Answer key
ECON 342: Midterm Exam

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Problem 1. We have,

$$f(x_i, y_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i - \mu_i)^2}{2\sigma^2}\right] \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_i - \mu_i)^2}{2\sigma^2}\right]$$

If we have observe a random sample of size n, the log-likelihood can be written as:

$$\mathcal{L}(\mu, \sigma) = -n \ln 2\pi - n \ln \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu_i)^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \mu_i)^2$$

First-order conditions are:

$$\frac{\mathcal{L}(\mu, \sigma)}{\partial \mu_i} = \frac{1}{\hat{\sigma}^2}[(x_i - \hat{\mu}_i) + (y_i - \hat{\mu}_i)] = 0$$

$$\frac{\mathcal{L}(\mu, \sigma)}{\partial \sigma^2} = -\frac{1}{\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4} \frac{1}{n} \sum_{i=1}^{n} [(x_i - \hat{\mu}_i)^2 + (y_i - \hat{\mu}_i)^2] = 0$$

The MLE estimator of μ_i equals,

$$\hat{\mu}_i = \frac{x_i + y_i}{2}$$

for $i = 1, \ldots, n$. Plug-in $\hat{\mu}_i$ into the foc for σ^2 and the MLE estimator of σ^2 equals:

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - y_i}{2}\right)^2$$

Note that $X_i - Y_i \sim N(0, 2\sigma^2)$ then,

$$\hat{\sigma}^2 = \frac{1}{2n} \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{2} \xrightarrow{p} \frac{1}{2}\sigma^2$$

so it is not consistent.

A consistent estimator is $2\hat{\sigma}^2$.

Problem 2. Let $\theta = (\alpha, \beta)'$, then the MLE estimator of θ is:

$$\hat{\theta} = \arg\max Q_n(\theta)$$

with,

$$Q_n(\theta) = \frac{1}{n}\sum_{n=i}^{n} y_i \ln \Phi(x_i'\theta) + (1 - y_i)\ln[1 - \Phi(x_i'\theta)]$$

Let

$$a(\theta) = \alpha * \beta - 1 = 0$$

from which we have $\frac{da(\theta)}{d\theta'} = A(\theta) = (\beta, \alpha)$. Then, the Wald statistic for the null hypothesis

$$H_0 : a(\theta_0) = 0$$

is equal,

$$W_n = na(\hat{\theta})'[A(\hat{\theta})\hat{\Sigma}^{-1}A(\hat{\theta})']^{-1}a(\hat{\theta}) \xrightarrow{d} \chi^2(1)$$

where,

$$\hat{\Sigma}^{-1} = -\left[\frac{1}{n}\sum_{n=i}^{n}\left\{-\left[\frac{y_i - \Phi(\mathbf{x}_i'\theta)}{(1 - \Phi(\mathbf{x}_i'\theta))\Phi(\mathbf{x}_i'\theta)}\right]^2 [\phi(\mathbf{x}_i'\theta)]^2 + \left[\frac{y_i - \Phi(\mathbf{x}_i'\theta)}{(1 - \Phi(\mathbf{x}_i'\theta))\Phi(\mathbf{x}_i'\theta)}\right][\phi'(\mathbf{x}_i'\theta)]\right\}\mathbf{x}_t'\mathbf{x}_t'\right]^{-1}$$

To derive the asymptotic distribution of W_n under the null $a(\theta_0) = 0$, using the MVT obtain:

$$a(\hat{\theta}) = a(\theta_0) + A(\tilde{\theta})(\hat{\theta} - \theta_0) = A(\tilde{\theta})(\hat{\theta} - \theta_0)$$

Given that the information matrix equality holds we have,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}\left(0, \Sigma^{-1}\right)$$

then,

$$\sqrt{n}a(\hat{\theta}) \xrightarrow{d} \mathcal{N}\left(0, A(\theta_0)\Sigma^{-1}A(\theta_0)'\right)$$

where $A(\theta_0)\Sigma^{-1}A(\theta_0)'$ is a scalar (i.e., rank equal to one). Therefore,

$$na(\hat{\theta})\left[A(\theta_0)\Sigma^{-1}A(\theta_0)'\right]^{-1}a(\hat{\theta}) \xrightarrow{d} \chi^2(1).$$

Problem 3. (a) The order condition requires that the number of predetermined variables is larger or equal to the number of regressors. For equation (1), we have as predetermined variables $\mathbf{x}_i' = (1, x_i, z_i)$ and as regressors $\mathbf{z}_i' = (1, p_i, x_i)$, then the order condition holds.

The rank condition for identification requires $\mathbb{E}(\mathbf{x}_i\mathbf{z}_i')$ is of full column rank. For (1) we have:

$$\mathbb{E}(\mathbf{x}_i\mathbf{z}_i') = \begin{pmatrix} 1 & \mathbb{E}(p_i) & \mathbb{E}(x_i) \\ \mathbb{E}(x_i) & \mathbb{E}(x_i p_i) & \mathbb{E}(x_i^2) \\ \mathbb{E}(z_i) & \mathbb{E}(z_i p_i) & \mathbb{E}(z_i x_i) \end{pmatrix}$$

which is satisfied if the $\det[\mathbb{E}(\mathbf{x}_i\mathbf{z}_i')] \neq 0$, which is satisfied if $\mathbb{E}(x_i p_i) \neq 0$ and $\mathbb{E}(z_i x_i) \neq 0$ (good instruments). The same reasoning follows for (2).

(b) Let $\mathbf{x}_i' = (1, x_i, z_i)$, $\mathbf{z}_{i1}' = (1, p_i, x_i)$, and $\mathbf{z}_{i2}' = (1, p_i, z_i)$, then:

$$\mathbf{g}_n(\delta) = \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i(q_i - \mathbf{z}_{i1}'\delta_1) \\ \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i(q_i - \mathbf{z}_{i2}'\delta_1) \end{pmatrix}$$

for a **W** symmetric positive definite matrix, the GMM estimator solves,

$$\hat{\delta} = \arg\min_{\delta} \mathbf{g}_n(\delta)'\mathbf{W}\mathbf{g}_n(\delta)$$

which equals,

$$\hat{\delta} = (\mathbf{S}_{xz}'\mathbf{W}\mathbf{S}_{xz})^{-1}\mathbf{S}_{xz}\mathbf{W}\mathbf{s}_{xy}$$

with,

$$\mathbf{s}_{\text{xy}} = \begin{pmatrix} \sum_{i=1}^n \mathbf{x}_i q_i \\ \sum_{i=1}^n \mathbf{x}_i q_i \end{pmatrix} \qquad \mathbf{S}_{xz} = \begin{pmatrix} \sum_{i=1}^n \mathbf{x}_i \mathbf{z}_{i1}' & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^n \mathbf{x}_i \mathbf{z}_{i2}' \end{pmatrix}$$

But, since $\mathbf{S}_{xz}$ is quadratic (i.e., the system is exactly identified) we can simplify this to,

$$\hat{\delta} = \mathbf{S}_{xz}^{-1}\mathbf{s}_{xy}$$

(c) The optimal weighting matrix for this problem is,

$$\mathbf{S}^{-1} = \left[\hat{\Sigma} \otimes \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i'\right]^{-1}$$

However, the system is exactly identified so there is no gain in efficiency setting W to be optimal.