Online Appendix of:

Through Scarcity to Prosperity: Toward a Theory of Sustainable Growth

Pietro F. Peretto*
Department of Economics, Duke University

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A Appendix

To facilitate the reader, all the equations from the text needed for the proofs are replicated in this document with self-contained numbering.

A.1 Derivation of the return to quality

The usual method of obtaining first-order conditions is to write the Hamiltonian for the optimal control problem of the firm. This derivation highlights the intuition. The firm undertakes R&D up to the point where the shadow value of the innovation, q_i, is equal to its cost,

$$1 = q_i \Leftrightarrow I_i > 0. \tag{A.1}$$

Since the innovation is implemented in-house, its benefits are determined by the marginal profit it generates. Thus, the return to the innovation must satisfy the arbitrage condition

$$r = \frac{\partial \Pi_i}{\partial Z_i}\frac{1}{q_i} + \frac{\dot{q}_i}{q_i}. \tag{A.2}$$

To calculate the marginal profit, observe that the firm's problem is separable in the price and investment decisions. Facing the isoelastic demand

$$X_i = \left(\frac{\theta}{P_i}\right)^{\frac{1}{1-\theta}} Z_i^{\alpha} Z^{1-\alpha} \frac{L^{\gamma} R^{1-\gamma}}{N^{1-\sigma}} \tag{A.3}$$

*Address: Department of Economics, Duke University, Durham, NC 27708. Phone: (919) 6601807. Fax: (919) 6848974. E-mail: peretto@econ.duke.edu

and a marginal cost of production equal to one, the firm sets $P_i = 1/\theta$. Substituting this result into the expression for the cash flow,

$$\Pi_i = \left[(P_i - 1) \left(\frac{\theta}{P_i} \right)^{\frac{1}{1-\theta}} \frac{L^\gamma R^{1-\gamma}}{N^{1-\sigma}} - \phi \right] Z_i^\alpha Z^{1-\alpha}, \tag{A.4}$$

differentiating with respect to Z_i, substituting into (A.2) and imposing symmetry yields

$$r = \alpha \frac{\Pi}{Z}. \tag{A.5}$$

A.2 Household behavior

The current value Hamiltonian is

$$\begin{aligned} CVH &= \mu \log C_M + \mu\eta \log M + (1-\mu) \log C_B + (1-\mu)\, \eta \log B \\ &\quad + \lambda_A \left[rA + wM + pR - (C_M M + C_B B) \right] + \lambda_M (B - \delta M) - \lambda_S R, \end{aligned}$$

where the λs denote the shadow value of respectively, financial assets, A, adult population, M, and the resource stock, S. The first order conditions for the control variables C_M, C_B, B, R are:

$$\frac{\mu}{C_M M} = \lambda_A = \frac{1-\mu}{C_B B}; \quad \frac{(1-\mu)\,\eta}{B} + \lambda_M = \lambda_A C_B; \quad \lambda_A p = \lambda_S.$$

The conditions for the state variables A, M, S are:

$$r + \frac{\dot{\lambda}_A}{\lambda_A} = \rho;$$

$$\frac{\eta + \lambda_A (wM - C_M M - C_B B)}{\lambda_M M} + \left(\frac{\dot{\lambda}_M}{\lambda_M} + \frac{\dot{M}}{M} \right) = \rho;$$

$$\frac{\dot{\lambda}_S}{\lambda_S} = \rho.$$

Associated to these are the transversality conditions that the value of each state variable times its shadow value converges to zero as $t \to \infty$.

Let $C \equiv C_M M + C_B B$ be aggregate consumption. The conditions for C_M and C_B yield $C = C_M M + C_B B = 1/\lambda_A$. Next, let the ratio of consumption to final output (consumption ratio) be $c \equiv C/Y$, births per adult (birth rate) be $b \equiv B/M$ and the shadow value of adult population be $h \equiv \lambda_M M$. The result $C = 1/\lambda_A$ and the first-order condition for financial wealth, A, yield the Euler equation for saving

$$r = \rho + \frac{\dot{C}}{C} = \rho + \frac{\dot{c}}{c} + \frac{\dot{Y}}{Y}. \tag{A.6}$$

The result $C = 1/\lambda_A$ and the conditions for fertility, B, financial wealth, A, and adult population size, M, yield the fertility rule

$$h = \frac{(1-\mu)(1-\eta)}{b} \tag{A.7}$$

and, recalling that $wM = \gamma(1-\theta)Y$, the asset-pricing-like equation

$$\dot{h} = \rho h - \eta - \frac{wM - C}{C} = \rho h - \eta - \frac{\gamma(1-\theta) - c}{(1-\mu)c}$$

that characterizes the evolution of the shadow value of adult population. Using (A.7), this can be rewritten

$$\frac{\dot{b}}{b} = \left[\frac{\gamma(1-\theta)}{c(1-\eta)} - 1\right]\frac{b}{1-\mu} - \rho, \tag{A.8}$$

which is the expression in the text.

Finally, the result $C = 1/\lambda_A$, the first-order conditions for the extraction flow R and the resource stock S plus the Euler equation (A.6) yield the Hotelling rule

$$\frac{p}{C} = \lambda_S \Rightarrow \frac{\dot{p}}{p} = \rho + \frac{\dot{C}}{C} = r. \tag{A.9}$$

A.3 Proof of Lemma 1

Log-differentiate the expression for GDP,

$$\frac{G}{M} = \theta^{\frac{2\theta}{1-\theta}}\left[1 - \theta^2\left(1 + \frac{\phi}{x}\right)\right]N^{\sigma}Z\left(\frac{R}{M}\right)^{1-\gamma}, \tag{A.10}$$

with respect to time and then use the Euler equation (A.6) and the extraction path,

$$\frac{\dot{R}}{R} = \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} = \frac{\dot{Y}}{Y} - r = -\left(\frac{\dot{c}}{c} + \rho\right), \tag{A.11}$$

to write:

$$r = \sigma n + z + \gamma\left(m + \frac{\dot{c}}{c} + \rho\right); \tag{A.12}$$

$$g = \sigma n + z + \xi(x)\frac{\dot{x}}{x} - (1-\gamma)\left(m + \frac{\dot{c}}{c} + \rho\right). \tag{A.13}$$

A.4 Proof of Lemma 2

Recall the expressions for the return to quality, equation (A.5), and to variety,

$$r = \frac{\Pi - I}{\beta X} + \frac{\dot{X}}{X}. \tag{A.14}$$

Use the expression for firms size,

$$x = \frac{X}{Z} = \frac{NX}{NZ} = \frac{\theta^2 Y}{NZ} = \theta^{\frac{2}{1-\theta}}\frac{M^{\gamma}R^{1-\gamma}}{N^{1-\sigma}}, \tag{A.15}$$

both directly and in log-differentiated form to rewrite the returns as:

$$r = \alpha\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right); \tag{A.16}$$

$$r=\frac{1}{\beta}\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)+\frac{\dot{x}}{x}+z. \tag{A.17}$$

Then log-differentiate the definition of x with respect to time and use the Euler equation (A.6) to obtain from these expressions

$$\frac{\dot{x}}{x}=\gamma m-(1-\gamma)\left(\frac{\dot{c}}{c}+\rho\right)-(1-\sigma)\,n. \tag{A.18}$$

A.5 Proof of Lemma 3

Recall the household budget

$$\dot{A}=rA+wM+pR-C_M M-C_B B. \tag{A.19}$$

Use the definition $C\equiv C_M M-C_B B$ to rewrite it as

$$\dot{A}=rA+D_Y+wM+pR-C,$$

where D_Y is the flow of dividends paid by the final sector. Under the paper's assumptions $D_Y=0$ and so omitting it from (A.19) does not change the analysis. However, for the purposes of this proof it is useful to include it and recognize that it is $D_Y=Y-wM-pR-N\cdot PX$ (recall that in equilibrium $L=M$). Dividing through by A yields

$$\frac{\dot{A}}{A}=r+\frac{D_Y+wM+pR-C}{A}.$$

When $n=0$ assets market equilibrium requires $A=NV$ but $V<\beta Y/N$ since by definition the free-entry condition does not hold. Differentiating with respect to time the expression for the value of the firm,

$$V(0)=\int_0^\infty e^{-\int_0^t r(s)ds}\left[\Pi(t)-I(t)\right]dt, \tag{A.20}$$

and substituting the result in the expression for the budget derived above yields

$$0=\frac{\Pi-I}{V}+\frac{Y-wM-pR-N\cdot PX+wM+pR-C}{NV},$$

which simplifies to

$$0=\frac{\Pi-I}{V}+\frac{Y-N\cdot PX-C}{NV}.$$

Using the definition of Π and rearranging terms yields

$$C=N\left[(P-1)\,X-\phi Z-I\right]+(1-\theta)\,Y.$$

The definitions of c and x, the R&D technology $\dot{Z}=I$, and the fact that $NX=\theta^2 Y$, then yield

$$c=\theta^2\left(\frac{1}{\theta}-1-\frac{\phi+z}{x}\right)+1-\theta.$$

Finally, set $z = 0$ in this expression, since it holds for $x \leq x_N < x_Z$, to obtain

$$c = \theta^2 \left(\frac{1}{\theta} - 1 - \frac{\phi}{x} \right) + 1 - \theta.$$

This is the top line of the expression in the text.

When $n > 0$ assets market equilibrium requires $A = NV = N\beta X = \beta\theta^2 Y$, which says that the wealth ratio A/Y is constant. Using the definition of c and the saving schedule (A.6), rewrite the budget constraint, after rearranging terms, as

$$c = \left(\rho + \frac{\dot{c}}{c} \right) \beta\theta^2 + 1 - \theta.$$

This unstable differential equation says that c jumps to its steady-state value $c^* = \rho\beta\theta^2 + 1 - \theta$, which is the value in the bottom line of the expression in the text.

A.6 Construction of the dynamical system

Rewrite equation (A.8) as

$$\frac{\dot{b}}{b} = \left[\frac{\gamma (1 - \theta)}{(1 - \eta) c} - 1 \right] \frac{b}{1 - \mu} - \rho,$$

where

$$c = \begin{cases} \theta^2 \left(\frac{1}{\theta} - 1 - \frac{\phi}{x} \right) + 1 - \theta & \phi / \left(\frac{1}{\theta} - 1 \right) < x \leq x_N \\ \rho\beta\theta^2 + 1 - \theta & x > x_N \end{cases}.$$

Lemma 2 yields

$$\frac{\dot{x}}{x} = \gamma m - (1 - \gamma) (\dot{c}/c + \rho) - (1 - \sigma) n. \tag{A.21}$$

In the region $x \leq x_N$, since $n = 0$ we have

$$\frac{\dot{x}}{x} = \gamma (b - \delta) - (1 - \gamma) \left(\frac{c'(x)\, x}{c(x)} \frac{\dot{x}}{x} + \rho \right) \Rightarrow \frac{\dot{x}}{x} = \frac{\gamma (b - \delta) - (1 - \gamma) \rho}{1 + (1 - \gamma) \frac{c'(x) x}{c(x)}}.$$

Noting that

$$\frac{c'(x)\, x}{c(x)} = \frac{\theta^2 \phi / x}{1 - \theta^2 (1 + \phi / x)} = \xi(x)$$

yields the expression in the text. In the region $x > x_N$, since $c = \rho\beta\theta^2 + 1 - \theta$ we have

$$\frac{\dot{x}}{x} = \gamma (b - \delta) - (1 - \gamma) \rho - (1 - \sigma) n.$$

Moreover, the fertility rate jumps to its own steady state

$$b^* = \frac{\rho (1 - \mu)}{\frac{\gamma(1-\theta)}{(1-\eta)\left(\rho\beta\theta^2 + 1 - \theta\right)} - 1}.$$

Finally, integrating the equilibrium extraction path yields

$$R(t) = R_0 e^{-\rho t}.$$

Substituting this result in the constraint

$$S_0 = \int_0^{\infty} R(t)\, dt$$

yields $R_0 = \rho S_0$. Therefore:

$$\begin{aligned} R(t) &= \rho S_0 e^{-\rho t}; \\ S(t) &= S_0 e^{-\rho t}. \end{aligned}$$

A.7 Proof of Lemma 4

Equation (A.12) and Lemma 1 allow one to rewrite the return to variety innovation in (A.17) as

$$n = \begin{cases} \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi+z}{x}\right) - \rho & z > 0 \\ \frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) - \rho & z = 0 \end{cases} \tag{A.22}$$

and the return to quality innovation in (A.16) as

$$z = \begin{cases} \alpha\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right) - \sigma n - \gamma\left(m^* + \rho\right) & n > 0 \\ \alpha\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right) - \gamma\left(m + \dot{c}/c + \rho\right) & n = 0 \end{cases}. \tag{A.23}$$

The threshold x_N follows directly from (A.22), which says that when agents anticipate $z = 0$ entry is positive for

$$x > x_N \equiv \frac{\phi}{\frac{1}{\theta} - 1 - \rho\beta}.$$

Solving (A.22) and (A.23) for z then yields

$$z = \frac{\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right]\left(\alpha - \frac{\sigma}{\beta x}\right) - \gamma\left(\rho + m^*\right) + \sigma\rho}{1 - \frac{\sigma}{\beta x}},$$

which says that in the region $x > x_N$ quality innovation is positive for

$$\left(\left(\frac{1}{\theta} - 1\right)x - \phi\right)\left(\alpha - \frac{\sigma}{\beta x}\right) > \gamma\left(m^* + \rho\right) - \sigma\rho \quad \text{and} \quad 1 - \frac{\sigma}{\beta x}.$$

Under the assumption $\beta x > \sigma \ \forall x > \phi/\left(\frac{1}{\theta} - 1\right)$, i.e., $\beta\phi/\left(\frac{1}{\theta} - 1\right) > \sigma$, these joint conditions say that there is the unique value

$$x_Z = \arg\operatorname{solve}\left\{\left[\left(\frac{1}{\theta} - 1\right)x - \phi\right]\left(\alpha - \frac{\sigma}{\beta x}\right) < \gamma\left(m^* + \rho\right) - \sigma\rho\right\},$$

such that $z > 0$ for $x > x_Z$. (The numerator of the expression for z suggests two zeros, but the one on the left is ruled out because it occurs at $x = \alpha\sigma/\beta < \sigma/\beta$.) The analytical solution for x_Z is

$$x_Z = \frac{\left(\frac{1}{\theta} - 1\right)\frac{\sigma}{\beta} + \phi\alpha + \gamma\left(m^* + \rho\right) - \sigma\rho + \sqrt{\left[\left(\frac{1}{\theta} - 1\right)\frac{\sigma}{\beta} + \phi\alpha + \gamma\left(m^* + \rho\right) - \sigma\rho\right]^2 - 4\left(\frac{1}{\theta} - 1\right)\alpha\phi\frac{\sigma}{\beta}}}{2\left(\frac{1}{\theta} - 1\right)\alpha}.$$

Finally, $x_Z > x_N$ if

$$\left(\left(\frac{1}{\theta}-1\right)x_N-\phi\right)\left(\alpha-\frac{\sigma}{\beta x_N}\right)<\gamma\left(m^*+\rho\right)-\sigma\rho,$$

which yields

$$\phi\alpha\frac{\rho\beta}{\frac{1}{\theta}-1-\rho\beta}<\gamma\left(m^*+\rho\right).$$

A.8 Proof of Proposition 1

Refer to Figure 1. In the region $x\in\left[\phi/\left(\frac{1}{\theta}-1\right),x_N\right]$, the $\dot{b}=0$ and $\dot{x}=0$ loci are, respectively:

$$b=\frac{\rho\left(1-\mu\right)}{\frac{\gamma(1-\theta)}{(1-\eta)c(x)}-1};$$

$$b=\frac{\left(1-\gamma\right)\rho}{\gamma}+\delta.$$

The $\dot{b}=0$ locus is increasing and concave in x. Three cases are possible. (i) The $\dot{x}=0$ locus is below the $\dot{b}=0$ locus for all $x\in\left[\phi/\left(\frac{1}{\theta}-1\right),x_N\right]$. (ii) The $\dot{x}=0$ locus intersects the $\dot{b}=0$ locus for some value $\tilde{x}\in\left[\phi/\left(\frac{1}{\theta}-1\right),x_N\right]$. (iii) The $\dot{x}=0$ locus is above the $\dot{b}=0$ locus for all $x\in\left[\phi/\left(\frac{1}{\theta}-1\right),x_N\right]$. Case (i) requires that at $x=\phi/\left(\frac{1}{\theta}-1\right)$ the $\dot{x}=0$ locus be lower than the $\dot{b}=0$ locus, that is,

$$\frac{\rho\left(1-\mu\right)}{\frac{\gamma}{1-\eta}-1}>\frac{\left(1-\gamma\right)\rho}{\gamma}+\delta.$$

This is condition C1, which ensures that the economy leaves the region $x\in\left[\phi/\left(\frac{1}{\theta}-1\right),x_N\right]$ and activates horizontal innovation because the only trajectories originating in the no innovation region that do not violate boundary conditions are those that enter the innovation region.

Note now that, of all the trajectories that enter the innovation region, those that connect with an explosive trajectory cannot be equilibria because they eventually violate boundary conditions. Consequently, there exists only one trajectory leaving the no innovation region that satisfies *all* boundary conditions: this is the trajectory that connects with smooth pasting with the saddle path that applies for $x>x_N$. To see this, note two things. First, the ratio of the $\dot{b}$ and $\dot{x}$ equations yields

$$\frac{db\left(x_N\right)}{dx}=\frac{b}{x_N}\left[1+\left(1-\gamma\right)\xi\left(x\right)\right]\frac{\left[\frac{\gamma(1-\theta)}{(1-\eta)c(x_N)}-1\right]\frac{b^*}{1-\mu}-\rho}{\gamma\left(b^*-\delta\right)-\left(1-\gamma\right)\rho}=0.$$

Second, recall that the economy chooses an initial pair (x_0,b_0), so that

$$x_0=\theta^{\frac{2}{1-\theta}}\frac{M_0^{\gamma}\left(\left[\int_0^{\infty}e^{-\varepsilon(t)t}dt\right]^{-1}S_0\right)^{1-\gamma}}{N_0^{1-\sigma}},$$

where

$$\varepsilon\left(t\right)\equiv\frac{1}{t}\int_0^t\left(\dot{c}\left(s\right)/c\left(s\right)+\rho\right)ds.$$

This choice incorporates the fact that by choosing the initial value of the extraction flow, R_0, to satisfy the lifetime constraint

$$S_0 = \int_0^\infty R(t)\, dt,$$

the economy in fact chooses the initial value of firm size subject to this constraint.

After the economy leaves to innovation region and starts riding the saddle path $b = b^*$, what happens next depends on condition C2. It the condition holds, the economy crosses the threshold x_Z and activates vertical innovation. It then converges to the steady state x^*, which exists and is positive if:

$$\frac{(1-\sigma)(1-\alpha)\phi}{\gamma(m^*+\rho)-\sigma\rho} - 1 > 0;$$

$$\frac{(1-\sigma)(1-\alpha)\left(\frac{1}{\theta}-1\right)}{\gamma(m^*+\rho)-\sigma\rho} - \beta > 0.$$

Combining these two inequalities and observing that $\frac{\beta}{\theta(1-\theta)}\phi > 1$ yields the existence condition C3:

$$\frac{(1-\sigma)(1-\alpha)}{\gamma(m^*+\rho)-\sigma\rho} > \frac{\beta}{\frac{1}{\theta}-1} > \frac{1}{\phi}.$$

Condition C4 ensures that steady-state growth at $x = x^*$ is positive:

$$g^* = \alpha\left(\left(\frac{1}{\theta}-1\right)x^* - \phi\right) - m^* - \rho > 0 \Rightarrow \alpha\frac{\phi\beta - \left(\frac{1}{\theta}-1\right)}{\frac{(1-\sigma)(1-\alpha)}{\gamma(m^*+\rho)-\sigma\rho}\left(\frac{1}{\theta}-1\right) - \beta} > m^* + \rho.$$

Finally, observe that

$$\frac{d(\dot{x}/x)}{dx} > 0$$

follows from the fact that, from the phase diagram,

$$\frac{db(x)}{dx} > 0,$$

that is the saddle path is upward sloping, and $\xi'(x) < 0$ since as firm size grows static economies of scale are gradually exhausted.

A.9 The transition: analytical details

Condition C1 is the restriction on the parameters that ensures that the economy crosses the threshold x_N and activates Schumpeterian innovation. Using the rates of innovation in Lemma 4, the law of motion of firm size becomes

$$\frac{\dot{x}}{x} = \begin{cases} \gamma(m^*+\rho) - \sigma\rho - (1-\sigma)\frac{1}{\beta}\left(\frac{1}{\theta} - 1 - \frac{\phi}{x}\right) & x_N < x \le x_Z \\ \gamma(m^*+\rho) - \sigma\rho - (1-\sigma)\frac{(1-\alpha)\left(\left(\frac{1}{\theta}-1\right)x-\phi\right)-\rho\beta x+\gamma(m^*+\rho)}{\beta x-\sigma} & x > x_Z \end{cases}. \tag{A.24}$$

Let:

$$\bar{\nu} \equiv (1-\sigma)\frac{\frac{1}{\theta}-1}{\beta} - \gamma(m^*+\rho) + \sigma\rho; \quad \bar{x}^* \equiv \frac{\phi}{\frac{1}{\theta} - 1 - \frac{\gamma(m^*+\rho)-\sigma\rho}{1-\sigma}\beta}.$$

The first line of (A.24) can be written $\dot{x} = \bar{\nu} \cdot (\bar{x}^* - x)$. Let T_N be the date when $x = x_N$. Solving this linear differential equation and then integrating between time T_N and time t yields

$$x(t) = x_N e^{-\bar{\nu}(T_N - t)} + \bar{x}^* \left(1 - e^{-\bar{\nu}(T_N - t)}\right).$$

There thus exists a value T_Z such that

$$x(T_Z) = x_N e^{-\bar{\nu}(T_N - T_Z)} + \bar{x}^* \left(1 - e^{-\bar{\nu}(T_N - T_Z)}\right) = x_Z,$$

which yields the date when the economy turns on quality growth:

$$T_Z = T_N + \frac{1}{\bar{\nu}} \log \left(\frac{\bar{x}^* - x_N}{\bar{x}^* - x_Z} \right). \tag{A.25}$$

This date is finite if and only if $\bar{x}^* > x_Z$. Using the definitions of $\bar{x}^*$ and x_Z yields condition C2, which, intuitively, says that the parameters are such that $z(\bar{x}^*) > 0$. An equivalent interpretation of the condition is that the parameters are such that $\dot{x}(x_Z) > 0$, that is, that firm size is strictly increasing in the whole range $[x_N, x_Z]$. Thereafter the economy follows the nonlinear differential equation in the second line of (A.24) and converges to x^*. Condition C3 ensures that this value exists.

Under the mild approximation discussed in Peretto (2015), one can characterize the dynamics of the last phase as a linear process. Let:

$$\nu \equiv (1-\sigma) \frac{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \rho\beta}{\beta} - \gamma(m^* + \rho) + \sigma\rho; \quad x^* \equiv \frac{(1-\alpha)\phi - \gamma(m^* + \rho)}{(1-\alpha)\left(\frac{1}{\theta} - 1\right) - \rho\beta - \frac{\gamma(m^*+\rho) - \sigma\rho}{1-\sigma}\beta}.$$

The second line of (A.24) can be written $\dot{x} = \nu \cdot (x^* - x)$. Solving this linear differential equation and then integrating between time T_Z and time t yields

$$x(t) = x_Z e^{-\nu(T_Z - t)} + x^* \left(1 - e^{-\nu(T_Z - t)}\right).$$

The main advantage of this simplification is that the in the second and third phases the dynamics of firm size follow a piece-wise linear differential equation. Accordingly, the model yields a closed-form solution for all endogenous variables as functions of time, t. This is not essential to the point made in the paper but it is a nice property to have in mind for future applications of the approach.

I have tried to develop a similar analytical characterization of the dynamics in the first phase. While the saddle path identifies a well-defined function $b(x)$, all attempts to obtain its analytical expression have failed.