1-\frac{1}{\theta}\left(\frac{\tilde{c}(x)}{1-\theta}-1\right)\right]=\phi\right\}
\end{gathered}
$$

The function $z(x)$ has zero derivative at $x=x_{Z}$, is increasing and has positive derivative at $x_{N}$.
According to these results, the only difference between the two cases is the middle region. With the functions $z(x)$ and $n(x)$ in hand, I can now prove the main result.

Step 3: Existence After some algebra, the equation

$$
\frac{\dot{x}}{x}=\lambda+(\kappa-1) z(x)-(1-\sigma) n(x)=0
$$

yields

$$
a_{1} x^{2}+a_{2} x+a_{3}=0,
$$

where:

$$
\begin{gathered}
a_{1}(\kappa) \equiv(\kappa-1) \alpha \pi ; \\
a_{2}(\kappa) \equiv-(1+\phi \pi)(\kappa-1)-(1-\sigma)(1-\alpha)+[(1-\sigma) \rho+\sigma \lambda] \pi \\
a_{3}(\kappa) \equiv \phi(\kappa-1)-[(1-\sigma) \rho+\sigma \lambda]
\end{gathered}
$$

It is immediate to check that these are in fact the coefficients of Proposition 2 with $\eta=1$. It follows that existence of the steady state is already established.

Step 4: Stability Figures 6-7 in the text illustrate the dynamics. Consider first the case $x_{N}<$ $x_{Z}$, in which the economy activates first variety innovation. For $x \leq x_{N}<x_{Z}$ the growth rate of profitability is $\dot{x} / x=\lambda$ and the economy crosses the threshold for entry in finite time. For $x_{N}<x<x_{Z}$ the growth rate is

$$
\frac{\dot{x}}{x}=\lambda-(1-\sigma)\left(\frac{x-\phi}{\pi x}-\rho+\lambda\right) .
$$

This expression identifies a steady-state value

$$
x_{N}^{*} \equiv \arg \text { solve }\left\{(1-\sigma) \frac{x-\phi}{\pi x}=\rho-\sigma(\rho-\lambda)\right\}=\frac{\phi}{1-\frac{\rho(1-\sigma)+\sigma \lambda}{1-\sigma} \pi} .
$$

The condition for $x_{Z}<x_{N}^{*}$ is thus

$$
\frac{\phi+\rho(1-\sigma)+\sigma \lambda+\frac{\sigma}{\pi}+\sqrt{\left[\phi+\rho(1-\sigma)+\sigma \lambda+\frac{\sigma}{\pi}\right]^{2}-4 \alpha \frac{\sigma \phi}{\pi}}}{2 \alpha}<\frac{\phi}{1-\frac{\rho(1-\sigma)+\sigma \lambda}{1-\sigma} \pi} .
$$

Interestingly, this condition does not depend on $\kappa$, since we are looking for parameter combinations that boost incentives to variety growth when quality growth is still zero. The intuition for this condition is that it prevents premature market saturation.

The case where $x_{N}<x_{Z}$ features an acceleration of the rate of growth of profitability at $x=x_{Z}$ so that the economy crosses the threshold $x_{N}$ in finite time. One concludes, therefore, that the condition stated in the proposition is sufficient for convergence to the steady state $x^{*}$ for any initial condition $x(0) \in(\phi, \bar{x})$.

## B An extension: difficulty index

It is often argued that to match the evidence models of endogenous innovation must allow for rising difficulty of innovation. To address such claims, it is useful to generalize the model by allowing for a richer cost structure in innovation.

Specifically, recall that $I_{i}$ is the firm's total expenditure on purchasing the inputs required to support a growth rate (a.k.a. rate of innovation) $z_{i}$, while $\phi Z_{i}$ is the firm's total expenditure on purchasing the inputs required to stay in operation (a.k.a. fixed operating costs or, equivalently, management costs). In both cases, total expenditure is the product of the price/cost per unit of the input times the number of units purchased to carry out the activity. It is then natural to think that the unit cost is the same if we think of $R \& D$ and management as activities using the same inputs. As one looks at the expressions, it is also natural to think that there is no compelling reason why the unit cost should exhibit any particular returns to scale. It follows that one can think of a generic function common to the two activities. As written, the model extends naturally to:

$$
\begin{aligned}
& \text { unit cost in } \mathrm{R} \& \mathrm{D}=Z_{i} \cdot D\left(Z_{i} ; Z, N\right) \\
& \text { unit cost in management }=Z_{i} \cdot D\left(Z_{i} ; Z, N\right) .
\end{aligned}
$$

These expressions say that the unit cost consists of an internal (firm-specific) component due to $Z_{i}$ and an external component due to $Z$ and $N$.

It is useful to be more specific and write:

$$
\text { unit cost in } \mathrm{R} \& \mathrm{D} \text { and management }=Z_{i} \cdot D\left(Z_{i} ; Z, N\right)=Z_{i} \cdot Z_{i}^{\delta_{1}} Z^{\delta_{2}} N^{\delta_{3}}
$$

The parameters $\delta_{1}, \delta_{2}$ and $\delta_{3}$ are unrestricted for now. Proceeding as in the previous analysis, the typical firm's Hamiltonian is:

$$
C V H_{i}=\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} Z_{i}^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}-\phi Z_{i} D\left(Z_{i} ; Z, N\right)-I_{i}+q_{i} I_{i} \frac{1}{D\left(Z_{i} ; Z, N\right)}
$$

This yields:

$$
r=\frac{\partial \Pi_{i}}{\partial Z_{i}} \frac{1}{q_{i}}-\frac{\partial D\left(Z_{i} ; Z, N\right)}{\partial Z_{i}} \frac{Z_{i}}{D\left(Z_{i} ; Z, N\right)} \frac{I_{i}}{Z_{i} D\left(Z_{i} ; Z, N\right)}+\frac{\dot{q}_{i}}{q_{i}}, \quad q_{i}=D\left(Z_{i} ; Z, N\right),
$$

where

$$
\begin{aligned}
\frac{\partial \Pi_{i}}{\partial Z_{i}}= & \frac{\partial\left(Z_{i}^{\alpha} Z^{\kappa-\alpha}\right) / \partial Z_{i}}{Z_{i}^{\alpha} Z^{\kappa-\alpha}} \cdot \underbrace{\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} Z_{i}^{\alpha} Z^{\kappa-\alpha} \frac{L}{N^{1-\sigma}}}_{\left(P_{i}-1\right) X_{i}} \\
& -\phi\left(D\left(Z_{i} ; Z, N\right)+Z_{i} \frac{\partial D\left(Z_{i} ; Z, N\right)}{\partial Z_{i}}\right) \\
= & \alpha \frac{\left(P_{i}-1\right) X_{i}}{Z_{i}}-\phi\left(D\left(Z_{i} ; Z, N\right)+Z_{i} \frac{\partial D\left(Z_{i} ; Z, N\right)}{\partial Z_{i}}\right) .
\end{aligned}
$$

With the functional form

$$
D\left(Z_{i} ; Z, N\right)=Z_{i}^{\delta_{1}} Z^{\delta_{2}} N^{\delta_{3}}
$$

I have:

$$
\begin{gathered}
r=\alpha \kappa \frac{\left(P_{i}-1\right) X_{i}}{D\left(Z_{i} ; Z, N\right)}-\delta_{1} z_{i}-\phi\left(1+\delta_{1}\right)+\frac{\dot{q}_{i}}{q_{i}}, \quad q_{i}=D\left(Z_{i} ; Z, N\right) ; \\
r=\frac{\left(P_{i}-1\right) X_{i}-\phi Z_{i} D\left(Z_{i} ; Z, N\right)-I_{i}}{V_{i}}+\frac{\dot{V}_{i}}{V_{i}}, \quad V_{i}=\frac{\beta Y}{N} .
\end{gathered}
$$

The price-dividend ratio in the return to entry can be written:

$$
\begin{aligned}
\frac{\left(P_{i}-1\right) X_{i}-\phi Z_{i} D\left(Z_{i} ; Z, N\right)-I_{i}}{V_{i}} & =\frac{\frac{\left(P_{i}-1\right) X_{i}}{Z_{i} D\left(Z_{i} ; Z, N\right)}-\frac{F_{i}+I_{i}}{Z_{i} D\left(Z_{i} ; Z, N\right)}}{\frac{\beta Y}{N}} Z_{i} D\left(Z_{i} ; Z, N\right) \\
& =\frac{\frac{\left(P_{i}-1\right) X_{i}}{Z_{i} D\left(Z_{i} ; Z, N\right)}-\phi-z_{i}}{\frac{\beta Y}{N}} Z_{i} D\left(Z_{i} ; Z, N\right)
\end{aligned}
$$

Imposing symmetry and recalling that $N P X=\theta Y$ yields:

$$
\begin{gathered}
r=\alpha \frac{(P-1) X}{Z D(Z ; Z, N)}-\delta_{1} z-\phi\left(1+\delta_{1}\right)+\frac{\dot{q}}{q}, \quad q=D(Z ; Z, N)=Z^{\delta_{1}+\delta_{2}} N^{\delta_{3}} \\
r=\left[1-\frac{\phi+z}{\frac{(P-1) X}{Z D(Z ; Z, N)}}\right] \frac{\theta(P-1)}{\beta P}+\frac{\dot{V}}{V}, \quad V=\frac{\beta Y}{N} .
\end{gathered}
$$

These show that we have the same mechanism as the basic model with the only difference that we now define

$$
x \equiv \frac{(P-1) X}{Z D(Z ; Z, N)}=\frac{(P-1) X}{Z^{\delta_{1}+\delta_{2} N^{\delta_{3}}}} .
$$

Concavity of the revenue function holds for

$$
\frac{\partial^{2}}{\partial Z_{i}^{2}}\left[\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} \frac{L Z^{\kappa-\alpha}}{N^{1-\sigma}} \cdot Z_{i}^{\alpha}-\phi Z^{\delta_{2}} N^{\delta_{3}} \cdot Z_{i}^{1+\delta_{1}}\right]<0
$$

that is, for

$$
\left(\frac{1}{\theta}-1\right) \theta^{\frac{2}{1-\theta}} \frac{L Z^{\kappa-\alpha}}{N^{1-\sigma}} \cdot \alpha(\alpha-1) Z_{i}^{\alpha-2}-\phi Z^{\delta_{2}} N^{\delta_{3}} \cdot\left(1+\delta_{1}\right) \delta_{1} Z_{i}^{\delta_{1}-1}<0 .
$$

We thus get the sufficient condition:

$$
\alpha \leq 1
$$

Quasi-convexity of the innovation plus management cost component holds for $\delta_{1} \geq 0$. Nothing else is needed.

We can now focus on the expressions for the rates of return to innovation in the symmetric equilibrium:

$$
\begin{gathered}
r=\alpha \kappa \frac{(P-1) X}{Z D(Z ; Z, N)}-\delta_{1} z-\phi\left(1+\delta_{1}\right)+\frac{\dot{q}}{q}, \quad q=D(Z ; Z, N) ; \\
r=\left[1-\frac{\phi+z}{\frac{(P-1) X}{Z D(Z ; Z, N)}}\right] \frac{\theta(P-1)}{\beta P}+\frac{\dot{V}}{V}, \quad V=\frac{\beta Y}{N} .
\end{gathered}
$$

Inspecting these expressions, it is evident that we have the same mechanism as the basic model with the only difference that we now define

$$
x \equiv \frac{(P-1) X}{Z D(Z ; Z, N)} .
$$

With the functional form posited above, this becomes:

$$
x \equiv \frac{(P-1) X}{Z^{1+\delta_{1}+\delta_{2}} N^{\delta_{3}}} .
$$

Endogenous growth is now possible for $\kappa \geq 1+\delta_{1}+\delta_{2}$, that is, if social increasing returns to quality in final production more than compensate for the rising difficulty of innovation, which occurs for $\delta_{1}>0 \mathrm{and} /$ or $\delta_{2}>0$. Accordingly, the relevant region of parameter space where steady-state exponential endogenous growth is feasible is $1+\delta_{1}+\delta_{2} \leq \kappa \leq \kappa^{\max }$. Note that for $\delta_{3}>0$, the restriction $\sigma<1$ needed to ensure the dominant market share effect is relaxed.

An interesting aspect of the structure proposed here is that nothing dictates that both $\delta_{1}$ and $\delta_{2}$ be positive so that one is free to believe in the difficulty of innovation rising in the firm's own knowledge $Z_{i}$ but decreasing in average knowledge $Z$. Alternatively one can believe that the difficulty of innovation rises in both $Z_{i}$ and $Z$. Or one can believe that both $Z_{i}$ and $Z$ reduce the cost of innovation. The core mechanism is robust to all such alternatives.

The only difference of substance between this extension and the basic model developed in the paper is that the analysis of the dynamics in this case is considerably more algebra-intensive. The reason is that we no longer work with $q=1$ but with $q=D(Z ; Z, N)=Z^{\delta_{1}+\delta_{2}} N^{\delta_{3}}$ since to allow for rising difficulty of innovation we sacrifice the transparency and tractability of the one-sector structure. The mechanism and the core insight concerning the conditions under which endogenous growth is a robust proposition, however, do not change. From the perspective of this paper, therefore, one can reasonably argue that extensions such as this are expensive - in the sense that
they cost a ton of extra calculations - while they add next to nothing to the main point of the analysis. Under the one-sector structure of the basic model, in fact, a rising cost of innovation that makes endogenous growth unfeasible can be captured in a straightforward manner by setting $\kappa<1$. There is no need to complicate matters by pursuing such a property in the roundabout fashion inherent to the difficulty of innovation index.

