A Unified Analysis of Rational Voting with Private Values and Cost Uncertainty*

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Abstract

We provide a unified analysis of the canonical rational voting model with privately known political preferences and costs of voting. Focusing on type-symmetric equilibrium, we show that for small electorates, members of the minority group vote with a strictly higher probability than do those in the majority, but the majority is strictly more likely to win the election. As the electorate size grows without bound, equilibrium outcome is completely determined by the individuals possessing the lowest cost of voting in each political group. We relate our equilibrium characterization to Myerson's Poisson games, and examine the rate of convergence as well as the potential uniqueness of equilibrium.

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1 Introduction

Rational voting theory, originally proposed by Downs (1957) in decision-theoretic terms, and later formulated by Ledyard (1981, 1984), and Palfrey and Rosenthal (1983, 1985) in game-theoretic terms, lays out the most basic incentives to vote and assumes that each agent trades off the net benefit of winning discounted by the probability of casting the pivotal

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vote against the cost of voting.[1] Despite severely underestimating turnout rate in large electorates [Palfrey and Rosenthal (1985)], rational voting theory is still widely believed to yield empirically reasonable comparative statics even in its purest form.[2] Perhaps this is why there is a renewed interest in the theory's applications. Most notably, Campbell (1999) applies the theory to rationalize election upsets by demonstrating that the minority group is likely to win the election if the electorate size is sufficiently large and if the minority is composed of agents with relatively low cost-benefit ratios. In a small electorate with *ex ante* symmetric agents, Börgers (2004) shows that voluntary participation may lead to too much turnout from the social viewpoint. Krasa and Polborn (2007) extend Börgers' analysis to asymmetric groups and large electorates, and point to the potential benefits of mandatory voting policies. In two related papers, Goeree and Grosser (2007) and Taylor and Yildirim (2005) examine the impact of releasing information about the distribution of political preferences through pre-election polls, political stock markets, etc. on equilibrium electoral outcomes and welfare.

While providing valuable insights, these papers have also recorded some important – and at times startling – theoretical results. For instance, Goeree and Grosser (2007) for small electorates, and Krasa and Polborn (2007), and Taylor and Yildirim (2005) for large electorates have noted that even in the presence of a clear majority, each alternative is *equally likely* to win the election in a type-symmetric equilibrium, which is also the basis for Campbell's finding. In addition, there seems to be a common understanding that an agent's vote becomes less pivotal as electorate size grows and/or others vote with a greater probability. Finally, despite the inherent coordination problem among the supporters of each alternative, Börgers (2004) establishes the uniqueness of type-symmetric equilibrium.

It is, however, often difficult to discern what factors drive these results and how robust they are, given that each paper employs a costly voting model with a varying degree of generality. The present paper aims to fill in this gap by providing a unified analysis while taking the celebrated "paradox of not voting" as given. The model we analyze is a generalization of Börgers (2004) and a slight variation of Palfrey and Rosenthal (1985). There are n agents divided randomly into two groups: supporters of alternative A and supporters of alternative B. These political preferences are distributed independently across agents.

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[1] See Aldrich (1997), Blais (2000), Feddersen (2004), and Merlo (2006) for excellent overviews of the literature.

[2] See, Hansen, Palfrey and Rosenthal (1987) for empirical evidence, and Levine and Palfrey (2007) for experimental evidence in favor of this model.

Moreover, within each group the costs of voting are also independently distributed. Each agent privately knows both his realized preference and voting cost.[3] Most significantly, this generalization allows for group-specific cost distributions with potentially different supports. An agent receives a net benefit normalized to 1 if his preferred alternative wins, and 0 otherwise. Agents decide whether to vote or abstain simultaneously, and ties are broken by a fair coin toss. As is common in the literature, we focus on type-symmetric equilibrium.

Our main result pertaining to small electorates formalizes the "underdog effect": given the same cost distribution, the members of the minority group vote with a strictly higher probability than do those in the majority.[4] Nonetheless, the majority never completely loses its initial advantage. This contrasts with the political neutrality findings of Goeree and Grosser (2007), and Taylor and Yildirim (2005) when voting costs are assumed fixed and equal for all agents.[5] As electorate size grows without bound, consistent with Campbell (1999), and Krasa and Polborn (2007), we show that only the agents with the lowest possible costs vote, independent of the distributions of preferences and costs, though we also show that the rate of convergence does depend on these distributions. Moreover, unlike Krasa and Polborn, by allowing for different cost supports across the two political groups, we discover that each alternative is equally likely to win the election if and only if the lower bounds of the supports are equal. Otherwise, the group with a cost advantage (in the sense of the lowest possible cost) is strictly more likely to win, as intuition suggests.

Our equilibrium characterization of large elections also bridges a gap between the costly voting model with a fixed population size and Myerson's Poisson games with a random population [Myerson (1998, 2000)]. We demonstrate that a large election can be considered a Poisson game – in Myerson's sense – where the population mean is the sum of equilibrium limit turnouts for each group and an appropriately defined probability of voting for each alternative in terms of these limits, which is, in general, different from the initial distribution of preferences.[6]

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[3] As in the Ledyard-Palfrey-Rosenthal model, agents in our private-values setup are also differentiated by their intrinsic preferences over political alternatives. Hence, we do not study the information aggregation problem that is the focus of common-value models such as Feddersen and Pesendorfer (1997), Krishna and Morgan (2008), and Razin (2003).

[4] To be sure, the underdog effect has been articulated in several empirical and experimental studies, the most recent being Levine and Palfrey (2007); but, to our knowledge, it has not been formally shown in a framework as general as ours.

[5] Such an underdog effect is not present in Börgers (2004) due to *ex ante* symmetry.

[6] Myerson (1998, pp. 386-92) makes a similar point but within a numerical example with a fixed cost of voting.

Finally, we establish a sufficient condition for equilibrium uniqueness, which is satisfied if agents are sufficiently symmetric, or if the electorate size is sufficiently large. In doing so, we demonstrate that Börgers' (2004) uniqueness result with symmetric agents is robust to small perturbations, and large elections are likely to yield a unique type-symmetric equilibrium.

The rest of the paper is organized as follows. In the next section, we set up the formal model, followed by the equilibrium characterizations for small and large electorates in Sections 3 and 4, respectively. In Section 5, we examine the question of equilibrium uniqueness, and we gather some concluding remarks in Section 6. The proofs not appearing in the text have been relegated to the Appendix.

2 The Model

There are $n \geq 2$ agents who may cast a vote in an election between two alternatives, $r = A, B$. Each agent i privately knows his 2-dimensional type, $t_i = (r_i, c_i)$, consisting of his political preference $r_i \in \{A, B\}$ and his cost of voting c_i. Political preferences are independently drawn from a Bernoulli distribution with $\lambda_r \in (0, 1)$ representing the probability of alternative r, and conditional on these preferences, the agents who favor alternative r pick their voting costs independently from the differentiable distribution $G_r(c)$ where $G_r'(c) = g_r(c) > 0$ for all $c \in (\underline{c}_r, \overline{c}_r) \subset \mathbb{R}_+$. Note that we allow voting costs across the two political groups to differ not only in their densities but also in their supports.[7] Upon privately observing their types, agents simultaneously choose whether to vote for their preferred alternative or to abstain. The election is decided by majority rule and ties are broken by a fair coin toss. Agent i receives a gross payoff normalized to 1 if r_i wins; and 0 otherwise.

Action/Outcome	r_i wins	r_i loses
Abstain	1	0
Vote	$1 - c_i$	$-c_i$

Table 1: Ex Post Payoffs of Agent i

As is clear from Table 1, abstaining strictly dominates voting for one's less preferred alternative, resulting in "sincere" voting in this setup.[8] In order to rule out trivial equilibria in which it is a dominant strategy for all agents in some political group to abstain or for

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[7] If $G_A = G_B$, then supports must, of course, be the same, but the converse is not true.

[8] Unlike a private values election, sincere voting, in general, does not obtain in equilibrium with common-values. However, Krishna and Morgan (2008) have shown that if voting is costly, then there always exists an equilibrium with sincere voting.

all to vote with certainty, we assume $0 < \underline{c}_r < \frac{1}{2} < \overline{c}_r$. All aspects of the environment are common knowledge. In the analysis below, we frequently refer to political group r as the majority and r' as the minority if $\lambda_r > \lambda_{r'}$, because, with $\lambda_r > \lambda_{r'}$, the expected size of group r is strictly greater than that of group r'.[9]

3 Equilibrium in Small Electorates

As is standard in the costly-voting literature, we concentrate on type-symmetric Bayesian Nash Equilibrium (BNE) in which all agents preferring alternative r follow the same equilibrium strategy. It is straightforward to verify that in a type-symmetric BNE, agents adopt a cutoff strategy in which a player favoring r votes if and only if his cost is less than some critical level, c_r^*. In order to characterize such a BNE, denote the *ex ante* probability that a type r agent[10] votes by $\phi_r \equiv G_r(c_r^*)$, and the *ex ante* probability that an agent votes for alternative r by $\alpha_r \equiv \lambda_r \phi_r$. Hence, the *ex ante* probability that an agent abstains is $(1 - \alpha_r - \alpha_{r'})$. Now, recall that the number of ways k other agents can vote for r, k' can vote for r', and $n - 1 - k - k'$ can abstain is given by the trinomial coefficient

$$\binom{n-1}{k, k', n-1-k-k'} \equiv \frac{(n-1)!}{k!k'!(n-1-k-k')!}.$$

Given this, the net expected utility from voting to an agent with voting cost, c, and political preference r may be written (see the proof of Lemma 1 in the Appendix)

$$\Delta_r \equiv \frac{1}{2} P(\alpha_r, \alpha_{r'}, n) - c, \tag{1}$$

where

$$\begin{aligned} P(\alpha_r, \alpha_{r'}, n) \equiv & \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{k, k, n-1-2k} \alpha_r^k \alpha_{r'}^k (1 - \alpha_r - \alpha_{r'})^{n-1-2k} \\ & + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{k, k+1, n-2-2k} \alpha_r^k \alpha_{r'}^{k+1} (1 - \alpha_r - \alpha_{r'})^{n-2-2k}, \end{aligned} \tag{2}$$

for $r = A, B$, $r \neq r'$, and $\lfloor \cdot \rfloor$ is the usual operator that rounds a number to the lower integer when necessary.

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[9] This terminology is commonly used in the literature, e.g., Campbell (1999), Goree and Grosser (2007), and Krasa and Polborn (2007).

[10] To avoid repetition, we sometimes abuse terminology and say "type" r to refer to one's political type only, keeping in mind type also includes his cost.

To understand expression (1), observe that $P(\alpha_r, \alpha_{r'}, n)$ is the probability that a type r agent casts a decisive vote; i.e., that his vote is pivotal in determining the outcome when each of the other $n-1$ agents: votes for r with probability α_r, votes for r' with probability $\alpha_{r'}$, and abstains with probability $1-\alpha_r-\alpha_{r'}$. In particular, his vote may be pivotal for one of two reasons corresponding to the two summations in (2). First, if k of the other agents vote for r, k vote for r', and $n-1-2k$ abstain, then the agent in question will *break* a tie by voting. The first summation in (2) is, therefore, the probability that the agent breaks a tie that would otherwise occur. Second, if k agents vote for alternative r, $k+1$ vote for r', and $n-2-2k$ abstain, then the agent in question will *create* a tie by voting. The second summation in (2) is, therefore, the probability that the agent in question creates a tie when alternative r' would otherwise have won. When his vote breaks a tie, the probability that alternative r is implemented rises from 1/2 to 1, and when his vote creates a tie, the probability that r is implemented rises from 0 to 1/2. This accounts for the factor 1/2 in (1). Of course, when an agent votes, his net expected benefit must also account for his voting cost, c.

An important step in understanding equilibrium voting behavior now, and the possibility of equilibrium uniqueness later, is to derive some basic properties of the pivot probability. To our knowledge, these properties have not been recorded elsewhere, except for the special case of $\alpha_r = \alpha_{r'}$.

Lemma 1. *For $(\alpha_r, \alpha_{r'}) \in (0, \lambda_r) \times (0, \lambda_{r'})$ where $r = A, B$ and $r \neq r'$,*

(i) $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_{r'}, \alpha_r, n) =^{sign} \alpha_{r'} - \alpha_r$,

(ii) $\frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n) =^{sign} \begin{cases} 0 & if \quad n = 2 \\ \alpha_r - \alpha_{r'} & if \quad n > 2 \end{cases}$,

(iii) $\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) < 0$, *if* $\alpha_r \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}$,

(iv) $P(\alpha_r, \alpha_{r'}, n) > P(\alpha_r, \alpha_{r'}, n+2)$, *but* $P(\alpha_r, \alpha_r, n) > P(\alpha_r, \alpha_r, n+1)$.

To aid with discussion, it is worth repeating that $P(\alpha_r, \alpha_{r'}, n)$ is the probability that the vote of an isolated type r agent is pivotal given the voting probabilities of the other $n-1$ agents. Now, to understand part (i) of the lemma, suppose $\alpha_r > \alpha_{r'}$. In this case, it is likely that alternative r has more votes than r', and therefore a vote for r (which widens the expected lead) is less apt to be pivotal than a vote for r' (which narrows it). Part (ii) says that an increase in the probability of voting for alternative r' makes a vote for r more likely pivotal because it closes the gap in voting probabilities between the two alternatives. Hence, the pivot probability of a vote for r is *nonmonotonic* in the probability of voting for r', peaking at the point where $\alpha_{r'} = \alpha_r$.[11]

Part (iii) reveals that the vote of an isolated type r agent is less apt to be pivotal when the probability that all other type r agents vote increases, provided they vote with higher probability than type r' agents (i.e., when the gap in voting probabilities increases). The converse is, however, not necessarily true. In other words, if $\alpha_r < \alpha_{r'}$, it is not necessarily true that the vote of an isolated type r agent is more apt to be pivotal when α_r increases (i.e., the gap in voting probabilities decreases). Part (iii) implies that an agent views his vote as a *substitute* to the voting probability of others' who share his political preference, so long as this probability is not too far behind the one for the competing alternative, and as a *complement* otherwise.[12]

Finally, part (iv) reveals that a vote for r becomes less apt to be pivotal when the electorate size increases by two. Intuition suggests that as the electorate grows, the pivot probability should decrease for all n. This turns out not to be true in general. For some fixed pair $(\alpha_r, \alpha_{r'})$, a type r agent's vote may actually be more likely to be pivotal as n increases by one.[13] This nonmonotonicity is a consequence of the different ways ties can occur when n is odd or even, and seems to be especially relevant in small electorates. Nonetheless, the monotonicity of the pivot probability is restored, if one takes increments by two rather than one, or compute it on a particular path such as $\alpha_r = \alpha_{r'}$, as in Börgers (2004), Goeree and Grosser (2007), and Taylor and Yildirim (2005). The latter plays a crucial role in establishing equilibrium uniqueness for these papers – an issue we address in Section 5.

In a type-symmetric equilibrium, the net expected payoff of a type r agent with the cutoff cost, c_r^*, must satisfy

$$\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* \leq 0 \text{ and } \left[\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^*\right](c_r^* - \underline{c}_r) = 0 \ . \tag{3}$$

To understand why, note that if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* > 0$, then a type r agent would not be indifferent but would prefer to vote with certainty, violating the definition of c_r^* as a cost

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[11] This makes sense, because if $\alpha_{r'} = \alpha_r$, then each alternative is equally likely to win, making a vote for r decisive with the highest probability.

[12] Note that since $\frac{\partial}{\partial \alpha_r}P(\alpha_r, \alpha_r, n) < 0$ and $\frac{\partial}{\partial \alpha_{r'}}P(\alpha_r, \alpha_r, n) = 0$, it follows that $\frac{d}{d\alpha_r}P(\alpha_r, \alpha_r, n) < 0$, as found in Börgers (2004), Goeree and Grosser (2007), and Taylor and Yildirim (2005).

[13] As an example, let n be even and $\alpha_r + \alpha_{r'} = 1$. Then, $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) = \binom{n-1}{\frac{n}{2}}\alpha_r^{\frac{n}{2}-1}\alpha_{r'}^{\frac{n}{2}}(1 - 2\alpha_r)$. Hence, $P(\alpha_r, \alpha_{r'}, n) < P(\alpha_r, \alpha_{r'}, n+1)$ whenever $\alpha_r > \frac{1}{2}$.

cutoff. Conversely, if $\frac{1}{2}P(\alpha_r^*, \alpha_{r'}^*, n) - c_r^* < 0$, then the agent would prefer to abstain with certainty or would have $c_r^* = \underline{c}_r$. Finally, if $c_r^* > \underline{c}_r$, then the agent would, by the definition of c_r^*, vote for some cost realizations, but because $\frac{1}{2} < \overline{c}_r$, not for all. Thus, in equilibrium, he must be indifferent at the cutoff cost.

Given $\phi_r \equiv G_r(c_r^*)$, $\alpha_r \equiv \lambda_r \phi_r$, and defining

$$\Phi_r(\alpha_r, \alpha_{r'}) \equiv G_r\left(\frac{1}{2}P(\alpha_r, \alpha_{r'}, n)\right) - \frac{\alpha_r}{\lambda_r},$$

we can rewrite (3):

$$\Phi_r(\alpha_r^*, \alpha_{r'}^*) \leq 0 \text{ and } \alpha_r^* \Phi_r(\alpha_r^*, \alpha_{r'}^*) = 0. \tag{4}$$

Finding an equilibrium, therefore, amounts to finding a pair $(\alpha_A^*, \alpha_B^*) \in [0, \lambda_A] \times [0, \lambda_B]$ that satisfy (4).

PROPOSITION 1. *There exists a type-symmetric equilibrium, and every type-symmetric equilibrium has the following properties:*

(i) $\phi_r^* < 1$ *for all* r*; and* $\phi_r^* > 0$ *for some* r*.*

(ii) *If* $\phi_r^* = 0$*, then* $\underline{c}_r > \underline{c}_{r'}$*.*

(iii) *If* $G_A = G_B$ *and* $\lambda_A > \lambda_B$*, then* $0 < \phi_A^* < \phi_B^*$*;* $\alpha_A^* > \alpha_B^* > 0$*; and* $\frac{1}{2} < \Pr\{A$ *wins*$\} < 1$*.*

(iv) *If* $\lambda_A = \lambda_B$*, and* G_A *first-order stochastically dominates* G_B*, then* $\phi_A^* \leq \phi_B^*$*;* $\alpha_A^* \leq \alpha_B^*$*; and* $0 < \Pr\{A$ *wins*$\} \leq \frac{1}{2}$*.*

PROOF. Let $\Psi(\alpha_A, \alpha_B) \equiv (\lambda_A G_A(\frac{1}{2}P(\alpha_A, \alpha_B, n), \lambda_B G_B(\frac{1}{2}P(\alpha_B, \alpha_A, n))$. From (4), it is clear that an equilibrium pair (α_A^*, α_B^*) is a fixed point of Ψ. Since Ψ maps the compact and convex set $[0, \lambda_A] \times [0, \lambda_B]$ into itself, and it is continuous in this region, by Brouwer's fixed theorem, there exists a type-symmetric equilibrium. Next, we prove each part in turn.

(i) Suppose, on the contrary, $\phi_r^* = 1$, or equivalently $\alpha_r^* = \lambda_r(\neq 0)$ for some r. Then, since $\frac{1}{2} < \overline{c}_r$, we have $\Phi_r(\lambda_r, \alpha_{r'}^*) < 0$, which, from (4) implies $\alpha_r^* = 0$, yielding a contradiction. Hence, $\phi_r^* < 1$ for all r. Next, suppose $\phi_r^* = 0$, or $\alpha_r^* = 0$ for all r. Then, $\Phi_r(0, 0) = G_r\left(\frac{1}{2}\right) > 0$, contradicting (4). Thus, $\phi_r^* > 0$ for some r.

(ii) Let $\phi_r^* = 0$ for some r. Then, $\phi_{r'}^* > 0$ by part (i). By (3), this means $\frac{1}{2}P(0, \alpha_{r'}^*, n) - \underline{c}_r \leq 0$ and $\frac{1}{2}P(\alpha_{r'}^*, 0, n) - c_{r'}^* = 0$ where $c_{r'}^* > \underline{c}_{r'}$. From (2), note that $P(0, \alpha_{r'}^*, n) = (1 - \alpha_{r'}^*)^{n-1} + (n-1)\alpha_{r'}^*(1 - \alpha_{r'}^*)^{n-2}$ and $P(\alpha_{r'}^*, 0, n) = (1 - \alpha_{r'}^*)^{n-1}$, which together require $c_{r'}^* + \frac{n-1}{2}\alpha_{r'}^*(1 - \alpha_{r'}^*)^{n-2} - \underline{c}_r \leq 0$, and imply $c_{r'}^* < \underline{c}_r$ because $\alpha_{r'}^* \in (0, 1)$.

(iii) Suppose $G_A = G_B = G$ and $\lambda_A > \lambda_B$, but, on the contrary, $\alpha_A^* \leq \alpha_B^*$. We make two observations. First, $G_A = G_B$ implies $\underline{c}_A = \underline{c}_B$, and thus $\alpha_A^*, \alpha_B^* > 0$ by part (ii). Second, $\frac{\alpha_A^*}{\lambda_A} < \frac{\alpha_B^*}{\lambda_B}$. Together with (4), the latter requires $G(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) < G(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n))$, which, given $G' > 0$, implies $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$, and $\alpha_A^* > \alpha_B^*$, by part (i) of Lemma 1, yielding a contradiction. Hence, $\alpha_A^* > \alpha_B^*$.

Given $\alpha_A^* > \alpha_B^* > 0$, we have $P(\alpha_A^*, \alpha_B^*, n) < P(\alpha_B^*, \alpha_A^*, n)$ by Lemma 1, and $\Phi_A(\alpha_A^*, \alpha_B^*) = \Phi_B(\alpha_B^*, \alpha_A^*) = 0$ by (4). Since $G(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) < G(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n))$ and $\phi_r^* \equiv \frac{\alpha_r^*}{\lambda_r}$, (4) further reveals $\phi_A^* < \phi_B^*$. To complete the proof of part (iii), note that

$$\begin{aligned} \Pr\{r \text{ wins}\} &= \frac{1}{2}\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k, k, n-2k} (\alpha_r^*)^k (\alpha_{r'}^*)^k (1 - \alpha_r^* - \alpha_{r'}^*)^{n-2k} \\ &+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k'=0}^{k-1} \binom{n}{k, k', n-k-k'} (\alpha_r^*)^k (\alpha_{r'}^*)^{k'} (1 - \alpha_r^* - \alpha_{r'}^*)^{n-k-k'}. \end{aligned} \tag{5}$$

Given $\alpha_A^* > \alpha_B^*$, it is clear that $\Pr\{A \text{ wins}\} > \Pr\{B \text{ wins}\}$, and hence $\Pr\{A \text{ wins}\} > \frac{1}{2}$. Moreover, given $\alpha_A^* < 1$, $\Pr\{A \text{ wins}\} < 1$.

(iv) Suppose $\lambda_A = \lambda_B$, and G_A first-order stochastically dominates G_B, but, on the contrary, $\alpha_A^* > \alpha_B^*$. This means $\alpha_A^* > 0$. By (4), we thus have

$$G_B(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)) - \frac{\alpha_B^*}{\lambda_B} \leq 0 = G_A(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n)) - \frac{\alpha_A^*}{\lambda_A}. \tag{6}$$

Given $\lambda_A = \lambda_B$, (6) reveals that $G_B(\frac{1}{2}P(\alpha_B^*, \alpha_A^*, n)) < G_A(\frac{1}{2}P(\alpha_A^*, \alpha_B^*, n))$, which, because G_A first-order stochastically dominates G_B, requires that $P(\alpha_B^*, \alpha_A^*, n) \leq P(\alpha_A^*, \alpha_B^*, n)$. Then, by Lemma 1, we have $\alpha_A^* \leq \alpha_B^*$, yielding a contradiction. Hence, $\alpha_A^* \leq \alpha_B^*$. Since $\lambda_A = \lambda_B$, this implies $\phi_A^* \leq \phi_B^*$. Finally, note from (5) that $0 < \Pr\{A \text{ wins}\} \leq \frac{1}{2}$. ■

Proposition 1 highlights some basic properties of a type-symmetric equilibrium.[14] Part (i) indicates that in equilibrium, no individual votes with certainty. This is because the

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[14] The existence of a type-symmetric equilibrium is well-established in the literature, e.g., Ledyard (1984) and Palfrey and Rosenthal (1985). Nevertheless, this is – to the best of our knowledge – the first formal derivation of the equilibrium properties for a small electorate.

maximum benefit from voting is $\frac{1}{2}$ and $\frac{1}{2} < \overline{c}_r$. Part (i) also indicates that at least some individuals are expected to vote. However, even though we have ruled out abstentions due to high costs, i.e, $\underline{c}_r < \frac{1}{2}$, it is possible that members of *some* political group abstain altogether for strategic reasons. Part (ii) reveals that if such full abstention occurs, the main reason must be the individuals with low costs of voting in the rival group and not necessarily the distribution of political preferences. Another important implication of part (ii) is that if $\underline{c}_r = \underline{c}_{r'}$, then the expected probability of voting is strictly positive for all individuals irrespective of the cost and political preference distributions. Hence, the knife-edge case of equal cost lower bounds, often assumed in the literature, seems to rule out the interesting case of complete abstention by one group. In our analysis of large elections, this knife-edge case will also be the source of a strong "neutrality" result.

Part (iii) formalizes the "underdog effect" alluded to in the introduction: given identical cost distributions, an agent in the minority group is strictly more likely to vote. This is due to the well-known tension between one's incentives for winning the election and free-riding on his fellow group members. Not surprisingly, the latter incentive is less pronounced in a smaller group. Nonetheless, part (iii) shows that the underdog effect never outweighs the initial majority advantage, and hence the majority is strictly more likely to win in a small electorate. Part (iv) examines the counterpart of (iii). When each agent is equally likely to support either alternative, the group whose members are more likely to have higher voting costs is less likely to win the election.

The presence of the underdog effect raises two important questions: First, is the majority prone to lose some of its initial advantage? That is, is $\Pr\{A\ wins\} < \lambda_A$ when $G_A = G_B$ and $\lambda_A > \lambda_B$? Second, if we replicated the electorate, would this necessarily improve the majority's chances of winning? That is, is $\Pr\{A\ wins\}$ strictly increasing in n?

Although we have been unable to provide a fully analytical answer to these questions,[15] the following lemma along with the numerical example helps develop the intuition.

LEMMA 2. *Fix the ex ante probabilities of voting,* $(\alpha_A, \alpha_B) \in [0, \lambda_A) \times [0, \lambda_B)$, *and let* $\Pr\{A\ wins\} \equiv \pi(\alpha_A, \alpha_B, n)$. *Then,*

(i) $\pi(.)$ *is strictly increasing in* n *if and only if* $\alpha_A > \alpha_B$. *Moreover, if* $\alpha_A > \alpha_B$, *then* $\pi(.) \to 1$ *as* $n \to \infty$.

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[15] Because both the pivot and winning probabilities are of nth order polynomials, and n has both direct and strategic effects on winning.

(ii) $\pi(.)$ *is strictly increasing in* α_A *and strictly decreasing in* α_B. *Moreover,* $\pi(.) \to \frac{1}{2}$ *as* $\alpha_r \to 0$ *for all* r.

Lemma 2 is quite intuitive. Fixing agents' voting strategies, part (i) says that if each agent's *ex ante* probability of voting for alternative A is strictly greater than that for B, then the probability of winning for A strictly increases and converges to 1 as the population grows – because additional agents who vote are more likely to do so for A. Part (ii) states the fact that for a fixed electorate size, the probability of winning for an alternative increases as individuals are more likely to vote for this alternative and less likely to vote for the competing one. The winning probability for each alternative converges to $\frac{1}{2}$ as more individuals abstain.

Part (i) of Lemma 2 highlights the direct effect of electorate size on the majority's winning probability, because, by Proposition 1, we know $\alpha_A^* > \alpha_B^*$ for $G_A = G_B$ and $\lambda_A > \lambda_B$. However, since equilibrium voting strategies, α_A^* and α_B^* are themselves functions of n, electorate size also has a strategic effect. While it is difficult to determine analytically the direction of the strategic effect owing to the potential nonmonotonicity of the pivot probability in n (see Lemma 1), in the next section we show that both α_A^* and α_B^* become negligible as electorate size grows without bound. By part (ii) of Lemma 2, this means the strategic effect of n on the probability of winning is likely to overwhelm its direct (positive) effect for the majority in a large electorate. We now solve a numerical example that illustrates this observation.

Example 1. *Let each agent draw his cost of voting independently from a uniform distribution in* $(.02, 1)$, *and* $\lambda_A = .55$. *Solving numerically for equilibrium (unique in this case for each* n*), we find:*

n	5	50	100	1,000	10,000	100,000	1,000,000
α_A^*	.15661	.07063	.05454	.02104	.00641	.00141	.00019
α_B^*	.13578	.06207	.04805	.01876	.00588	.00136	.00018
Pr{A *wins*}	.53204	.56492	.57945	.64064	.68357	.62688	.52780

Table 2. Nonmonotonicity of winning probability in electorate size

Remark 1. *According to Example 1, the majority doesn't necessarily lose its initial advantage. Whereas* Pr{A *wins*} $< \lambda_A$ *for sufficiently small and sufficiently large electorates,* Pr{A *wins*} $> \lambda_A$ *for a medium-size one. To understand why, note that by*

Lemma 2, the direct effect of a population increase is always favorable to the majority. When the population is small, although the free-rider problem is also not as severe for the majority, members of the minority have a strong underdog incentive in this case, suppressing the majority's initial advantage. As the population grows, the underdog incentive softens, improving the majority's chances of winning. However, when the population becomes sufficiently large, the free-rider incentive takes over in both groups, leading to more balanced winning probabilities.

Proposition 1 puts a perspective on recent studies of the costly-voting model with a small electorate. As mentioned in the Introduction, Börgers (2004) examines the symmetric setup in which $G_A = G_B$ and $\lambda_A = \lambda_B$ so that the underdog effect does not emerge. Goeree and Grosser (2007), and Taylor and Yildirim (2005) allow for $\lambda_A \neq \lambda_B$, and show that each group is equally likely to win the election. Part (iii) of Proposition 1 indicates that their assumption of a fixed and equal voting cost for all agents plays a crucial role in this "neutrality" result, because when there is cost uncertainty, the majority is strictly more likely to win even if the cost distributions are identical.

Although many voting situations such as those in boards of directors and congressional committees involve small electorates, many others such as referendums are about large electorates, which we investigate next.

4 Equilibrium in Large Electorates

We have three main objectives in this section. First, we wish to identify conditions (if any) under which the advantage from being the majority group or the group with stochastically lower cost vanishes as the population becomes large. Second, we want to determine when the limit turnout does not depend on the initial distribution of political preferences. Third, we would like to know when large elections with fixed population size can be interpreted as Myerson's Poisson games with an appropriately assigned distribution of political preferences. We begin the analysis with the following well-known result:

Lemma 3. *In equilibrium,* $\lim_{n\to\infty} \alpha_r^*(n) = 0$ *and* $\lim_{n\to\infty}[n\alpha_r^*(n)] = m_r^* < \infty$ *for* $r = A, B$.

As first shown by Palfrey and Rosenthal (1985), Lemma 3 establishes that the individual probability of voting, and thus the turnout rate, becomes negligible in large elections.

Moreover, the expected limit turnout for each alternative is finite. If it were infinite for some alternative, then each vote would be negligible, and no individual would vote given a strictly positive cost. But then, each vote would become pivotal with probability 1, yielding a contradiction.

Lemma 3 implies that in large elections, the equilibrium cutoff for each alternative must be close to the lower bound of the cost distribution, which, together with (3), leads to:

LEMMA 4. $\lim_{n\to\infty}\left[\frac{1}{2}P(\alpha_r^*(n),\alpha_{r'}^*(n),n)\right]\leq \underline{c}_r$ *(*$=\underline{c}_r$ *whenever* $c_r^*(n)>\underline{c}_r$*) for* $r,r'=A,B$ *and* $r\neq r'$.

In order to determine expected voter turnout in the limit, consider the situation facing a representative agent favoring alternative r and suppose that the other $n-1$ agents vote if and only if their costs are less than the equilibrium cutoff $c_r^*(n)$. Let $X_{A,n-1}$ and $X_{B,n-1}$ be the number of votes for alternatives A and B, respectively. Furthermore, let $X_{0,n-1}=n-1-X_{A,n-1}-X_{B,n-1}$ be the number of abstentions. Using this notation, a type r agent's vote will be pivotal if and only if $X_{r',n-1}=X_{r,n-1}$ (he breaks a tie) or $X_{r',n-1}=X_{r,n-1}+1$ (he creates a tie). Hence, the equilibrium probability that his vote is pivotal can be written

$$P(\alpha_r^*(n),\alpha_{r'}^*(n),n)=\Pr\{X_{r',n-1}^*=X_{r,n-1}^*\}+\Pr\{X_{r',n-1}^*=X_{r,n-1}^*+1\}. \tag{7}$$

Next, observe that $(X_{r,n-1}^*,X_{r',n-1}^*,X_{0,n-1}^*)$ ~Multinomial$(\alpha_r^*(n),\alpha_{r'}^*(n),1-\alpha_r^*(n)-\alpha_{r'}^*(n)|n-1)$. Note that $X_{A,n-1}^*$ and $X_{B,n-1}^*$ are not independent for $n<\infty$, but the following result establishes independence in the limit.

LEMMA 5. *The limiting marginal distributions,* $X_{A,\infty}^*$ *and* $X_{B,\infty}^*$ *are independent Poisson distributions with means* m_A^* *and* m_B^**, respectively. Hence, the limiting distribution of* $X_{A,\infty}^*+X_{B,\infty}^*$ *is Poisson with mean* $m_A^*+m_B^*$.

In light of Lemma 5, let $f(k|\mu)$ be the *p.d.f.* for a Poisson distribution with mean μ. Recall that $f(k|\mu)=\frac{\mu^k e^{-\mu}}{k!}$ for $k=0,1,\ldots$ Combining (7) and Lemma 5, it follows that

$$\begin{aligned}\lim_{n\to\infty}P(\alpha_r^*(n),\alpha_{r'}^*(n),n)&=\Pr\{X_{r',\infty}^*=X_{r,\infty}\}+\Pr\{X_{r',\infty}^*=X_{r,\infty}^*+1\}]\\&=\sum_{k=0}^{\infty}f(k|m_r^*)f(k|m_{r'}^*)+\sum_{k=0}^{\infty}f(k|m_r^*)f(k+1|m_{r'}^*) \qquad (8)\\&\equiv Q(m_r^*,m_{r'}^*).\end{aligned}$$

Together with Lemma 4, the equilibrium limiting turnouts, m_A^* and m_B^*, must then satisfy

$$\frac{1}{2}Q(m_r^*, m_{r'}^*) - \underline{c}_r \ \leq 0 \ (= \underline{c}_r \text{ if } m_r^* > 0). \tag{9}$$

Proposition 2. *Without loss of generality, suppose $\underline{c}_B \leq \underline{c}_A$. Then,*

(i) *there is a unique and strictly increasing $d(\underline{c}_A) \in (0, \underline{c}_A)$ such that*

$$\begin{cases} m_B^* > m_A^* = 0 & \text{if} \quad \underline{c}_B \leq d(\underline{c}_A) \\ m_B^* > m_A^* \ > 0 & \text{if} \quad d(\underline{c}_A) < \underline{c}_B < \underline{c}_A \\ m_B^* = m_A^* > 0 & \text{if} \quad \underline{c}_B = \underline{c}_A. \end{cases}$$

(ii) *Given $\underline{c}_A$, m_B^* is strictly decreasing and m_A^* is weakly increasing in $\underline{c}_B$.*

(iii) *Given $\underline{c}_A$, the limiting probability, $\lim_{n\to\infty} \Pr\{B \text{ wins}\}$, is strictly decreasing in $\underline{c}_B$, and equal to $\frac{1}{2}$ for $\underline{c}_B = \underline{c}_A$.*

Proof. Without loss of generality, suppose $\underline{c}_B \leq \underline{c}_A$. Using the Poisson density,

$$Q(m_r^*, m_{r'}^*) = e^{-(m_A^* + m_B^*)} \left[\sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{(k!)^2} + m_{r'}^* \sum_{k=0}^{\infty} \frac{(m_A^* m_B^*)^k}{k!(k+1)!} \right].$$

Hence, (9) implies $m_B^* \geq m_A^*$. Given $Q(0,0) = 1$ and $\underline{c}_r < \frac{1}{2}$, (9) also implies $m_B^* > 0$. Moreover, $m_A^* = 0$ if and only if $\frac{1}{2}Q(0, m_B^*) - \underline{c}_A \leq 0$ and $\frac{1}{2}Q(m_B^*, 0) - \underline{c}_B = 0$. Since $Q(0, m_B^*) = e^{-m_B^*}(1 + m_B^*)$ and $Q(m_B^*, 0) = e^{-m_B^*}$, this means $m_A^* = 0$ if and only if $2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A$. Note that for $x \in (0,1)$, the function $\varphi(x) = x(1 - \ln x)$ satisfies: $\lim_{x\to 0^+} \varphi(x) = 0$, $\lim_{x \to 1^-} \varphi(x) = 1$, $\varphi(x) > x$ and $\varphi'(x) > 0$. Hence, there exists a unique and strictly increasing $d(\underline{c}_A) \in (0, \underline{c}_A)$ that solves $2d[1 - \ln(2d)] = 2\underline{c}_A$. Clearly, $2\underline{c}_B[1 - \ln(2\underline{c}_B)] \leq 2\underline{c}_A$ for all $\underline{c}_B \leq d(\underline{c}_A)$, and $m_A^* = 0$ as a result. For $\underline{c}_B \in (d(\underline{c}_A), \underline{c}_A]$, we have $m_A^* > 0$, and by (9), $m_B^* = m_A^*$ if and only if $\underline{c}_B = \underline{c}_A$, proving part (i).

Next, if $\underline{c}_B \leq d(\underline{c}_A)$, then $m_A^* = 0$ and $\frac{1}{2}e^{-m_B^*} = \underline{c}_B$ by part (i). Thus, m_B^* is strictly decreasing in $\underline{c}_B$. Now, suppose $\underline{c}_B \in (d(\underline{c}_A), \underline{c}_A)$. Then, by part (i), $m_B^* > m_A^* \ > 0$ that solve $\frac{1}{2}Q(m_B^*, m_A^*) = \underline{c}_B$ and $\frac{1}{2}Q(m_A^*, m_B^*) = \underline{c}_A$. Simple algebra shows that $\frac{\partial}{\partial m_B} Q(m_B^*, m_A^*) < 0$; $\frac{\partial}{\partial m_A} Q(m_B^*, m_A^*) > 0$; $\frac{\partial}{\partial m_A} Q(m_A^*, m_B^*) < 0$; and $\frac{\partial}{\partial m_B} Q(m_A^*, m_B^*) < 0$. From here, it follows that m_B^* is strictly decreasing and m_A^* is strictly increasing in $\underline{c}_B$.

Finally, note that $\lim_{n\to\infty} \Pr\{B \text{ wins}\} = \sum_{k=0}^{\infty} \sum_{k'=k+1}^{\infty} f(k'|m_B^*) f(k|m_A^*) + \frac{1}{2} \sum_{k=0}^{\infty} f(k|m_B^*) f(k|m_A^*)$. It is easy to verify that the r.h.s. is strictly increasing in m_B^* and strictly decreasing in m_A^*. Part (iii) then follows from part (ii). ■

Proposition 2 is a key result of this paper. The most important observation is that the limit turnouts and the probability of winning are completely determined by the individuals with the lowest cost of voting in each group – *not* by the distributions of voting costs, G_r *or* political preferences, λ_r. This is because the free-rider problem in each group is amplified as the electorate size grows, leaving only the lowest cost agents to vote. As part (i) indicates however, one group may abstain altogether if the cost differential is sufficiently large, and such a cost differential *always* exists. Nonetheless, because the limit turnout for each group is finite, there is still a significant probability that the abstaining group will win.[16]If the cost differential is not too high, Proposition 2 leads to the intuitive observation that the group with the lowest possible cost is expected to turn out in larger number and thus more likely to win the election. In addition, as the differential increases, so does the probability of winning for the low-cost group.

The distinction between a large and small election becomes most transparent when $\underline{c}_A = \underline{c}_B$. The advantage from being in the majority or from having a favorable cost distribution identified in Proposition 1 for a small election completely vanish in the limit, making each alternative *equally likely* to win. Moreover, two large elections one with $\lambda_r = .5$ and one with $\lambda_r \neq .5$ result in equal limiting turnouts.[17] Hence, the widely held intuition that elections with a more evenly split electorate should generate a greater expected turnout appears to be a property of small elections.

Proposition 2 also unifies various results pertaining to large electorates in the costly voting literature. For instance, it implies that an expected minority whose members are likely to have lower cost-benefit ratios may end up winning the election if the electorate size is sufficiently large. Indeed, this is what Campbell (1999), in rationalizing minority upsets in elections, finds, though he doesn't provide a complete asymptotic characterization. In a sense, it is the "quality" – not the "quantity" – of supporters that counts in order to win an election. In a more recent paper, Krasa and Polborn (2007), using a special case of the present model where $G_A = G_B$ and thus $\underline{c}_A = \underline{c}_B$, investigate socially optimal voting subsidies or nonvoting penalties, and have independently discovered that without such interventions, each alternative wins a large election with probability $\frac{1}{2}$. Proposition 2 shows that the assumption of equal cost lower bounds is both necessary and sufficient for

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[16] In fact, if $\underline{c}_B \leq d(\underline{c}_A)$, then $\lim_{n\to\infty} \Pr\{A \text{ wins}\} = \frac{1}{2}\Pr\{X_B^* = 0\} = \underline{c}_B$.

[17] In a previous version of this paper, we determined that if $\underline{c}_A = \underline{c}_B = \underline{c}$ and $\underline{c}$ is small, the aggregate limit turnout can be approximated by $\frac{1}{2\pi\underline{c}^2}$.

this $\frac{1}{2}$ result, pointing to its rather fragile knife-edge nature. If the cost lower bounds are not equal, then one still obtains the intuitive result that the group with a cost advantage is strictly more likely to win a large election. Finally, Taylor and Yildirim (2005) who analyze the impact of public information about the distribution of political preferences on election outcomes and welfare, also uncover the $\frac{1}{2}$ result to be the probability of winning in a large electorate where each agent has a fixed and equal cost of voting, i.e., a degenerate cost distribution.[18] Proposition 2 reveals that while the result would remain true with the introduction of a nondegenerate cost distribution, if voting costs are fixed, then they must be equal in order for neutrality to obtain. Said differently, in a large election, a model with equal fixed voting costs and a model with symmetric cost uncertainty are strategically equivalent if and only if the fixed voting cost in the first model equals the lowest possible cost in the second one.

In the next two subsections, we take advantage of the Poisson characterization to link the costly voting model to Myerson's Poisson games, and also investigate the rate of convergence to the limits found in Proposition 2.

4.1 Large Elections with Costly Voting and Myerson's Poisson Games

Inspired mostly by large elections, Myerson (1998, 2000) introduced the concept of Poisson games, where the number of players (the electorate size, here) is distributed according to a Poisson distribution with an exogenous mean, rather than being fixed. Our characterization in Proposition 2 allows us to interpret large elections as Poisson games. However, as Myerson (2000, p.27) notes, because abstentions occur in equilibrium with costly voting, this interpretation is nontrivial. In particular, the expected number of active players, m_r^*, is endogenously determined by asymptotic equilibrium strategies. Moreover, within the set of active players, it is *incorrect* to assume that the probability that an agent votes for alternative r is λ_r. Rather, large elections analyzed here correspond to a Poisson game where the mean population is $m_A^* + m_B^*$, and each agent favoring alternative r votes with probability $\frac{m_r^*}{m_A^*+m_B^*}$, which is $\frac{1}{2}$ in the special case of $\underline{c}_A = \underline{c}_B$.

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[18] As mentioned in the previous section, Goeree and Grosser (2007) also find the $\frac{1}{2}$ result, but only for a small electorate.

4.2 Rate of Convergence

A striking implication of Proposition 2 is that as $n \to \infty$, the expected turnout, and thus the probability of winning, is independent of political preferences and cost distributions except for their lower bounds. Nonetheless, it seems intuitive that if, all else equal, cost distributions put more weight on realizations around the lower bound, it should take a larger electorate size for most agents on both sides to abstain and the expected turnout to converge to the limit in Proposition 2. By the same token, convergence should be slower when there is a stronger majority, because, in this case, a larger electorate size is needed for the free-rider problem in the majority group to overwhelm its initial advantage. The following result formalizes these intuitions.

PROPOSITION 3. *Suppose $\lambda_A \geq \lambda_B$ and G_A is first-order stochastically dominated by G_B. Then, an election with electorate size n converges more slowly as $\lambda_A g_A(\underline{c}_A)$ increases.*

PROOF. Suppose $\lambda_A \geq \lambda_B$ and G_A is first-order stochastically dominated by G_B. By Proposition 1, this means $\alpha_B^*(n) \leq \alpha_A^*(n)$ and $\alpha_A^*(n) > 0$, which, by (4), imply $\lambda_B G(\frac{1}{2}P(\alpha_B^*(n), \alpha_A^*(n), n)) \leq \alpha_B^*(n)$ and $\lambda_A G(\frac{1}{2}P(\alpha_A^*(n), \alpha_B^*(n), n)) = \alpha_A^*(n)$. Now, recall from Lemma 3 that $\lim_{n\to\infty} \alpha_A^*(n) = \lim_{n\to\infty} \alpha_B^*(n) = 0$. Thus, for every $\varepsilon > 0$, there exist $N_A(\varepsilon) \geq N_B(\varepsilon)$ such that for all $n \geq N_A(\varepsilon)$, we have $|\alpha_r^*(n) - 0| < \varepsilon$ for $r = A, B$, or equivalently

$$\lambda_r G_r(\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n)) < \varepsilon. \tag{10}$$

Next, for a large n, we expand $G_r(\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n))$ around $\underline{c}_r$ using a first-order Taylor approximation:

$$G_r(\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n)) \approx g_r(\underline{c}_r)\left[\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n) - \underline{c}_r\right].$$

Inserting this fact into (10) implies that for all $n \geq N_A(\varepsilon)$:

$$\lambda_r g_r(\underline{c}_r)\left[\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n) - \underline{c}_r\right] < \varepsilon. \tag{11}$$

Since $\alpha_B^*(n) \leq \alpha_A^*(n)$, we have

$$\lambda_B g_B(\underline{c}_B)\left[\frac{1}{2}P(\alpha_B^*(n), \alpha_A^*(n), n) - \underline{c}_B\right] \leq \lambda_A g_A(\underline{c}_A)\left[\frac{1}{2}P(\alpha_A^*(n), \alpha_B^*(n), n) - \underline{c}_A\right].$$

Hence, the inequality in (11) is less likely to be satisfied as $\lambda_A g(\underline{c}_A)$ increases. ■

Proposition 3 indicates that closely contested elections and elections with relatively high cost to benefit ratio will converge to the limit in Proposition 2 faster.

5 On the Uniqueness of Type-Symmetric Equilibrium

In this section, we take existence of a type-symmetric equilibrium one step further and establish a sufficient condition for its uniqueness. The possibility of equilibrium uniqueness received some attention in the literature with private values. Börgers (2004) showed that when all agents are *ex ante* symmetric, i.e., $\lambda_A = \lambda_B$ and $G_A = G_B$, then the type-symmetric equilibrium is unique. Goeree and Grosser (2007), and Taylor and Yildirim (2005) proved the uniqueness of type-symmetric equilibrium in totally mixed strategies when $\lambda_A \neq \lambda_B$ and each agent has a fixed and equal cost of voting. Given the special structures in these investigations, however, it is difficult to understand what drives the uniqueness result and whether or not it is robust to (at least) small perturbations. In particular, all of these studies have utilized two observations: $\alpha_A = \alpha_B = \alpha$ at an equilibrium, and the pivot probability along this path, namely $P(\alpha, \alpha, n)$, is strictly decreasing in α. Neither of these observations is true in general, as we now know from Lemma 1 and Proposition 1 above. Intuitively though, the uniqueness result should continue to hold if α_A and α_B are sufficiently close in equilibrium.[19]

Lemma 6. *If, in a type-symmetric equilibrium,* $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B}{\alpha_A} \leq 1$*, then it is unique.*

Proof. Suppose, in equilibrium, $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\alpha_B}{\alpha_A} \leq 1$. Then, by Lemma 1, $\frac{\partial}{\partial \alpha_A} P(\alpha_A, \alpha_B, n) < 0$ and $\frac{\partial}{\partial \alpha_B} P(\alpha_A, \alpha_B, n) \geq 0$, which imply $\frac{\partial}{\partial \alpha_A} \Phi_A(\alpha_A, \alpha_B) < 0$ and $\frac{\partial}{\partial \alpha_B} \Phi_A(\alpha_A, \alpha_B) \geq 0$. Again, by Lemma 1, $\frac{\partial}{\partial \alpha_B} P(\alpha_B, \alpha_A, n) < 0$ and $\frac{\partial}{\partial \alpha_A} P(\alpha_B, \alpha_A, n) \leq 0$, and thus $\frac{\partial}{\partial \alpha_B} \Phi_B(\alpha_B, \alpha_A) < 0$ and $\frac{\partial}{\partial \alpha_A} \Phi_B(\alpha_B, \alpha_A) \leq 0$. Now, suppose there are two equilibria: (α_A^*, α_B^*) and $(\alpha_A^{**}, \alpha_B^{**})$. If $\alpha_A^* > \alpha_A^{**}$, then since, by definition of equilibrium, $\Phi_A(\alpha_A^*, \alpha_B^*) = \Phi_A(\alpha_A^{**}, \alpha_B^{**}) = 0$, we would have $\alpha_B^* > \alpha_B^{**}$. Moreover, since $\Phi_B(\alpha_B^*, \alpha_A^*) = \Phi_B(\alpha_B^{**}, \alpha_A^{**}) = 0$, we would also have $\alpha_B^* \leq \alpha_B^{**}$ – a contradiction. Thus, $\alpha_A^* \leq \alpha_A^{**}$. By an exact argument, it also follows that $\alpha_A^* \geq \alpha_A^{**}$. Hence, $\alpha_A^* = \alpha_A^{**}$, which implies $\alpha_B^* = \alpha_B^{**}$. ■

As alluded to above, the potential source of multiple equilibria is that members of *some* political group view their votes as complements rather than substitutes. In light of Lemma 1, such complementarity between the votes can occur only in the group whose members' *ex ante* probability of voting is far below the rival's so that the free-rider incentive is not

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[19] For $n = 2$, uniqueness obtains without any additional condition because, by Lemma 1, $P(\alpha_r, \alpha_{r'}, 2)$ is independent of $\alpha_{r'}$ and so is $\Phi_r(\alpha_r, \alpha_{r'}, 2)$.

strong enough to overwhelm the coordination incentive. Lemma 6 simply says that when equilibrium voting strategies are sufficiently symmetric across the groups, the free-rider incentive dominates for *all* individuals, facilitating equilibrium uniqueness. The following result provides some parameter conditions under which Lemma 6 holds.

Proposition 4. *Suppose $G_A = G_B$. Then, type-symmetric equilibrium is unique, if*

- $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A} \leq 1$*; or*
- n *is sufficiently large.*

Proof. Let $G_A = G_B$ and $\lambda_A \geq \lambda_B$. From Proposition 1, we have $0 < \phi_A^* \leq \phi_B^*$ and $\alpha_A^* \geq \alpha_B^*$ in a type-symmetric equilibrium. This implies

$$\frac{\alpha_B^*}{\alpha_A^*} = \frac{\lambda_B \phi_B^*}{\lambda_A \phi_A^*} \geq \frac{\lambda_B}{\lambda_A}.$$

Hence, if, in addition, $1 - \frac{1}{\lfloor \frac{n}{2} \rfloor} \leq \frac{\lambda_B}{\lambda_A}$, then we have $\alpha_B^* \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_A^*$, and uniqueness follows from Lemma 6. Next, note that since we have $\lim_{n \to \infty} [n\alpha_r^*(n)] = m_r^* < \infty$ by Lemma 3, and $m_A^* = m_B^* = m^* > 0$ by Proposition 2, it follows that $\alpha_r^*(n) \approx \frac{m^*}{n}$ for a large n, and thus $\alpha_B^* \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_A^*$ is satisfied.[20] Uniqueness again follows from Lemma 6. ■

The first part of Proposition 4 demonstrates that Börgers' uniqueness result derived under complete symmetry, i.e., $G_A = G_B$ and $\lambda_A = \lambda_B$ is robust to (at least) small perturbations. That is, if agents are sufficiently symmetric, then type-symmetric equilibrium remains unique. The second part shows that large elections are likely to yield a unique equilibrium as well; because each agent votes with a small probability in any case. The latter finding is significant because Palfrey and Rosenthal (1985) observe the possible multiplicity of type-symmetric equilibria, and then argue that the low turnout equilibrium is more likely to arise. Although we study a slight variation of their model, Proposition 4 provides formal support that the low turnout equilibrium is unique in large elections. Finally, our result complements that of Krishna and Morgan (2008) who establish equilibrium uniqueness for a large election with *common* values.

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[20] Put differently, whereas $\alpha_B^*(n)$ converges to 0 at the rate of $\frac{1}{n}$, $\left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_A^*(n)$ converges to 0 at the rate of $\frac{1}{n^2}$.

6 Concluding Remarks

There are two ways to interpret the contribution of this paper. First, it deepens our understanding of the rational choice theory of voting in its purest form, and second, by doing so, it allows for richer and better grounded empirical and experimental investigation. Some prominent recent developments in voting theory have been concentrated around a model involving "group-based ethical voters" who care not only about their own payoff but also the payoffs of others with similar political preferences (Feddersen (2004)). While we believe the group-based approach shows some promise, we also believe there are further directions in which rational voting theory can be fruitfully extended to better reflect reality.

For one, it would be useful to expand the notion of a pivotal vote to recognize the fact that the vote counting process is imperfect.[21] Hence, it would be edifying to extend the basic model to explicitly account for the vote counting technology, and to study its implications on voter behavior. Another important assumption of the basic theory that could be profitably relaxed is that costs of voting are independently distributed across citizens. There are many factors such as weather and security concerns that influence costs of voting for large groups of citizens. Hence, an extension that allows for cost correlation would also be valuable. We leave these extensions for future work and conclude by remarking simply that there is still a lot to discover in the context of rational voting theory.

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[21] In fact, in many states in the U.S. if a vote count is too close, then a recount is either triggered automatically or may be demanded by the potential loser.

A Appendix

Proof of Lemma 1: First, note that a type r agent's expected payoffs from voting and abstaining are given, respectively by

$$\begin{aligned} V_r^1 = & \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r} \\ & \times \left[\sum_{k_{r'}=0}^{k_r-1} \binom{n-1-k}{k_{r'}} \phi_{r'}^{k_{r'}} (1-\phi_{r'})^{n-1-k-k_{r'}} + \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right. \\ & \left. + \frac{1}{2} \binom{n-1-k}{k_r+1} \phi_{r'}^{k_r+1} (1-\phi_{r'})^{n-2-k-k_r} \right] - c, \end{aligned}$$

and

$$\begin{aligned} V_r^0 \; = \; & \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r} \\ & \times \left[\sum_{k_{r'}=0}^{k_r-1} \binom{n-1-k}{k_{r'}} \phi_{r'}^{k_{r'}} (1-\phi_{r'})^{n-1-k-k_{r'}} + \frac{1}{2} \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right]. \end{aligned}$$

To understand these expected payoffs, fix a type r agent, and let k_r be the number of votes for alternative r excluding his, and $k_{r'}$ be the number of all votes for alternative r'. Clearly, if $k_{r'} \leq k_r - 1$ and $k_{r'} \geq k_r + 2$, then alternative r respectively wins and loses with probability 1, regardless of the type r agent's action. If $k_{r'} = k_r$, alternative r wins with probability 1 if the type r agent in question votes, and wins with probability $\frac{1}{2}$ if he abstains and leaves the tie. Finally, if $k_{r'} = k_r + 1$, alternative r loses with probability 1 if the type r agent abstains; but may win with probability $\frac{1}{2}$ if he votes. These events explain the expressions in parentheses above. The first two summations in V_r^1 and V_r^0 account for the distribution of preferences.

Next, subtracting V_r^0 from V_r^1, the third summation inside parentheses cancel out, reducing the net expected payoff to

$$\begin{aligned} \Delta_r = V_r^1 - V_r^0 = & \sum_{k=0}^{n-1} \binom{n-1}{k} \lambda_r^k (1-\lambda_r)^{n-1-k} \sum_{k_r=0}^{k} \binom{k}{k_r} \phi_r^{k_r} (1-\phi_r)^{k-k_r} \qquad \text{(A-1)} \\ & \times \left[\frac{1}{2} \binom{n-1-k}{k_r} \phi_{r'}^{k_r} (1-\phi_{r'})^{n-1-k-k_r} \right. \\ & \left. + \frac{1}{2} \binom{n-1-k}{k_r+1} \phi_{r'}^{k_r+1} (1-\phi_{r'})^{n-2-k-k_r} \right] - c. \end{aligned}$$

Now, recall $\alpha_r = \lambda_r\phi_r$ and define $\beta_r = \lambda_r(1-\phi_r)$. By substituting for these terms into (A-1), and noting the following facts,

$$\begin{aligned}\lambda_r^k &= \lambda_r^{k_r}\lambda_r^{k-k_r}\\ (1-\lambda_r)^{n-1-k} &= (1-\lambda_r)^{k_r}(1-\lambda_r)^{n-1-k-k_r}\\ (1-\lambda_r)^{n-1-k} &= (1-\lambda_r)^{k_r+1}(1-\lambda_r)^{n-2-k-k_r},\end{aligned}$$

(A-1) further reduces to

$$\begin{aligned}\Delta_r &= \frac{1}{2}\sum_{k=0}^{n-1}\Bigg[\binom{n-1}{k}\sum_{k_r=0}^{k}\binom{k}{k_r}\alpha_r^{k_r}\beta_r^{k-k_r}\\ &\quad\times\left(\binom{n-1-k}{k_r}\alpha_{r'}^{k_r}\beta_{r'}^{n-1-k-k_r}+\binom{n-1-k}{k_r+1}\alpha_{r'}^{k_r+1}\beta_{r'}^{n-2-k-k_r}\right)\Bigg]-c\\ &= \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r}\\ &\quad+\frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r+1}\alpha_r^{k_r}\alpha_{r'}^{k_r+1}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r}-c.\end{aligned}$$

Using the following two combinatorial identities:

$$\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r}=\binom{n-1}{k_r,k_r,n-1-2k_r}\binom{n-1-2k_r}{k-k_r}$$

and

$$\binom{n-1}{k}\binom{k}{k_r}\binom{n-1-k}{k_r+1}=\binom{n-1}{k_r,k_r+1,n-2-2k_r}\binom{n-2-2k_r}{k-k_r},$$

Δ_r becomes

$$\begin{aligned}\Delta_r &= \frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{k_r,k_r,n-1-2k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r}\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r}\\ &\quad+\frac{1}{2}\sum_{k_r=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k_r,k_r+1,n-2-2k_r}\alpha_r^{k_r}\alpha_{r'}^{k_r+1}\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r}-c\\ &= \frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1}{k,k,n-1-2k}\alpha_r^{k}\alpha_{r'}^{k}(\beta_r+\beta_{r'})^{n-1-2k}\\ &\quad+\frac{1}{2}\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k+1,n-2-2k}\alpha_r^{k}\alpha_{r'}^{k+1}(\beta_r+\beta_{r'})^{n-2-2k}-c,\end{aligned}$$

where we also use the facts that

$$\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-k-k_r}=\sum_{k=0}^{n-1}\binom{n-1-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-1-2k_r-(k-k_r)}=(\beta_r+\beta_{r'})^{n-1-2k_r}$$

and

$$\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-k-k_r}=\sum_{k=0}^{n-1}\binom{n-2-2k_r}{k-k_r}\beta_r^{k-k_r}\beta_{r'}^{n-2-2k_r-(k-k_r)}=(\beta_r+\beta_{r'})^{n-2-2k_r}$$

and, without loss of generality, change index of summations to k in the last equality. The expressions in (1) and (2) then follows by simply observing that $\beta_A+\beta_B=1-\alpha_A-\alpha_B$.

Next, we proceed to Lemma 1. Let $(\alpha_r,\alpha_{r'})\in(0,\lambda_r)\times(0,\lambda_{r'})$. Using the definition of $P(\alpha_r,\alpha_{r'},n)$ in (2) and canceling the first terms, part (i) follows because

$$P(\alpha_r,\alpha_{r'},n)-P(\alpha_{r'},\alpha_r,n)=(\alpha_{r'}-\alpha_r)\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\binom{n-1}{k,k+1,n-2-2k}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}.$$

Differentiating $P(\alpha_r,\alpha_{r'},n)$ with respect to $\alpha_{r'}$,

$$\begin{aligned}\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n) &= \sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^{k+1}\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-3-2k}\\ &\quad-\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{(n-1)!}{k!k!(n-2-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}\\ &\quad+\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\frac{(n-1)!}{k!k!(n-2-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-2-2k}\\ &\quad-\sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^k\alpha_{r'}^{k+1}(1-\alpha_r-\alpha_{r'})^{n-3-2k}.\end{aligned}$$

Note that the second and third terms on the r.h.s. cancel out. The remaining two terms can be rewritten

$$\frac{\partial}{\partial\alpha_{r'}}P(\alpha_r,\alpha_{r'},n)=(\alpha_r-\alpha_{r'})\sum_{k=0}^{\lfloor\frac{n-3}{2}\rfloor}\frac{(n-1)!}{k!(k+1)!(n-3-2k)!}\alpha_r^k\alpha_{r'}^k(1-\alpha_r-\alpha_{r'})^{n-3-2k},$$

where the summation is 0 for $n=2$ for all $(\alpha_r,\alpha_{r'})$.

To prove part (iii), suppose n is odd and differentiate $P(\alpha_r, \alpha_{r'}, n)$ with respect to α_r to obtain

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(k-1)!k!(n-1-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\ & - \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ & + \sum_{k=1}^{\frac{n-3}{2}} \frac{(n-1)!}{(k-1)!(k+1)!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ & - \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-3-2k)!} \alpha_r^{k} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-3-2k}.\end{aligned}$$

Note that the first and the last terms cancel out. Separating the term for $k=0$, the second term can be re-written $\sum_{k=0}^{\frac{n-3}{2}} (.) = (n-1)(1-\alpha_r-\alpha_{r'})^{n-2} + \sum_{k=1}^{\frac{n-3}{2}} (.)$. This implies

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\frac{n-3}{2}} \left(\frac{k}{k+1} \alpha_{r'} - \alpha_r \right) \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ & -(n-1)(1-\alpha_r-\alpha_{r'})^{n-2}.\end{aligned}$$

A similar line of derivation shows that for an even n, only the upper bound in the above summation switches to $\frac{n-2}{2}$. Hence, for any $n \geq 2$,

$$\begin{aligned}\frac{\partial}{\partial \alpha_r} P(\alpha_r, \alpha_{r'}, n) &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\frac{k}{k+1} \alpha_{r'} - \alpha_r \right) \frac{(n-1)!}{k!k!(n-2-2k)!} \alpha_r^{k-1} \alpha_{r'}^{k} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\ & -(n-1)(1-\alpha_r-\alpha_{r'})^{n-2}.\end{aligned}$$

Note that if $\alpha_r \geq \left(1 - \frac{1}{\lfloor \frac{n}{2} \rfloor}\right) \alpha_{r'}$, then $\frac{k}{k+1}\alpha_{r'} - \alpha_r \leq 0$ for each $k \in \{1, ..., \lfloor \frac{n}{2} \rfloor - 1\}$. Together with $1 - \alpha_r - \alpha_{r'} \neq 0$, it follows that $\frac{\partial}{\partial \alpha_{r'}} P(\alpha_r, \alpha_{r'}, n) < 0$.

To prove part (iv), we begin by noting that

$$P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \tag{A-2}$$

$$+ \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}$$

$$- \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$- \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k}.$$

Before signing this expression, we suppose that n is odd, and re-write the third summation:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= \sum_{k=0}^{\frac{n-1}{2}} \left[1 + \frac{2k}{n-2k}\right] \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= (1-\alpha_r-\alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k}$$

$$+ 2 \sum_{k=1}^{\frac{n-1}{2}} \frac{(n-1)!}{(k-1)!k!(n-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-2k}$$

$$= (1-\alpha_r-\alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k}$$

$$+ 2 \sum_{k=0}^{\frac{n-1}{2}-1} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^{k+1} \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}.$$

Inserting this into (A-2) and canceling terms yield

$$\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
= \; & (\alpha_r + \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& +(1-2\alpha_r) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k} \\
& - \sum_{k=0}^{\frac{n-1}{2}} \frac{n!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k}.
\end{aligned}$$

Now, noting $\frac{n!}{k!(k+1)!(n-1-2k)!} = \left(1 + \frac{k}{n-1-2k} + \frac{k+1}{n-1-2k}\right) \frac{(n-1)!}{k!(k+1)!(n-2-2k)!}$, we re-write the last summation in three terms. Moreover, we expand the first and second summations by multiplying with $(\alpha_r + \alpha_{r'})$ and $(1-2\alpha_r)$, respectively. Canceling and collecting terms then reveal

$$\begin{aligned}
& P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) \\
= \; & \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{k!(k+1)!(n-1-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& +(\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-1}{2}} \frac{(n-1)!}{(k!)^2(n-1-2k)!} \alpha_r^k \alpha_{r'}^k (1-\alpha_r-\alpha_{r'})^{n-1-2k} \\
& -(\alpha_r - \alpha_{r'}) \sum_{k=0}^{\frac{n-3}{2}} \frac{(n-1)!}{k!(k+1)!(n-2-2k)!} \alpha_r^k \alpha_{r'}^{k+1} (1-\alpha_r-\alpha_{r'})^{n-2-2k}.
\end{aligned}$$

For $\alpha_r = \alpha_{r'}$, clearly $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1) > 0$. For $\alpha_r \neq \alpha_{r'}$, note that

$$\begin{aligned}
P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) &= [P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+1)] \\
&\quad +[P(\alpha_r, \alpha_{r'}, n+1) - P(\alpha_r, \alpha_{r'}, n+2)]
\end{aligned}$$

Performing similar decompositions to those above, it follows that $P(\alpha_r, \alpha_{r'}, n) - P(\alpha_r, \alpha_{r'}, n+2) > 0$. ■

Before proving Lemmas 2 and 3, we note the following useful result.

LEMMA A1. *Fix a pair* $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ *such that* $(\alpha_A, \alpha_B) \neq (0, 0)$*. Then,* $\lim_{n\to\infty} P(\alpha_A, \alpha_B, n) = \lim_{n\to\infty} P(\alpha_A, \alpha_B, n) = 0$.

PROOF OF LEMMA A1. Fix a pair $(\alpha_A, \alpha_B) \in [0, \lambda_A] \times [0, \lambda_B]$ such that $(\alpha_A, \alpha_B) \neq (0, 0)$. Let $X_{A,n}$ and $X_{B,n}$ be the number of votes for alternatives A and B, respectively, and $X_{0,n} = n - X_{A,n} - X_{B,n}$ be the number abstentions. Clearly, $(X_{A,n}, X_{B,n}, X_{0,n}) \sim$Multinomial$(\alpha_A, \alpha_B, 1-\alpha_A - \alpha_B | n)$. By definition of the pivot probability in (2), this means

$$P(\alpha_A, \alpha_B, n) = \Pr\{W_{BA,n} = 0\} + \Pr\{W_{BA,n} = 1\},$$

where $W_{BA,n} \equiv X_{B,n} - X_{A,n}$ such that $E[W_{BA,n}] = n(\alpha_B - \alpha_A)$ and $Var[W_{BA,n}] = n[\alpha_A(1-\alpha_A) + \alpha_B(1 - \alpha_B) + 2\alpha_A\alpha_B]$. It is well-known (see, e.g., Arnold (1990), Th. 5.8) that

$$\frac{W_{BA,n} - E[W_{BA,n}]}{\sqrt{Var[W_{BA,n}]}} \to^D N(0, 1),$$

which implies $\Pr\{W_{BA,n} = 0\} \to 0$ and $\Pr\{W_{BA,n} = 1\} \to 0$ as $n \to \infty$. Hence, $P(\alpha_A, \alpha_B, n) \to 0$. Re-labeling, it also follows that $P(\alpha_B, \alpha_A, n) \to 0$. ■

PROOF OF LEMMA 2. Fix a pair $(\alpha_A, \alpha_B) \in [0, \lambda_A) \times [0, \lambda_B)$. Using the expression in (5) for $\pi(\alpha_A, \alpha_B, n)$, a simple algebra shows that $\pi(\alpha_A, \alpha_B, n+1) - \pi(\alpha_A, \alpha_B, n) = \frac{1}{2}(\alpha_A - \alpha_B) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2(n-2k)!} \alpha_A^k \alpha_B^k (1 - \alpha_A - \alpha_B)^{n-2k}$. Hence, $\pi(\alpha_A, \alpha_B, n+1) > \pi(\alpha_A, \alpha_B, n)$ if and only if $\alpha_A > \alpha_B$. To prove the limit result, we use the characterization in the proof of Lemma A1. Note that $\pi(\alpha_A, \alpha_B, n) = \Pr\{W_{BA,n} < 0\} + \frac{1}{2}\Pr\{W_{BA,n} = 0\}$. If $\alpha_A > \alpha_B$, then $\Pr\{W_{BA,n} < 0\} \to 1$ and $\Pr\{W_{BA,n} = 0\} \to 0$, and thus $\pi(\alpha_A, \alpha_B, n) \to 1$ as $n \to \infty$.

To prove part (ii), we simply differentiate $\pi(\alpha_A, \alpha_B, n)$ with respect to α_A, and after some tedious algebra, find that $\pi(\alpha_A, \alpha_B, n)$ is strictly increasing in α_A. Since, by symmetry, $\pi(\alpha_A, \alpha_B, n) + \pi(\alpha_B, \alpha_A, n) = 1$ for all α_A and α_B, it also follows that $\pi(\alpha_A, \alpha_B, n)$ is strictly decreasing in α_B. Finally, from the expression in (5), we obtain $\lim_{\alpha_A \to 0} \lim_{\alpha_B \to 0} \pi(\alpha_A, \alpha_B, n) = \lim_{\alpha_B \to 0} \lim_{\alpha_A \to 0} \pi(\alpha_A, \alpha_B, n) = \pi(0, 0, n) = \frac{1}{2}$. ■

PROOF OF LEMMA 3. Suppose, on the contrary, $\lim_{n\to\infty} \alpha_r^*(n) > 0$. Since $\alpha_r^*(n) \in [0, \lambda_r]$, by Bolzano-Weierstrass theorem, there is a subsequence $\widehat{\alpha}_r^*(n)$ that converges to some $\ell > 0$. This implies: $\widehat{\alpha}_r^*(n) > 0$ for a sufficiently large n, and together with Lemma A1, $P(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n), n) \to 0$ as $n \to \infty$. Using (4), the latter further implies $\Phi_r(\widehat{\alpha}_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n, and thus $\widehat{\alpha}_r^*(n) = 0$ – a contradiction. Hence, $\lim_{n\to\infty} \alpha_r^*(n) = 0$.

To prove the second part, suppose, on the contrary, $\lim_{n\to\infty}[n\alpha_r^*(n)] = \infty$. Then, clearly $\alpha_r^*(n) > 0$ for a large n and thus $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) = 0$. Moreover, for a fixed n, we can use a multinomial decomposition for the pivot probability as in Lemma A1 above, and find that $P(\alpha_r^*(n), \alpha_{r'}^*(n), n)$ becomes arbitrarily small as n gets large. In particular,

$\frac{1}{2}P(\alpha_r^*(n), \alpha_{r'}^*(n), n) < \underline{c}_r$ and $\Phi_r(\alpha_r^*(n), \alpha_{r'}^*(n)) < 0$ for a sufficiently large n, yielding a contradiction. Hence, $\lim_{n\to\infty}[n\alpha_r^*(n)] < \infty$. ■

Proof of Lemma 4. Immediately follows from Lemma 3 and eq.(3). ■

Proof of Lemma 5. Note first that the marginal distribution of $X_{A,n-1}^*$ conditional on X_B is $X_{A,n-1}^*|X_B \sim$Binomial$(n-1-X_B, \frac{\alpha_A^*(n)}{1-\alpha_B^*(n)})$. Since, by Lemma 4, $\alpha_r^*(n) \to 0$ and $n\alpha_r^*(n) \to m_r^* < \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} E[X_{A,n-1}^*|X_B] = m_A^*.$$

Hence, (see, Arnold (1990), Th. 5.5)

$$X_{A,n-1}^*|X_B \xrightarrow{D} \text{Poisson}(m_A^*),$$

which is independent of X_B. The same argument shows

$$X_{B,n-1}^*|X_A \xrightarrow{D} \text{Poisson}(m_B^*).$$

As a result, the limiting distributions, of $X_{A,\infty}^*$ and $X_{B,\infty}^*$ are independent Poissons, and

$$\left(X_{A,\infty}^* + X_{B,\infty}^*\right) \sim \text{Poisson}(m_A^* + m_B^*). \quad ■$$

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