Asymmetric Inventory Dynamics and Product Market Search

Linxi Chen*

October 13, 2017

Abstract

Why does inventory investment accounts for on average 72% of GDP decline during recessions but only 8% during expansions? Why does inventory-sales ratio ceases to be countercyclical since 1990s but continue to lag GDP for 4 quarters? These newly documented stylized facts pose challenges to existing macroeconomic models with inventories and cast doubts on important conclusions drawn from standard models. In this paper I show that incorporating product market search friction in a standard inventory model can address these stylized facts. Product market search enhances firms' asymmetric trade-off between accumulating inventory and adjusting markup, and thus generates strong asymmetric share of inventory investment in GDP movements. Product market search also generates the lagging inventory-sales ratio because in expansions(recessions) households' procyclical effort to search for varieties increases(decreases) sales and inventory holding at the early stage of expansions(recessions). Its effects, however, are later eclipsed by heightened(lowered) return to holding inventory which only increases(decreases) inventory stock. The model is empirically disciplined with micro evidence provided by recent empirical studies and its behavior is consistent with inventory and business cycle stylized facts in the U.S., in addition to explaining the two newly documented inventory stylized facts and being broadly consistent with observed business cycle asymmetries.

JEL classification: E32; E22

Keywords: Inventory; Product Market; Business Cycle Asymmetry; Search and match friction

[1] I thank my advisors and family for their support and tolerance.

* Duke Univeristy, Economics Deparment. Email: linxi.chen@duke.edu

1 Introduction

Since the key contribution of Metzler (1941) inventory behavior is at the center of business cycle research. It is generally agreed that even though inventory behavior is *stabilizing* at the microeconomic level it is *destabilizing* at the macroeconomic level. A key observation is that aggregate inventory investment is positively correlated with final sales thus rendering GDP to be more volatile than final sales. For excellent reviews of the vast macroeconomic inventory research the reader is referred to Blinder and Maccini (1991); Ramey and West (1999). However modern incarnations to uncover the role played by inventory yield mixed conclusion. Some authors, for example Khan and Thomas (2007); Kryvtsov and Midrigan (2012), find that inventory matters little for business cycle volatility while others (Kahn, McConnell, Perez-Quiros et al. (2002); Davis and Kahn (2008); Kahn (2008)) link improvements in inventory management to the Great Moderation.

I argue that one important piece of inventory behavior is missing in the debates: inventory behaves differently in booms and busts. It is well recognized that inventory de-cumulation accounts for large share of GDP decline during recessions and this fact is commonly used to argue the importance to inventory research. However a less well known fact is that inventory accumulation accounts for little GDP growth during economic expansions. This asymmetry is strikingly because it doesn't apply to other major components of GDP (for example, consumption, fixed investment, and net exports). More importantly, most existing models are focusing on explaining the *unconditional* behavior of inventory, which is arguably dominated by its behavior during expansions, even though inventory is considerably more destabilizing in recessions.

Using a Markov-switching vector autoregression (MS-VAR) model, I document substantial differences in the joint dynamics of inventory investment, sales, and GDP. In particular, it suggests that inventory plays less of a role of stock-out avoidance and more of production smoothing in recessions, and vice versa in expansions. Additionally, the strong asymmetry is the joint product of transitional dynamics from one regime to the other and the change in conditional (on regimes) joint dynamics. The MS-VAR reveals rich dynamics that existing models fail to account for entirely thus explaining their inadequacy in the explanation of asymmetric inventory dynamics

Why inventory investment accounts for drastically larger share of GDP change during recessions than that during expansions? Why this asymmetry features only in inventory investment but not in other components of GDP? Without a model to adequately explain the asymmetry of inventory dynamics, we can't completely understand the asymmetric inventory dynamics and more broadly business cycles asymmetries and fluctuations. I augmented a off-the-shelf real business cycle (RBC) model with search and match friction in goods market to investigate the cause of asymmetric inventory dynamics and the implications. The model generates strong asymmetric inventory dynamics as seen in the data. My model is also successful in matching traditional inventory stylized facts. What's more, the model fares better than most models with inventory in matching business cycle moments. Some direct evidences from micro-level data are presented to support the main mechanism.

The idea behind the model is simple. Search and match friction prevents all goods from successfully selling, therefore only a fraction of goods would become sales. The exact fraction and the profitability of becoming sales are determined by order relative to supply of good. During economic expansions, especially when it reaches the peak of the cycle, firms can be incentivized to produce more when order is strong relative to supply. However, the marginal incentive is sharply decreasing in demand relative to supply therefore to reach the peak of business cycle, order has to well exceed final demand for goods. If an adverse shock takes place at the peak of business cycle, the decline in final demand for goods resulting in a much larger decline in orders for goods. This large decline in order reduces incentive to hold inventory therefore an inventory de-cumulation happens.

The key to understand this asymmetry is that while inventories serve as buffering stock against demand fluctuations during economic expansions, this buffering role vanishes during recessions. It follows that the targeted level of inventory stock declines substantially during recessions. Combining with the fact that final sales of goods is declining, to achieve lower level of inventory stock the contraction in production has to be significant. In other words, during recessions, much of the fall in production is attributable to the need to de-cumulate inventory stock. In contrast, during expansions, not only sales is growing but also the need for buffer stock is substantial thus the change in production is due to changes in sales, not the need to adjust inventory stock level.

The model also indicates that aggregate inventory data provides good measures for goods market friction and thus they are very useful for differentiating responses to and transmissions of demand and supply shocks. In terms of policy implications, my model suggests that during recessions government policies that prop up demand for goods can be more effective than that during expansions.

2 Empirical Asymmetric Inventory Dynamics

2.1 Summary Statistics

Inventory de-cumulation, in other words decline in the aggregate inventory stock, coincides with periods of economic recessions. In Figure 1 recession periods, as dated by NBER, are shown as shaded areas while the quarterly growth rates of inventory stock are shown in units of percentage.

The figure shows clearly recessions are associated with negative inventory investments.

To see this more formally, I count periods of negative growth conditional on being in recession or expansion and present results in Table 1. Apparently 61% of quarters in recession witnessed inventory de-cumulation in contrast to only 8% of quarters in expansion. Similar message can be deduced from a admittedly naive logistic regression:

$$\text{odds(de-cumulation)}_t = 0.087^{***} + 17.825^{***} \times \text{RecessionDummy}_t$$

where the dependent variable is the odds ratio of inventory de-cumulation happening , and RecessionDummy is an indicator of NBER recessions (1 when the economy is in recession and 0 otherwise). The odds ratio of inventory de-cumulation, defined as the ratio of probability of de-cumulation happening to not happening, is predicted to drastically increase from 0.087 to 17.912 when the economy is in recession.

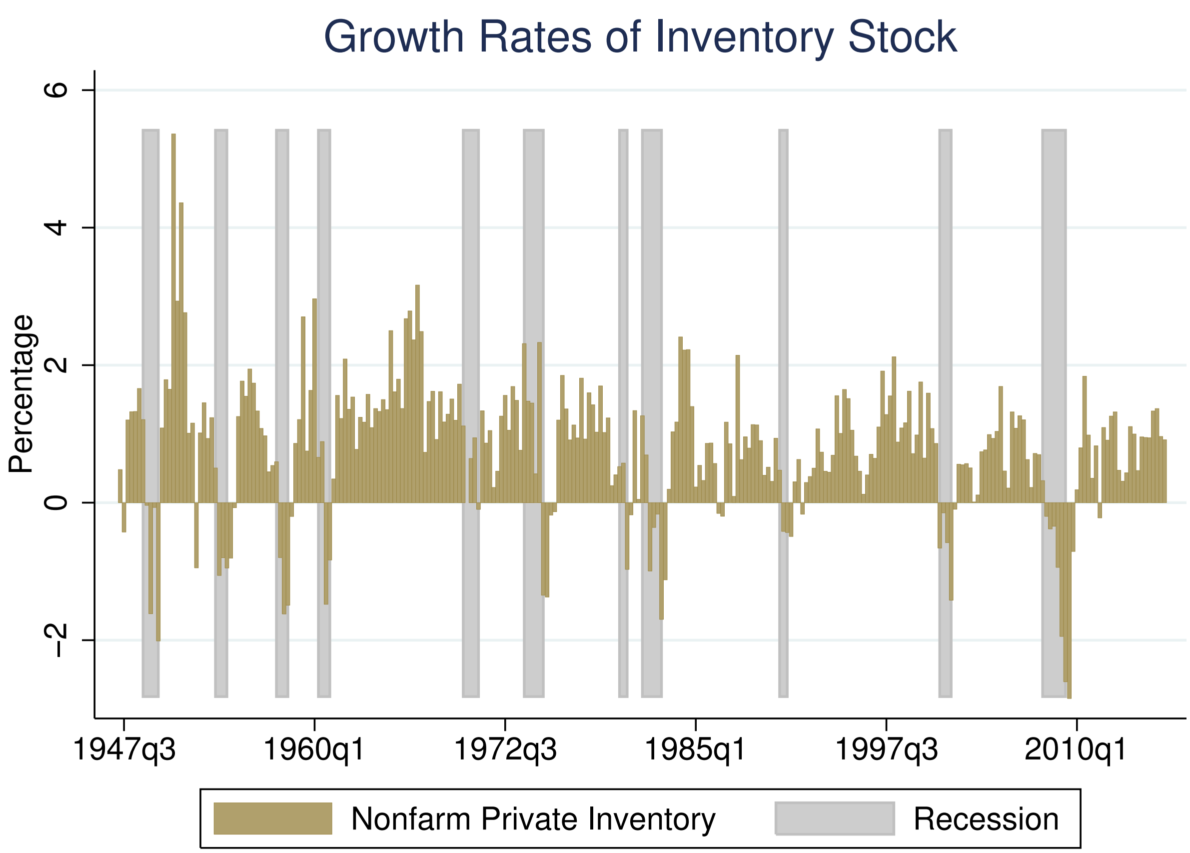


Figure 1: Time Series of Inventory Stock Growth

Uniquely, inventory investment exhibits asymmetric dynamics over phases of business cycle.

Table 1: Description of Fractions of De-cumulations Over Business Cycles

Recession?	Sample Size	# of Inventory De-cumulation	Fraction
No	224	18	8%
Yes	51	31	61%

While major components of GDP accounts for similar shares of GDP changes over phases of business cycle, movements in inventory relative to GDP are Drastically larger in recessions than that in expansions. When inventory de-cumulation happens during recessions, the peak-to-trough decline in inventory investment accounts on average 72% share of peak-to-trough GDP decline in recent recessions. This is striking for two reasons. First, as we can see in the left and middle bar chart in Figure 2, inventory investment accounts for less than 1% o GDP level and GDP growth rate on average. Second, trough-to-peak increase in inventory investment accounts for on average 8% of trough-to-peak increase in GDP, as is made evident by comparisons in Table 2.

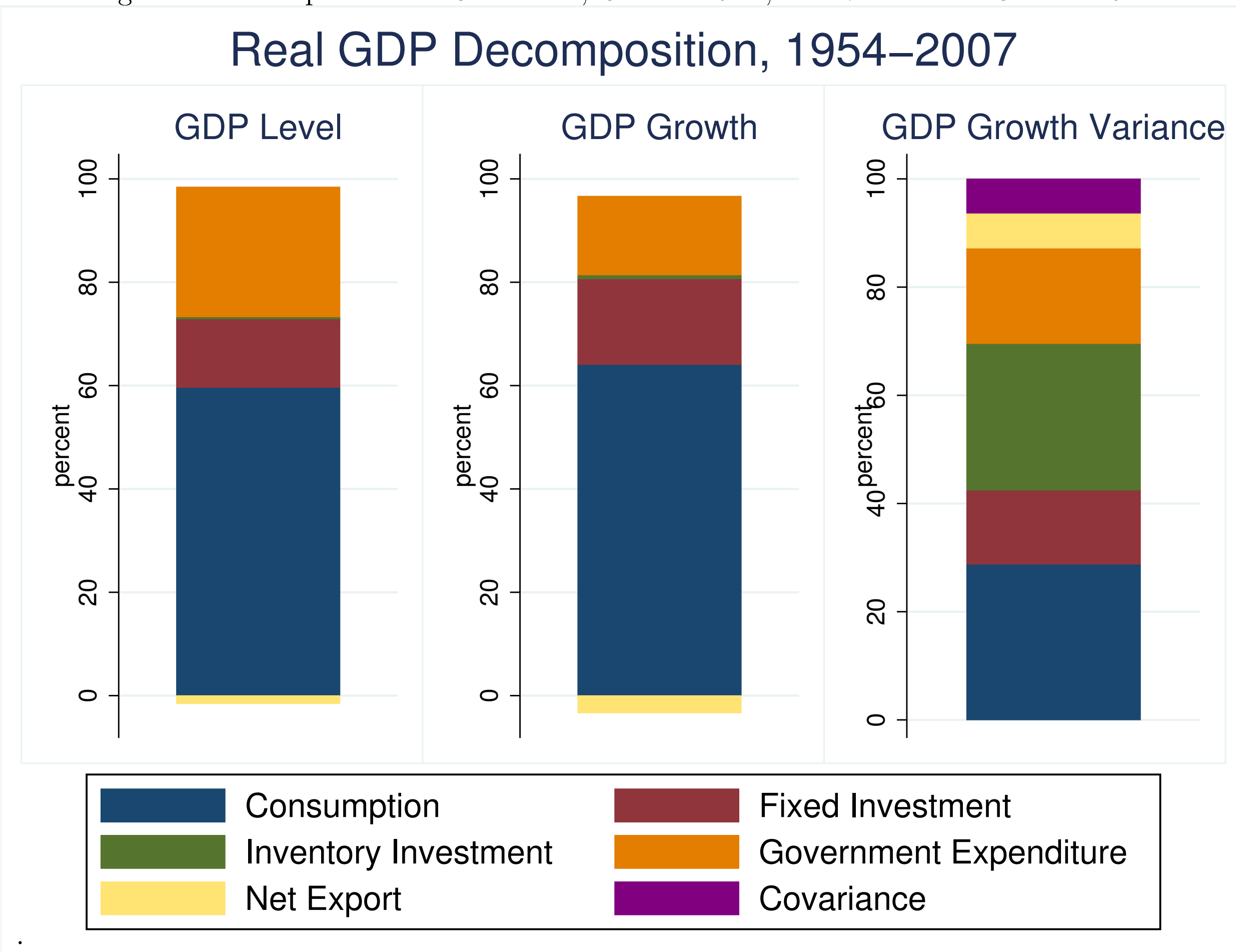
In summary, inventory de-cumulation coincides with economic recession and accounts for a large share of decline in economic activity. This phenomenon stands oddly with the fact that inventory accumulation accounts for only a small share of economic activity during economic expansions. Additionally this asymmetry is not observed in other major components of GDP.

2.2 MS-VAR Analysis

2.2.1 The Regime-Switching VAR Model

I summarize the asymmetric inventory dynamics with a regime-switching vector autoregression (VAR) that allows the intercepts and covariance matrix of reduced form prediction errors to dependent on an underlying variable that conventionally called the "regime" of the economy. This class of models is made popular by Hamilton (1989) and proven useful to capture asymmetry present in the business cycles. For a comprehensive review on the concepts and methodology the reader is referred to the textbook treatment of Hamilton (1994); Kim et al. (1999).

Figure 2: Decomposition of GDP level, Growth Rate, and Variance of Growth Rate



Note: This figure extends the one produced in McMahon (2012)

Consider the following specification of MS-VAR:

$$y_t - \mu_t = \sum_{j=1}^{p} \Phi_{j,t} \left(y_{t-1} - \mu_{t-j} \right) + e_t \tag{1}$$

$$e_t \sim N(0, \Sigma_t). \tag{2}$$

where y_t is a $N \times 1$ vector of variables that we are interested in, μ_t is the time-varying regime-specific mean, and $\Phi_{j,t}$ is the time-varying regime-specific autoregressive coefficients at the j-th lag.

Two independent Markov chains $S_{1,t}$ and $S_{2,t}$ control the autoregressive regimes and the co-

Table 2: Contributions of Inventory Investment to GDP Changes in Post-war U.S. Business Cycles

(a) Peak to Trough Declines, Annualized Billions of Dollars

Date	Inventory Investment	GDP	Share
1948:4-1949:4	-40.66	-30.68	132%
1953:2-1954:2	-24.47	-62.77	39%
1957:3-1958:2	-21.19	-84.98	25%
1960:2-1961:1	-21.38	-9.06	236%
1969:4-1970:4	-35.84	-7.19	498%
1973:4-1975:1	-80.06	-169.95	47%
1980:1-1980:3	-67.26	-142.02	47%
1981:3-1982:4	-120.51	-169.73	71%
1990:3-1991:1	-46.87	-118.38	39%
2001:1-2001:4	-24.09	-40.20	60%
2007:4-2009:3	-213.07	-636.23	33%

(b) Trough to Peak Increases, Annualized Billions of Dollars

Date	Inventory Investment	GDP	Share
1949:4-1953:2	33.28	588.80	6%
1954:2-1957:3	20.94	345.25	6%
1958:2-1960:2	22.89	320.36	7%
1961:1-1969:4	30.15	1613.21	2%
1970:4-1973:4	76.25	754.12	10%
1975:1-1980:1	33.01	1232.47	3%
1980:3-1981:3	115.93	279.97	41%
1982:4-1990:3	84.35	2490.81	3%
1991:1-2001:2	7.16	3844.74	0.1%
2001:4-2007:4	120.34	2286.52	5%

[1] NOTE: The entries in this table denote cumulative decline(increase) from the peak(trough) to the trough(peak) of each cycle in units of annualized billions of 2009 dollars. The last column simply counts the ratio of the second to the third column.

[2] NOTE:

variance regimes independently:

$$\mu_{j,t} = \sum_{m_1}^{M_1} \left[\mu_{m_1} + \tilde{\mu}_{m_1} \mathbf{1}(t > \tau)\right] \mathbf{1}(S_{1,t} = m_1) \tag{3}$$

$$\Phi_{j,t} = \sum_{m_1}^{M_1} \Phi_{m_1} \mathbf{1}(S_{1,t} = m_1) \tag{4}$$

$$\Sigma_t = \sum_{m_2}^{M_2} \Sigma_{m_2} \mathbf{1}(S_{2,t} = m_2) \tag{5}$$

where I allow for a one-time structural break ($\tilde{\mu}_{m_1}$)in the mean that happens at period τ such that the structural break is placed at the first quarter of 1984.

This specification, by allowing the intercepts and covariance matrix to differ across regimes, is designed to capture asymmetries in growth rates due to switches in intercepts (i.e. asymmetry driven by reverting to different means) and asymmetries due to switches in the propagation of exogenous shocks (i.e. differences in Σ_m across regimes). Note that I allow the MS-VAR to identify the distinct regimes on its own without in any way forcing it to match pre-specified dates.

2.2.2 Characterizing Asymmetric Inventory Dynamics

The variable of interest is the real change in inventory normalized by potential GDP[1] $\Delta E/GDP^{pot}$, the growth rate of real final sales of goods by total U.S. businesses $\Delta \log(Sales)$ and the growth rate of U.S. real GDP. I used E_t to denote the beginning of period t level of aggregate inventory holdings. For the details of data construction the reader is referred to the appendix.

To facilitate structural analysis of the VAR system, the ordering of the variables is assumed to be:

$$y_t = \left[\Delta \log(Sales_t), \Delta \log(GDP), \Delta E_{t+1}/GDP_t^{pot}\right]'.$$

The real change in inventory is ordered last because it is plausible to be a function of both sales and production during that period as the variable represents unsold production of goods. Sales growth is ordered earlier than GDP growth but this is not consequential for the results.

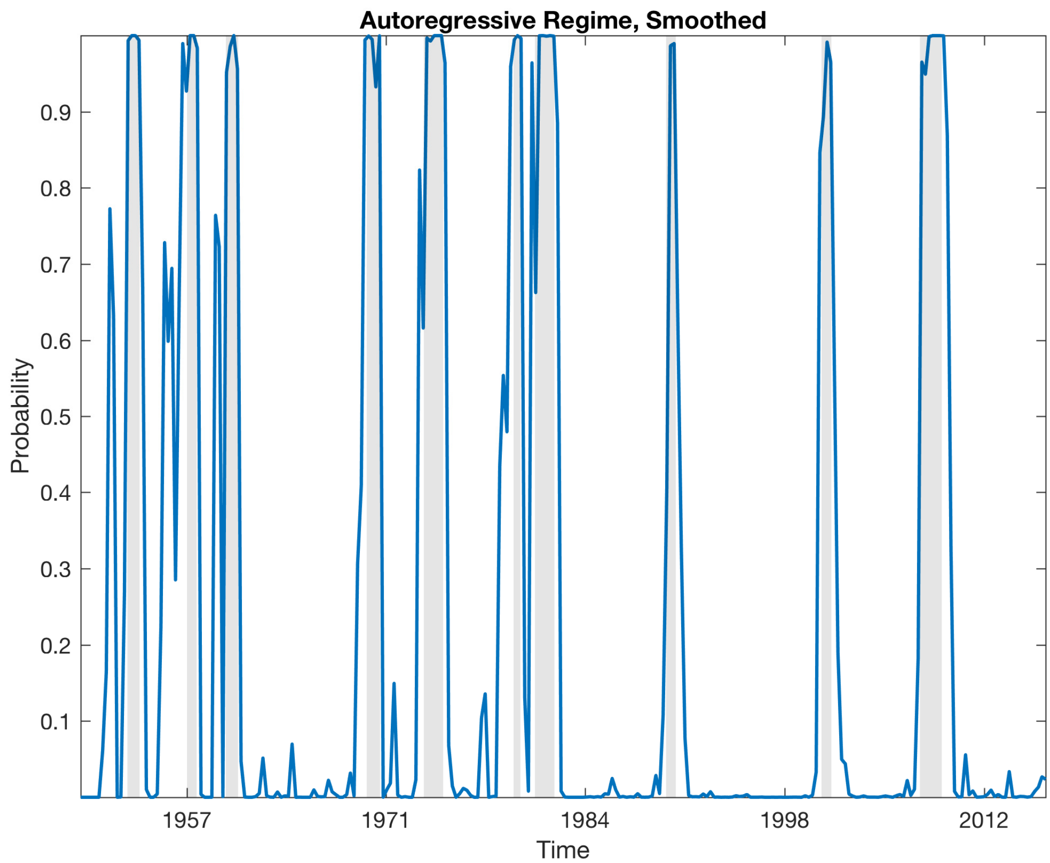
In the following exercise I uses 3 lags of autoregression ($p = 3$) due to the fact the inventories are well known to be lagging the overall aggregate activity (see B for more details). The number of regimes are set to be two in each chain with the intention to capture the key differences between economic expansions and recessions. Currently the switches in all parameters are governed by the same regime which I intent to relax in the future. The estimation and inference of the model is carried out in the Bayesian framework via Markov Chain Monte Carlo techniques, in particular the Gibbs sampler. Generally speaking the estimation and inference is a straightforward multivariate generation of techniques made popular by Kim and Yoo (1995); Kim and Nelson (1998, 1999); Kim et al. (1999). For more details please see A in the Appendix for the derivation for the Gibbs sampling algorithm.

First to validate the usage of regime-switching model we note the recession regime identified

[1] This normalization is necessary because 1) inventory investment frequently turn negative therefore we can't take logs 2) inventory investment exhibits growing variance over time, plausibly due to the growing size of economy therefore it's not stationary. Similar normalizations are carried out in Wen (2005); Khan and Thomas (2007); Benati and Lubik (2014)

by the estimation procedure in Figure 3. The model does an excellent job in matching the NBER recession dates, denoted as gray bars in the figure. Second the model also identifies one covariance regimes (see Figure 4) that accords generally with the pre-1984 periods and one with post-1984 periods.

Figure 3: Autoregressive Regimes Identified By Regime-Switching VAR



NOTE: Blue solid lines represented the probability of being in the recession regime conditional on the full sample (i.e. $Pr\left[S_{1,t} = 2 \mid y^T\right]$). Gray bars denote NBER recession dates.

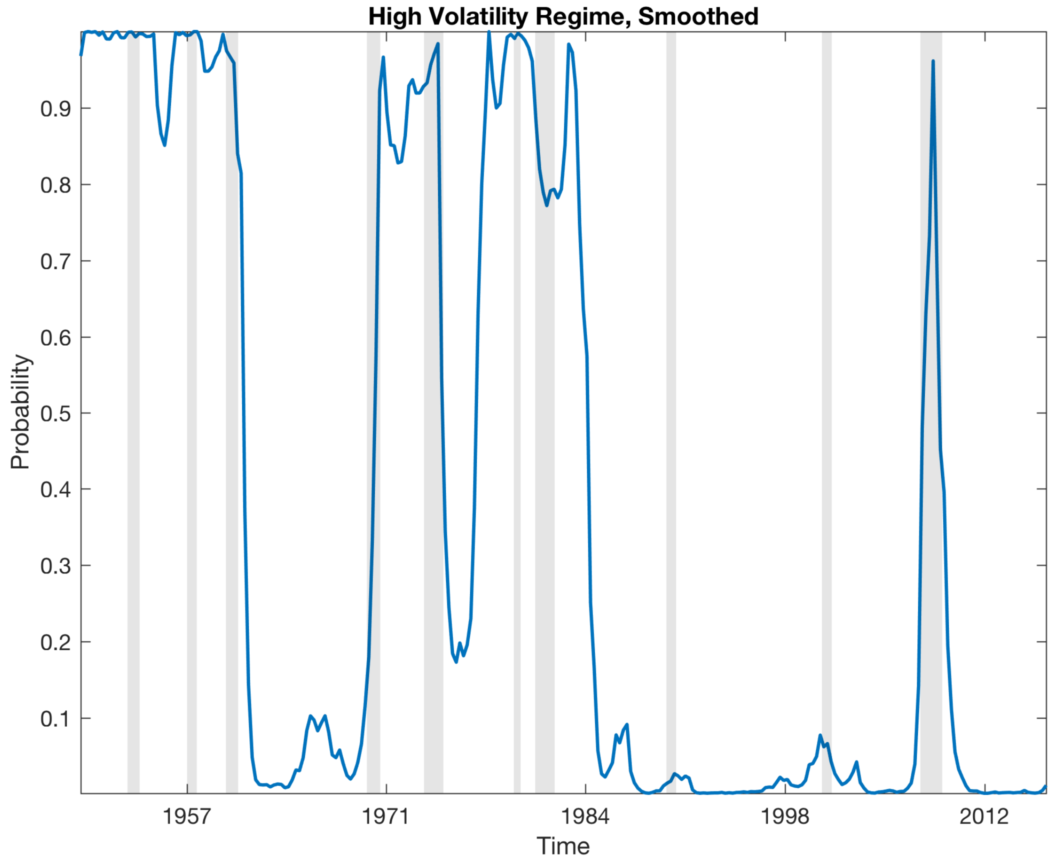
Second we note that two observations emerge from Table 3 and Table 4:

1. Inventory investment is procyclical in expansions but mildly countercyclical in recessions.

2. Inventory investment comoves more negatively with sales growth in recessions than expansions.

These observations suggest the role that inventories play might change over the business cycle. In particular, the stock-out avoidance behavior seems weakened in recessions. The trademark behavior

Figure 4: Covariance Regimes Identified By Regime-Switching VAR



Note: Blue solid lines represented the probability of being in the high volatility regime conditional on the full sample. Gray bars denote NBER recession dates.

of stockout-out avoidance, when demand is persistent, is that firms would increase inventory holding when sales is strong. This is because a strong sales today indicates strong sales in the recent future thus it is optimal to increase inventory holdings. This is also how stock-out motive explains the puzzle of production being more volatile than sales.

Additionally, inventory investment being more negatively correlated with sales growth in recessions indicates that the production-smoothing motive is stronger in recessions. The intuition is simple: Imagine the extreme case of constant production level, and since production is the sum of inventory-investment and sales, this results in perfect negative correlation. The weaker the production smoothing motive is, the larger the correlation between inventory investment and sales is.

Table 3: Standard Deviation and Correlation, Low Volatility

(a) Low Volatility - Expansion Regime

	Sales Growth	GDP Growth	CIPI/GDP Pot
Sales Growth	1.00 (0.95,1.06)		
GDP Growth	0.66 (0.61,0.70)	0.60 (0.57,0.63)	
CIPI/GDP Pot	-0.16 (-0.23,-0.09)	0.26 (0.20,0.33)	0.32 (0.30,0.35)

(b) Low Volatility - Recession Regime

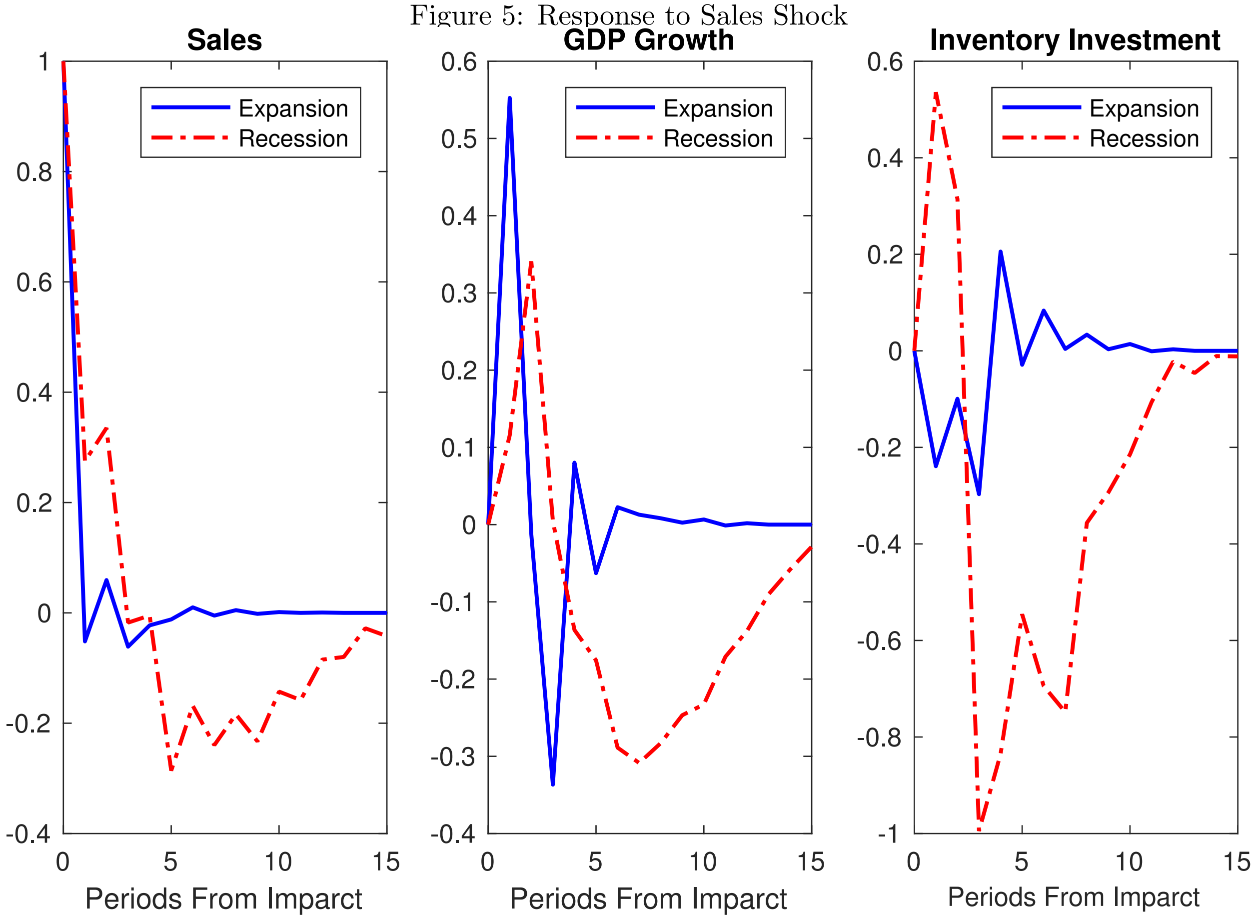
	Sales Growth	GDP Growth	CIPI/GDP Pot
Sales Growth	1.51 (1.22,2.19)		
GDP Growth	0.69 (0.51,0.84)	1.06 (0.86,1.52)	
CIPI/GDP Pot	-0.36 (-0.56,-0.16)	-0.10 (-0.33,0.08)	0.86 (0.65,1.40)

NOTE: Entries on the diagonal show median standard deviation in units of percentage points. Off-diagonal entries show the correlations between variables in the respective columns and rows. Parenthesis contains 16% and 84% percentiles of corresponding statistics.

2.2.3 Stock-Out Avoidance or Production Smoothing?

[Under Revision]

Figure 5: Response to Sales Shock



NOTE: Blue solid lines represented the response in expansions while red dashed lines recessions. Units in percentage points.

Table 4: Standard Deviation and Correlation, High Volatility

(a) High Volatility - Expansion Regime

	Sales Growth	GDP Growth	CIPI/GDP
Sales Growth	2.03 (1.85,2.23)		
GDP Growth	0.65 (0.56,0.72)	1.19 (1.08,1.32)	
CIPI/GDP Pot	-0.18 (-0.30,-0.06)	0.29 (0.18,0.40)	0.56 (0.51,0.63)

High Volatility - Recession Regime

	Sales Growth	GDP Growth	CIPI/GDP
Sales Growth	3.01 (2.41,4.43)		
GDP Growth	0.68 (0.48,0.84)	2.12 (1.70,3.08)	
CIPI/GDP Pot	-0.36 (-0.56,-0.15)	-0.09 (-0.32,0.10)	1.68 (1.23,2.79)

NOTE: Entries on the diagonal show median standard deviation in units of percentage points. Off-diagonal entries show the correlations between variables in the respective columns and rows. Parenthesis contains 16% and 84% percentiles of corresponding statistics.

3 Model

In this section I describe the benchmark dynamic stochastic general equilibrium (DSGE) model to shed lights on the asymmetry in inventory dynamics and the lagging property documented in previous sections. The first order conditions and their derivations can be found in the appendix.

3.1 Environment

Time is discrete and denoted by $t \in \mathbb{N}$.

Agents. The model economy is populated with four types of agents: households, intermediate good producers, variety good producers, and final good producers. All agents are normalized to be measure one.

Markets. Households are endowed with one unit of hours that they can supply to the intermediate producers in exchange for wage income. Only intermediate producers can utilize labor for production and the labor market is perfectly competitive. Intermediate goods are homogeneous and sold in a perfectly competitive market where variety good producers purchase intermediate goods in order to produce differentiated goods indexed by variety $i \in [0, 1]$. To fix ideas, we assume the variety good producers simply paint the otherwise identical intermediate goods with different colors, indexed by i, to create product differentiation. The market for variety goods is monopolistic competitive as in the classic formulation of Dixit and Stiglitz (1977). Each variety can only be produced by one firm thus we will index both of them by variety i and use the terms variety/variety producer interchangeably. Final good producers purchase varieties from these monopolistic competitive markets and sell final goods, which are made from a number of varieties, to households in a perfectly competitive market.Finally, we assume a single corporation owns all producers and one unit of equity is issued and traded on a stock market. The price of this stock is used as numeraire.

Product Market Search. Households exert efforts to search for variety in the final consumption bundle. However they can't reach all varieties (measure one) due to search and match friction (for a textbook treatment of the friction see Pissarides (2000)). Denote the time t aggregate measure of search effort by households to be D_t then measure of effort-variety matches is given by a matching function:

$$x_t = G(D_t, 1) \tag{6}$$

where the second argument is one because all variety producers participate in the search and match process. We assume the matches are uniformly distributed on each household, therefore each unit of search effort acquires $\Psi_{D,t}$ units of varieties:

$$\Psi_{D,t} \equiv \frac{x_t}{D_t}. \tag{7}$$

3.2 Household

There is an unit measure of identical households in the economy. The representative household maximizes the expected lifetime utility and solves the following problem take as given initial stock holding a_{-1}, sequences of prices $\{w_t, \bar{P}_t, \Psi_{D,t}\}_{t=0}^{\infty}$:

$$\max_{\{c_t, a_{t+1}, d_t, n_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t x_t^{\rho}, d_t, n_t)$$

$$s.t. \quad \bar{p}_t x_t c_t + a_{t+1} \leq a_t(1 + \Pi_t) + w_t n_t \tag{8}$$

$$x_t = \Psi_{D,t} d_t \tag{9}$$

where $U(\cdot)$ is the period felicity function and c_t the period t consumption per variety index, x_t the number of variety in the consumption bundle, $\bar{p}_t$ the per variety consumption good price index. Lastly Π_t denotes the profit flow returned from ownership of the corporation.

Households decide how much variety x_t they would consume according to (9). Search effort d_t incurs dis-utility but brings about matches with variety producers at the rate of $Psi_{D,t}$ which the household take as given. The degree of "love of variety", or the inverse of elasticity of substitution across varieties, is controlled by the parameter $\rho > 1$. Final consumption bundle $c_t x_t^{\rho}$ consists of x_t varieties and each variety averages c_t amount of consumption.

This formulation of household's problem highlights the choice for varieties and search effort similar to that in Huo and Ríos-Rull (2013). An alternative formulation, where the household and final good producer are lumped into one agent and choose consumption of each variety along with the number of varieties, is located in the appendix. These two formulations yield identical optimality conditions and interpretations, but for the simplicity of exposition we separate the problems of households and final good producers.

3.3 Final Good Producer

Final good producers pack x_t measures of varieties, dictated by the household, into final goods. For tractability we assume all final good producers always purchase from varieties with index from 0 to x_t, but this fact is unknown to the variety good producers. The representative final good producer solves the following problem:

$$
\begin{aligned}
\max_{c_t, \{c_{i,t}\}_{i=0}^{x_t}} \quad & \bar{p}_t x_t c_t - \int_0^{x_t} p_{i,t} c_{i,t} di \\
s.t. \quad & c_t = \left(\frac{1}{x_t} \int_0^{x_t} v_i^{1-\frac{1}{\rho}} c_{i,t}^{\frac{1}{\rho}} di \right)^{\rho} \qquad (10) \\
& c_{i,t} \leq z_{i,t}. \qquad (11)
\end{aligned}
$$

Taking as given the competitive price $\bar{p}_t$ of final goods, prices $\{p_{i,t}\}_{i=0}^{x_t}$ of variety goods, available quantities $\{z_{i,t}\}_{i=0}^{x_t}$, and the required measure of varieties x_t, the final good producer decides the quantities it will purchase for each variety $\{c_{i,t}\}_{i=0}^{x_t}$. It produces final goods with technology described by (10) using different varieties $\{c_{i,t}\}_{i=0}^{x_t}$. The productivity of variety i is influenced by identically and independently distributed (i.i.d) idiosyncratic shifter v_i which is known to the final good producer at the time of decision. For tractability we assume all Importantly, the purchase of each variety cannot exceed z_i, the amount made available by the variety i producer. Stock-out happens when the price of variety i is low enough such that the availability constraint (11) becomes binding.

The decisions for $c_{i,t}$ generate demand curves for variety i:

$$
c_{i,t} = \min \left\{ z_{i,t}, v_{i,t} c_t \left(\frac{p_{i,t}}{\bar{p}_t} \right)^{\frac{\rho}{1-\rho}} \right\} \qquad (12)
$$

which the variety i producers take as given (more details in the following subsections). For a visual representation of this curve see Figure 6. Final good producers purchase variety i goods according to the price *price* relative to an average price index and an average consumption index c_t, which we can interpret as a measure of market size, until the demand reaches maximum availability $z_{i,t}$.

The average price index satisfies:

$$
\bar{p}_t = \left[\frac{1}{x_t} \int_0^{x_t} v_{i,t} (p_{i,t} + \mu_{i,t})^{\frac{1}{1-\rho}} \right]^{1-\rho} \qquad (13)
$$

where $\mu_{i,t}$ is the Lagrange multiplier associated with constraint (11). The term $(p_{i,t} + \mu_{i,t})$ is the reservation price that final good producers are willing to pay. In the case where stock-out happens, $\mu_{i,t}$ is nonnegative indicating the market price is lower than the reservation price thus demand is larger than what's made available. On the other hand when stock-out doesn't happen constraint (11) does not bind and thus $\mu_{i,t} = 0$. Price index $\bar{p}_t$ captures the average reservation price after adjusted for productivity shifter v_i therefore is the relevant quantity in the determination of variety i demand, Equation 12.

3.4 Variety Producer

At the beginning of a time period, the representative variety producer starts with e_i amounts of inventory stock decide on the price of its own good (variety i), new order y_i, and inventory at the end of period $e_i^{'}$ to maximize its value $\mathscr{V}$. The goods available for sale z_i is the sum of existing inventory e_i and new orders y_i. Inventory stock next period is simply the amount of goods available minus sales and depreciation (Equation (15)). The price of new orders (intermediate goods) is simply P_M.

With probability x, it matches with final good producers and can make sales according to final good producers' demans schedule (12). With probability $1-x$ the variety producer is unmatched and can't make any sales at all. If the variety producer is matched, then the idiosyncratic demand shock v_i[2] is revealed and sales c_i is determined.

Similar to Wen (2005, 2011); Kryvtsov and Midrigan (2012) we assume the producer has to decide on price and new orders before knowing whether it is matched with consumers and the realization of v_i. Proposed first by Wen (2005), having to make decisions before observing v_i generates incentive to hold inventories to guard against situations when demand shock v_i is so high that stock-out happens. What's new here is that the variety producer faces another layer of uncertainty in decision-making: it may not even be able make any sale at all! When the variety good producer is not matched with buyers, the realization of demand shock v_i ceases to matter.

For the ease of exposition I formulate the variety producer's problem as a Bellman's equation:

$$\mathscr{V}(e_i) = \max_{y_i, p_i, e_i^{'}} -P_M y_i + x \int \left\{ c_i p_i + \mathbb{E}m^{'} \mathscr{V}(e_i^{'}) \right\} F_v(dv_i) + (1-x)\mathbb{E}m^{'}\mathscr{V}(e_i^{'})$$

$$s.t. \quad c_i = \min\left\{ z_i, v_i \left(\frac{p_i}{\overline{p}}\right)^{\frac{\rho}{1-\rho}} \overline{c} \right\}$$

$$z_i = e_i + y_i \tag{14}$$

$$e_i^{'} = \begin{cases} (1-\delta_e)\left[e_i + y_i - c_i\right] & \text{“matched”} \\ (1-\delta_e)\left[e_i + y_i\right] & \text{“unmatched”} \end{cases} \tag{15}$$

[2] It is same the variable called "productivity shifter" in Section 3.3. However from the perspective of variety good producer it is a demand shifter because v_i shifts the demand curve faced by variety producer.

which yields the following characterization of pricing and stocking decisions:

$$p_i = \frac{\epsilon_i}{\epsilon_i - 1}(1 - \delta_e)\mathbb{E}m' P_M^{'} \tag{16}$$

where the price elasticity of expected sales is given by:

$$\epsilon_i = \frac{\rho}{1-\rho} \frac{\int_0^{v_i^*} c_i(p_i, n_i, v_i) F^v(dv_i)}{\int_0^{v_i^*} c_i(p_i, n_i, v_i) F^v(dv_i) + [1 - F^v(v_i^*)] \, [e_i + F(n_i)]} \tag{17}$$

3.5 Retailer

There is a fringe of retailers wanting to enter the goods market in order to match with one unit of available good and sell to buyers of goods. In other words, these retailers serve as "middle-men" between final demand and supply of goods produced.

To enter, retailers have to pay κ^S units of final good as sunk cost and pay κ^F units of transaction cost once they successfully matched with a unit of available good. Once matched, a retailer pay P_t^M to the producer and can sell the good to buyers. I assume the final goods market to be perfectly competitive and because we use final good as numeraire, the price of final good is 1. The probability of matching is given by q_t thus free entry dictates the following zero-profit condition:

$$\kappa^S + \kappa^F q_t = q_t(1 - P_t^M).$$

3.6 Nash Bargaining

In this section I will describe the Nash bargaining problem that determines the transaction price P_t^M. Once a retailer is matched with a unit of inventory, the bargaining involves the retailer to propose a price and then the inventory holder to either accept or decline the offer. Once the proposed price is declined, the match is destroyed therefore there's no chance of re-negotiation.

If the proposed price, say $\tilde{P}$, is accepted then the retailer enjoys a surplus of $1 - \tilde{P}$. This is the difference between profit $1 - P_t^M$ once the deal is made (proposed price accepted) and the zero profit once the deal is broken (proposed price declined). For the inventory holder, the total surplus is given by $\tilde{P} - \mathbb{E}_t m_{t+1} U_{t+1}$ which is the difference between realizing revenue P_t^M and continuing to hold the unit inventory $\mathbb{E}_t m_{t+1} U_{t+1}$.

It follows that the total surplus from agreeing at a price of transaction is given by $1 - \mathbb{E}_t m_{t+1} U_{t+1}$. I follow the standard approach in search and match literature in assuming the total surplus is split in fixed proportion. In particular, the retailer gets fraction $\tau \in [0, 1]$ of the total surplus and thus the price of transaction is implicitly given by:

$$1 - P_t^M = \tau \left(1 - \mathbb{E}_t m_{t+1} U_{t+1}\right).$$

4 Quantitative Analysis

[Under revision]

5 Mechanism

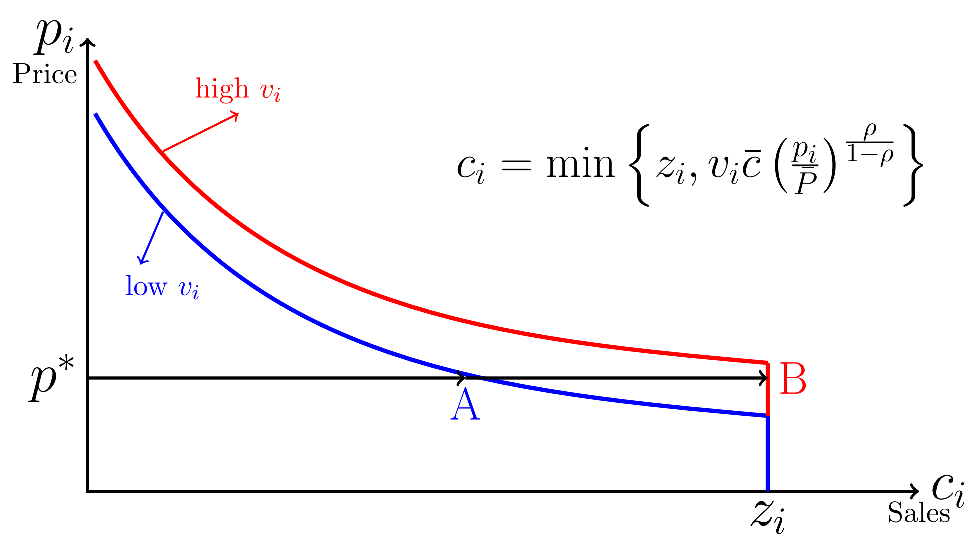


Figure 6: Demand Curve for Variety i

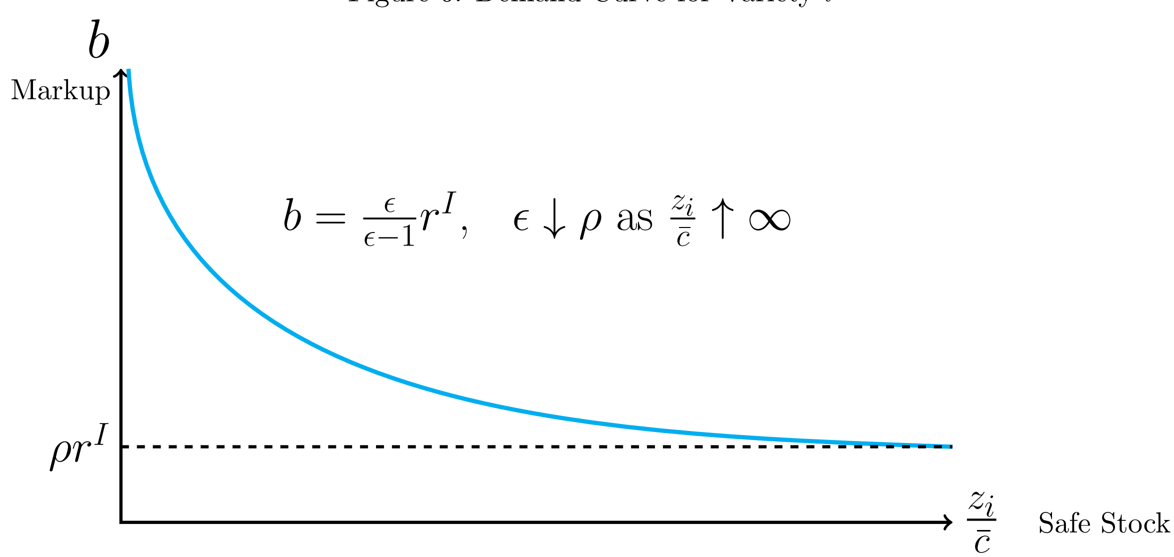


Figure 7: Optimal Choices For Markup and Safe Buffer

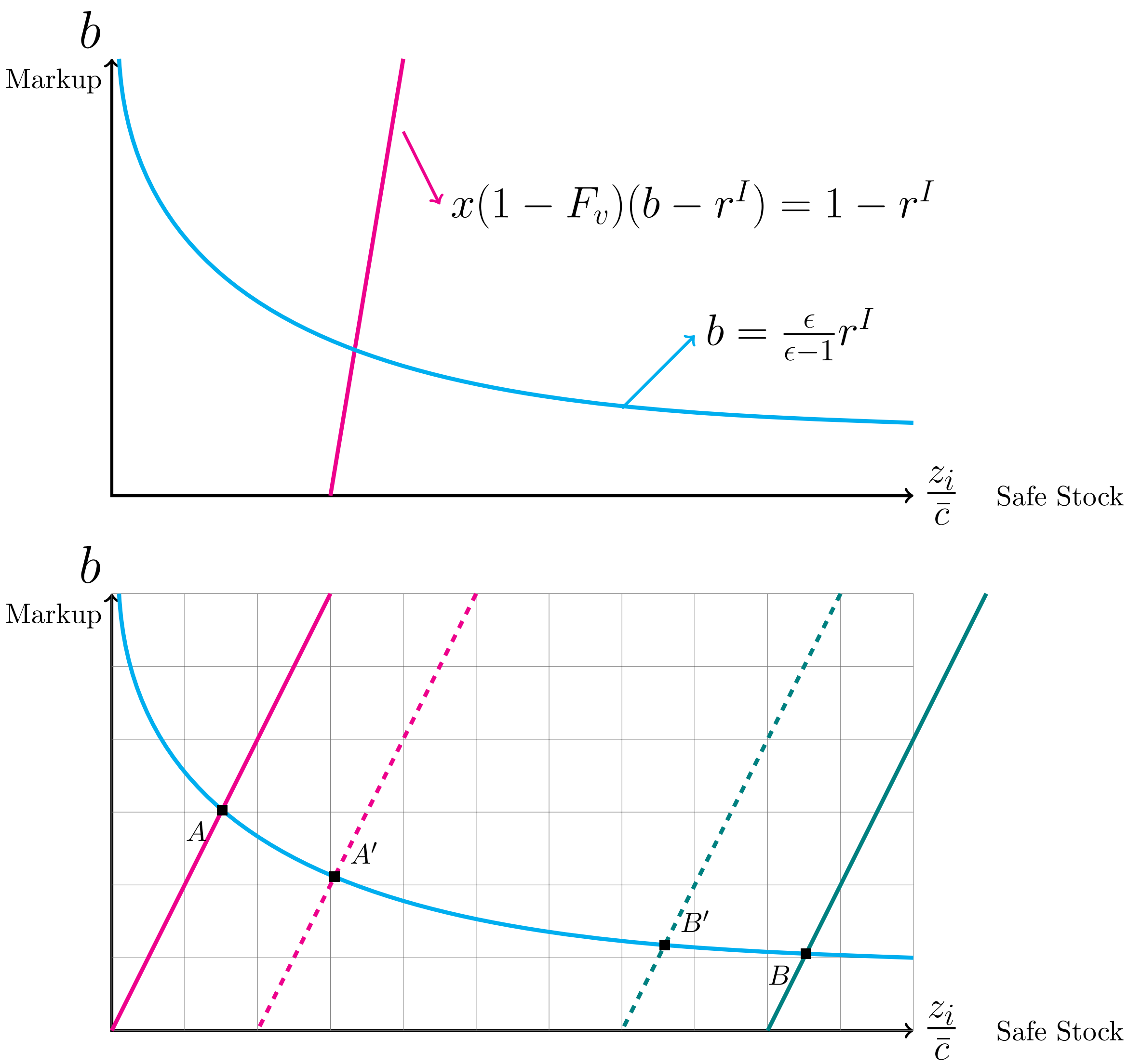


Figure 8: Optimal Choices For Markup and Safe Buffer

5

References

Benati, L. and T. A. Lubik (2014): “Sales, Inventories And Real Interest Rates: A Century Of Stylized Facts,” *Journal of Applied Econometrics*, 29, 1210–1222.

Blinder, A. S. and L. J. Maccini (1991): “The resurgence of inventory research: what have we learned?” *Journal of Economic Surveys*, 5, 291–328.

Bry, G. and C. Boschan (1971): “Cyclical Analysis of Time Series: Selected Procedures and Computer Programs,” .

Davis, S. J. and J. A. Kahn (2008): “Interpreting the Great Moderation: Changes in the Volatility of Economic Activity at the Macro and Micro Levels,” *The Journal of Economic Perspectives*, 22, 155–180.

Dixit, A. K. and J. E. Stiglitz (1977): “Monopolistic competition and optimum product diversity,” *The American Economic Review*, 67, 297–308.

Hamilton, J. D. (1989): “A new approach to the economic analysis of nonstationary time series and the business cycle,” *Econometrica: Journal of the Econometric Society*, 57, 357–384.

——— (1994): *Time series analysis*, vol. 2, Princeton university press Princeton.

Huo, Z. and J.-V. Ríos-Rull (2013): “Paradox of thrift recessions,” Tech. rep., National Bureau of Economic Research.

Kahn, J. A. (2008): “Durable goods inventories and the Great Moderation,” *FRB of New York Staff Report*.

Kahn, J. A., M. M. McConnell, G. Perez-Quiros, et al. (2002): “On the Causes of the Increased Stability of the US Economy,” *Federal reserve Bank of new york Economic Policy review*, 8, 183–202.

Khan, A. and J. K. Thomas (2007): “Inventories and the Business Cycle: An Equilibrium Analysis of (S, s) Policies,” *American Economic Review*, 97, 1165–1188.

Kim, C.-J. and C. R. Nelson (1998): “Business cycle turning points, a new coincident index, and tests of duration dependence based on a dynamic factor model with regime switching,” *Review of Economics and Statistics*, 80, 188–201.

——— (1999): "Has the US economy become more stable? A Bayesian approach based on a Markov-switching model of the business cycle," *Review of Economics and Statistics*, 81, 608–616.

Kim, C.-J., C. R. Nelson, et al. (1999): *State-space models with regime switching: classical and Gibbs-sampling approaches with applications*, vol. 2, MIT press Cambridge, MA.

Kim, M.-J. and J.-S. Yoo (1995): "New index of coincident indicators: A multivariate Markov switching factor model approach," *Journal of Monetary Economics*, 36, 607–630.

Kryvtsov, O. and V. Midrigan (2012): "Inventories, Markups, and Real Rigidities in Menu Cost Models*," *The Review of Economic Studies*, rds028.

McKay, A. and R. Reis (2008): "The brevity and violence of contractions and expansions," *Journal of Monetary Economics*, 55, 738–751.

McMahon, M. F. (2012): "Inventories in motion: a new approach to inventories over the business cycle," .

Metzler, L. A. (1941): "The nature and stability of inventory cycles," *The Review of Economics and Statistics*, 23, 113–129.

Mönch, E. and H. Uhlig (2005): "Towards a Monthly Business Cycle Chronology for the Euro Area," *Journal of Business Cycle Measurement and Analysis*, 2005, 43–69.

Pissarides, C. A. (2000): *Equilibrium unemployment theory*, MIT press.

Ramey, V. A. and K. D. West (1999): "Inventories," *Handbook of macroeconomics*, 1, 863–923.

Wen, Y. (2005): "Understanding the inventory cycle," *Journal of Monetary Economics*, 52, 1533–1555.

——— (2011): "Input and output inventory dynamics," *American Economic Journal: Macroeconomics*, 3, 181–212.

A Gibbs Sampler for Reduced Form MS-VAR

A.1 General Bayesian Multivariate Regressions (VAR(p) as an exampe)

We first consider a special case where the coefficients are time-invariant. This case serves as a building block for many steps in the later described Gibbs sampling algorithm. Now assume the hyperparameters are time-invariant, i.e. $B_t = B, \quad \forall t$, we can easily stack the data sample in the simultaneous regression form, or table form, as:

$$Y = XB + E$$

where

$$\underbrace{Y}_{T\times n} = \begin{pmatrix} \underbrace{y_1'}_{1\times n} \\ \vdots \\ y_T' \end{pmatrix}, \quad \underbrace{X}_{T\times(np+1)} = \begin{pmatrix} \underbrace{x_1'}_{1\times(np+1)} \\ \vdots \\ x_T' \end{pmatrix}, \quad \underbrace{B}_{(np+1)\times n} = \begin{pmatrix} c' \\ \Phi_1' \\ \vdots \\ \Phi_p' \end{pmatrix}$$

and most importantly:

$$\underbrace{E}_{T\times n} = \begin{pmatrix} \underbrace{e_1'}_{1\times n} \\ \vdots \\ e_T' \end{pmatrix}$$

An equally useful representation is the vectorization of the above equation:

$$\begin{aligned} \tilde{y} &= (I_n \otimes X)\,\tilde{b} + \tilde{e} \\ &= \tilde{Z}\tilde{b} + \tilde{e} \end{aligned}$$

where $\tilde{x}$ denotes $vec(X)$ and similarly

$$\tilde{e} \equiv vec(E) \sim N(0, \Sigma \otimes I_{T\times T})$$

and the covariance takes some mind-bending to crank out.

It can be showed that the likelihood function can be decompose into the following two part (see this pdf by Gary Koop and friends monograph):

$$\tilde{b}|\Sigma, \tilde{y}, X \sim N(b^{OLS}, \Sigma \otimes \left(X'X\right)^{-1})$$

where $b^{OLS} = vec(B^{OLS})$, $B^{OLS} = (\tilde{Z}'\tilde{Z})^{-1}(\tilde{Z}'\tilde{y})$ and

$$\Sigma^{-1}|y, X \sim Wishart(S^{-1}, T - n(p+1) - 2)$$

where $S = (Y - XB^{OLS})'(Y - XB^{OLS})$

A.2 Independent Normal-Wishart, Time-Invariant VAR, possibly restricted

Following 2.2.3 in the Gary Koop PDF, I allow for restricted reduced form VAR by the following formulation. Let the equation for the n-th variable at time t be:

$$y_{n,t} = z'_{n,t}\beta_n + \epsilon_{n,t}$$

where $z_{n,t}$ is the k_n-vector of explanatory variable and β_n the accompanying $k_n \times 1$ parameter vector. We then stack

$$y_t = \underbrace{\begin{pmatrix} y_{1,t} \\ \vdots \\ y_{N,t} \end{pmatrix}}_{N\times 1}, \quad \epsilon_t = \underbrace{\begin{pmatrix} \epsilon_{1,t} \\ \vdots \\ \epsilon_{N,t} \end{pmatrix}}_{N\times 1}, \quad \beta = \underbrace{\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}}_{\sum_{n=1}^{N} k_n \equiv K \times 1}$$

with the assumption that $\epsilon_t \sim N(0_n, \Sigma)$ and also

$$Z_t = \underbrace{\begin{pmatrix} \underbrace{z'_{1,t}}_{1\times k_1} & 0 & \cdots & 0 \\ 0 & \underbrace{z'_{2,t}}_{1\times k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underbrace{z'_{N,t}}_{1\times k_N} \end{pmatrix}}_{N\times K \equiv \sum_{n=1}^{N} k_n}.$$

It follows that the possibly restricted VAR can be written as:

$$y_t = Z_t\beta + \epsilon_t.$$

Again we stack the all time observations:

$$y = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix}}_{TN\times 1}, \quad \epsilon = \underbrace{\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_T \end{pmatrix}}_{TN\times 1}, \quad Z = \underbrace{\begin{pmatrix} \underbrace{Z_1}_{N\times K} \\ \vdots \\ Z_T \end{pmatrix}}_{TN\times K}$$

and the whole observation table is:

$$y = Z\beta + \epsilon$$

If we use the very general prior (could be Minnesota, or very non-informative with $\underline{v} = \underline{S}^{-1} = \underline{V_\beta}^{-1} = 0$):

$$\tilde{\beta} \sim N(\underline{\beta}, \underline{V_\beta})$$

and

$$\Sigma^{-1} \sim Wishart(\underline{S}^{-1}, \underline{\nu})$$

then the full conditional posteriors would be

$$\tilde{\beta} \mid \tilde{y}, X, \Sigma \sim N(\bar{\beta}, \overline{V_\beta})$$

where

$$\begin{aligned} \overline{V_\beta} &= \left(\underline{V_\beta}^{-1} + \sum_{t=1}^{T} Z_t' \Sigma^{-1} Z_t\right)^{-1} \\ &= \left(\underline{V_\beta}^{-1} + Z'(I_{T\times T} \otimes \Sigma^{-1}) Z\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} \bar{\beta} &= \overline{V_\beta}\left(\underline{V_\beta}^{-1}\underline{\beta} + \sum_{t=1}^{T} Z_t' \Sigma^{-1} y_t\right) \\ &= \overline{V_\beta}\left(\underline{V_\beta}^{-1}\underline{\beta} + Z'(I_{T\times T} \otimes \Sigma^{-1}) y\right) \end{aligned}$$

also

$$\Sigma^{-1} \mid \beta, y \sim W(\overline{S}^{-1}, \overline{\nu})$$

where $\overline{\nu} = T + \underline{\nu}$ and

$$\overline{S} = \underline{S} + \sum_{t=1}^{T} (y_t - Z_t\beta)(y_t - Z_t\beta)'$$

Note that this is a very general result that I am gonna use repeatedly in the Gibbs sampling involving dummy VAR.

A.3 Reduced Form VAR(p) Model with Switching Means with Multiple Chains and Structural Breaks

Consider this general time-varying reduced form VAR(p) with endogenous variables y_t and $n_z \times 1$ exogenous variables (for example constant and indicator variables):

$$\begin{aligned} \underbrace{y_t}_{n\times 1} &= \underbrace{\mu_t}_{n\times n_z} \underbrace{z_t}_{n_z\times 1} + \underbrace{\Phi_{1,t}}_{n\times n} (y_t - \mu_{t-1}) + \cdots + \Phi_{p,t} (y_{t-p} - \mu_{t-p}) + \underbrace{e_t}_{n\times 1} \\ e_t &\sim N(0, \Sigma_t) \end{aligned} \tag{18}$$

where the parameters are functions of latent discrete state variable $S_{1,t}$ that has M_1 and M_2 states respectively:

$$\begin{aligned} \mu_t &= \sum_{m_1=1}^{M_1} \mathbf{1}(S_{1,t} = m_1) \left[\mathbf{1}(t < \tau) \mu_{m_1}^{pre} + \mathbf{1}(t \geq \tau) \mu_{m_1}^{post} \right] \\ \Phi_{i,t} &= \sum_{m_1=1}^{M_1} \Phi_{i,m_1} \mathbf{1}(S_{1,t} = m_1), \quad i = 1, 2, \ldots, p \\ \Sigma_t &= \sum_{m_2=1}^{M_2} \Sigma_{m_2} \mathbf{1}(S_{2,t} = m_2). \end{aligned}$$

We note that the regression model can be written as:

$$\underbrace{y_t'}_{1\times n} = \begin{pmatrix} z_t' & y_{t-1}' - \mu_{t-1}' & \cdots & y_{t-p}' - \mu_{t-p}' \end{pmatrix} \begin{pmatrix} \underbrace{\mu_t'}_{1\times n} \\ \underbrace{\Phi_{1,t}'}_{n\times n} \\ \vdots \\ \Phi_{p,t}' \end{pmatrix} + e_t'$$

$$\equiv \underbrace{x_t}_{1\times(np+n_z)} \underbrace{B_t}_{(np+n_z)\times n} + e_t'$$

For quick reference we see $B_t = \begin{pmatrix} \mu_t - \sum_{j=1}^p \Phi_{j,t}\mu_{t-j} & \Phi_{1,t} & \cdots & \Phi_{p,t} \end{pmatrix}'$, and $x_t = \begin{pmatrix} z_t' & y_{t-1}' & \cdots & y_{t-p}' \end{pmatrix}$.

A.3.1 Generating μ

Suppose we know everything except for $\mu_{m_1}, \forall m_1$, we can rewrite (**??**) as:

$$\begin{aligned} y_t - \sum_{j=1}^p \Phi_{j,t} y_{t-j} &= \mu_t - \sum_{j=1}^p \Phi_j \mu_{t-j} + e_t \\ &= \sum_{m_1=1}^{M_1} \mathbf{1}(S_{1,t} = m_1) \left[\mathbf{1}(t \le \bar{\tau})\mu_{m_1}^{pre} + \mathbf{1}(t > \bar{\tau})\mu_{m_1}^{post}\right] \\ &- \sum_{m_1=1}^{M_1} \sum_{j=1}^p \mathbf{1}(S_{1,t-j} = m_1)\Phi_{j,t} \left[\mathbf{1}(t-j \le \bar{\tau})\mu_{m_1}^{pre} + \mathbf{1}(t-j > \bar{\tau})\mu_{m_1}^{post}\right] + e_t \\ &\equiv \sum_{m_1=1}^{M_1} \underbrace{\begin{bmatrix} \underbrace{S_{m_1,t}^{pre}}_{N\times N} & \underbrace{S_{m_1,t}^{post}}_{N\times N} \end{bmatrix}}_{N\times 2N} \times \underbrace{\begin{pmatrix} \mu_{m_1}^{pre} \\ \mu_{m_1}^{post} \end{pmatrix}}_{2N\times 1} + e_t. \end{aligned}$$

Defining $y_t^* \equiv y_t - \sum_{j=1}^p \Phi_{j,t} y_{t-j}$, $\mu_{m_1} \equiv \begin{bmatrix} \mu_{m1}^{pre'} & \mu_{m_1}^{post'} \end{bmatrix}'$, and the following two key definitions:

$$S_{m_1,t}^{pre} \equiv \mathbf{1}(S_{1,t} = m_1, t < \bar{\tau}) I_N - \sum_{j=1}^p \mathbf{1}(S_{1,t-j} = m_1, t-j < \bar{\tau})\Phi_{j,t}$$

$$S_{m_1,t}^{post} \equiv \mathbf{1}(S_{1,t} = m_1, t \ge \bar{\tau}) I_N - \sum_{j=1}^p \mathbf{1}(S_{1,t-j} = m_1, t-j \ge \bar{\tau})\Phi_{j,t}$$

for matrix-intensive construction, we can use the following formula if we want to avoid for-loops in code:

$$S_{m_1,t}^{pre} = \left\{ \underbrace{\begin{bmatrix} \mathbf{1}(S_{1,t} = m_1, t < \bar{\tau}) & \mathbf{1}(S_{1,t-1} = m_1, t-1 < \bar{\tau}) & \cdots & \mathbf{1}(S_{1,t-p} = m_1, t-p < \bar{\tau}) \end{bmatrix}}_{1 \times p+1} \otimes I_N \right\}$$

$$\times \underbrace{\begin{bmatrix} I_N \\ -\Phi_{1,t} \\ \vdots \\ -\Phi_{p,t} \end{bmatrix}}_{(Np+N) \times N}$$

$$= \begin{bmatrix} I_N & -\Phi_{1,t} & \cdots & -\Phi_{p,t} \end{bmatrix} \times \left\{ \begin{bmatrix} \mathbf{1}(S_{1,t} = m_1, t < \bar{\tau}) \\ \mathbf{1}(S_{1,t-1} = m_1, t-1 < \bar{\tau}) \\ \vdots \\ \mathbf{1}(S_{1,t-p} = m_1, t-p < \bar{\tau}) \end{bmatrix} \otimes I_N \right\}$$

then we can more compactly rewrite the above as:

$$y_t^* = \sum_{m_1=1}^{M_1} \underbrace{\begin{bmatrix} S_{m_1,t}^{pre} & S_{m_1,t}^{post} \end{bmatrix}}_{N \times 2N} \underbrace{\mu_{m_1}}_{2N \times 1} + e_t. \tag{19}$$

$$= \sum_{m_1=1}^{M_1} \underbrace{S_{m_1,t}^*}_{N \times 2N} \underbrace{\mu_{m_1}}_{2N \times 1} + e_t \tag{20}$$

Again we can use matrix notation to represent $S_{m_1,t}^*$ to be:

$$S_{m_1,t}^* = \begin{bmatrix} S_{m_1,t}^{pre} & S_{m_1,t}^{post} \end{bmatrix}$$

$$= \begin{bmatrix} I_N & -\Phi_{1,t} & \cdots & -\Phi_{p,t} \end{bmatrix} \times \left\{ \begin{bmatrix} \mathbf{1}(S_{1,t} = m_1, t < \bar{\tau}) & \mathbf{1}(S_{1,t} = m_1, t \geq \bar{\tau}) \\ \mathbf{1}(S_{1,t-1} = m_1, t-1 < \bar{\tau}) & \mathbf{1}(S_{1,t-1} = m_1, t-1 \geq \bar{\tau}) \\ \vdots & \vdots \\ \mathbf{1}(S_{1,t-p} = m_1, t-p < \bar{\tau}) & \mathbf{1}(S_{1,t-p} = m_1, t-p \geq \bar{\tau}) \end{bmatrix} \otimes I_N \right\}$$

Now since we also observe Σ_t due to the fact we observe $\{S_{2,t}\}_{t=1}^T, \{\Sigma_{m_2}\}_{m_2=1}^{M_2}$. We create a regression with homoskedasticity by first doing Chloesky decomposition to get lower triangular matrix $L_t = chol(\sum_{m_2=1}^{M_2} \Sigma_{m_2} \mathbf{1}(S_{2,t} = m_2))$ such that:

$$\Sigma_t = L_t L_t^{'}$$

where and pre-multiply **??** by L_t^{-1}:

$$L_t^{-1} y_t^* = \sum_{m_1=1}^{M_1} L_t^{-1} S^*_{m_1,t} \mu_{m_1} + L_t^{-1} \epsilon_t$$

it follows that

$$\begin{aligned} y_t^{**} &= \sum_{m_1=1}^{M_1} S^{**}_{m_1,t} \mu_{m_1} + \epsilon_t^{**} \\ &= \underbrace{\begin{pmatrix} \underbrace{S^{**}_{1,t}}_{N\times 2N} & \cdots & S^{**}_{M_1,t} \end{pmatrix}}_{N\times 2M_1 N} \underbrace{\begin{pmatrix} \mu_1 \\ \vdots \\ \mu_{M_1} \end{pmatrix}}_{2M_1 N\times 1} + \epsilon_t^{**} \\ &\equiv \underbrace{S_t^{**}}_{N\times 2M_1 N} \underbrace{\mu}_{2M_1 N\times 1} + \epsilon_t^{**} \end{aligned}$$

where $\epsilon_t^{**} \sim N(0, I_N)$, $S^{***}_{m,t} \equiv L_t^{-1} S^{**}_{m,t}$, and $y_t^{**} = L_t^{-1} y_t^*$.

Again for the convenience (don't despair yet) in computation we note that:

$$\begin{aligned} S_t^{**} &= \begin{pmatrix} \underbrace{S^{**}_{1,t}}_{N\times N} & \cdots & S^{**}_{M_1,t} \end{pmatrix} \\ &= L_t^{-1} \times \begin{bmatrix} I_N & -\Phi_{1,t} & \cdots & -\Phi_{p,t} \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} \mathbf{1}(S_{1,t} = 1, t < \bar{\tau}) & \mathbf{1}(S_{1,t} = 1, t \geq \bar{\tau}) & \cdots & \mathbf{1}(S_{1,t} = M_1, t < \bar{\tau}) & \mathbf{1}(S_{1,t} = M_1, t \\ \mathbf{1}(S_{1,t-1} = 1, t-1 < \bar{\tau}) & \mathbf{1}(S_{1,t-1} = 1, t-1 \geq \bar{\tau}) & \cdots & \mathbf{1}(S_{1,t-1} = M_1, t-1 < \bar{\tau}) & \mathbf{1}(S_{1,t-1} = M_1, t - \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{1}(S_{1,t-p} = 1, t-p < \bar{\tau}) & \mathbf{1}(S_{1,t-p} = 1, t-p \geq \bar{\tau}) & \cdots & \mathbf{1}(S_{1,t-p} = M_1, t-p < \bar{\tau}) & \mathbf{1}(S_{1,t-p} = M_1, t - \end{bmatrix} \right. \end{aligned} \tag{21}$$

$$\begin{aligned} &= L_t^{-1} \times \begin{bmatrix} I_N & -\Phi_{1,t} & \cdots & -\Phi_{p,t} \end{bmatrix} \\ &\times \{([AR_{dummy}(t:t-p,:) \otimes ones(1, \text{break periods})] \times [ones(1, M_1) \otimes break_{dummy}(t:t-p,:)]) \otimes I_N\} \end{aligned}$$

where AR_{dummy} and $break_{dummy}$ are $T \times M_1$ and $T \times$number of breaks periods tables of dummy variables (more details...). If there's one break say happening in 1984 then there are two break periods in total over the sample.

Now if we assume the prior to be $\mu \mid \{\Sigma_{m_2}\}, \{\Phi_{j,m_1}\} \sim N(\underline{\mu}, \underline{V_\mu})\mathbf{1}(\mu_{GDP,1} < \mu_{GDP,2} < \cdots < \mu_{GDP,M})$, in other words, I impose the identifying assumption that the regimes are ranked by GDP growth rates. From the results in subsection A.2, we can see quickly that the conditional posterior for $\underbrace{\mu}_{MN\times 1}$ is given by:

$$\mu \mid \{y_t\}, \{S_t\}, \{\Sigma_m\}, \{\Phi_j\} \sim N(\overline{\mu}, \overline{V_\mu})$$

where

$$\overline{\mu} = \overline{V_\mu}\left(\underline{V_\mu}^{-1}\underline{\mu} + (S^{**})'Y^{**}\right), \quad \overline{V_\mu} = \left(\underline{V_\mu}^{-1} + (S^{**})'S^{**}\right)^{-1}.$$

and

$$S^{**} = \begin{bmatrix} S_1^{**'} \\ S_2^{**'} \\ \vdots \\ S_T^{**'} \end{bmatrix}$$

After drawing the vector of μ we discard the draws that doesn't satisfy the GDP ranking restrictions $\mu_{GDP,1} < \mu_{GDP,2} < \cdots < \mu_{GDP,M}$.

A.3.2 Generating $\{\Phi_j\}$

I am gonna re-use the * variables. Define $y_t^* = y_t - \mu_t$. We can rewrite Equation 18 as:

$$\begin{aligned} y_t^* &= \sum_{j=1}^{p} \Phi_{j,t} y_{t-j}^* + \epsilon_t \\ &= \sum_{j=1}^{p} \left[\sum_{m_1=1}^{M_1} \mathbf{1}(S_{1,t} = m_1)\Phi_{j,m_1}\right] y_{t-j}^* + \epsilon_t \\ &= \sum_{j=1}^{p} \sum_{m_1=1}^{M_1} \mathbf{1}(S_{1,t} = m_1)\Phi_{j,m_1} y_{t-j}^* + \epsilon_t \end{aligned}$$

then vectorize both sides of the equation and reverse summation order we get:

$$\begin{aligned} y_t^* &= \sum_{m_1=1}^{M_1} \mathbf{1}(S_{1,t} = m_1) \sum_{j=1}^{p} \left[\left(y_{t-j}^*\right)' \otimes I_N\right] \phi_{j,m_1} + \epsilon_t \\ &\equiv \sum_{m_{1=1}}^{M_1} z_{m_1,t}\phi_{m_1} + \epsilon_t \end{aligned}$$

where

$$\underbrace{\phi_{m_1}}_{pN^2\times 1} \equiv \begin{bmatrix} vec(\Phi_{1,m_1}) \\ vec(\Phi_{2,m_1}) \\ \vdots \\ vec(\Phi_{p,m_1}) \end{bmatrix}$$

and

$$\underbrace{z_{m_1,t}}_{N\times pN^2} \equiv \begin{bmatrix} z_{m_1,t-1} & z_{m_1,t-2} & \cdots & z_{m_1,t-p} \end{bmatrix}$$

$$\underbrace{z_{m_1,t-j}}_{N\times N^2} \equiv \mathbf{1}(S_{1,t}=m_1)\left[\left(y^*_{t-j}\right)'\otimes I_N\right]$$

we can decompose $\Sigma_t = L_t L_t'$ and then premultiply both sides by L_t^{-1} to get:

$$\begin{aligned} y_t^{**} &= \sum_{m_{1=1}}^{M_1} z_{m_1,t}^{**}\phi_{m_1} + \epsilon_t^{**}, \quad \epsilon_t^{**}\sim N(0,I_N) \\ &= \underbrace{\begin{pmatrix} z_{1,t} & \cdots & z_{M_1,t}\end{pmatrix}}_{N\times pM_1N^2} \underbrace{\begin{pmatrix}\phi_1 \\ \vdots \\ \phi_{M_1}\end{pmatrix}}_{pM_1N^2\times 1} + \epsilon_t^{**} \\ &\equiv z_t\phi + \epsilon_t^{**} \end{aligned}$$

We then stack vertically to get:

$$Y^{**} = Z\phi + E^{**}$$

where

$$Y** = \underbrace{\begin{pmatrix} y_1^{**} \\ \vdots \\ y_T^{**}\end{pmatrix}}_{TN\times 1}, \quad Z = \underbrace{\begin{pmatrix} z_1 \\ \vdots \\ z_T\end{pmatrix}}_{TN\times pN^2}.$$

Again using the result from **??** with independent normal prior we can draw ϕ:

$$\phi \mid \{y_t\},\{S_t\},\{\Sigma_m\},\{\mu_m\} \sim N(\overline{\phi},\overline{V_\phi})$$

where

$$\overline{\phi} = \overline{V_\phi}\left(\underline{V_\phi}^{-1}\underline{\phi} + Z'Y^{**}\right), \quad \overline{V_\phi} = \left(\underline{V_\phi}^{-1} + Z'Z\right)^{-1}.$$

Note that this is coming from a prior of the form:

$$\phi \mid \{\Sigma_m\}, \{\mu_m\} \sim N(\underline{\phi}, \underline{V_\phi})\mathbf{1}(I - \Phi(L) \text{ is unstable}).$$

We only keep those draws that such that the roots of $I - \sum_{j=1}^{p} \Phi(z)$ are outside the unit circle so that the draws are guaranteed to be a stable VAR.

Alternatively, we can simply select the observations that belongs to regimes m_1 can simulate the ϕ_{m1} vector regime-by-regime and lastly stack them together.

A.3.3 Generating Σ_m

Now that we condition on data and other hyperparameters, we are directly observing the realizations of ϵ_t in some sense! Therefore for each state m we can collect those error terms that actually are in state m in E_m and then the conditional posterior for Σ_m is:

$$\Sigma_m^{-1} \mid \{y_t\}, \{S_t\}, \{\mu_m\} \sim Wishart(\overline{S}^{-1}, \overline{\nu})$$

where

$$\overline{\nu} = \underline{\nu} + T$$

and

$$\overline{S} = \underline{S} + E_m E_m'$$

or equivalently

$$\Sigma_{m_2} \mid \{y_t\}, \{S_t\}, \{\mu_m\} \sim Inverse - Wishart(\overline{S}, \overline{\nu})$$

B Is Inventory Lagging Business Cycle?

I characterized previously the asymmetry of inventory dynamics base on the dates for U.S. business cycle peaks and troughs announced by the National Bureau of Economic Research (NBER). Even though these dates are accepted as "gold standards" in dating business cycles, they incorporated subjective judgments of economists and thus makes it impossible for me to date the business cycles generated from the simulations of my quantitative model in the exact same way as the NBER do. Without reliable ways to compare the asymmetries of inventory dynamics in my model and in the actual data, it is difficult to assess the quantitative performance of my. To this end, I follow McKay and Reis (2008) in performing algorithm-based dating procedures (see for details Bry and Boschan (1971); Mönch and Uhlig (2005)) to determine peaks and troughs in actual U.S. data and model generated business cycles.

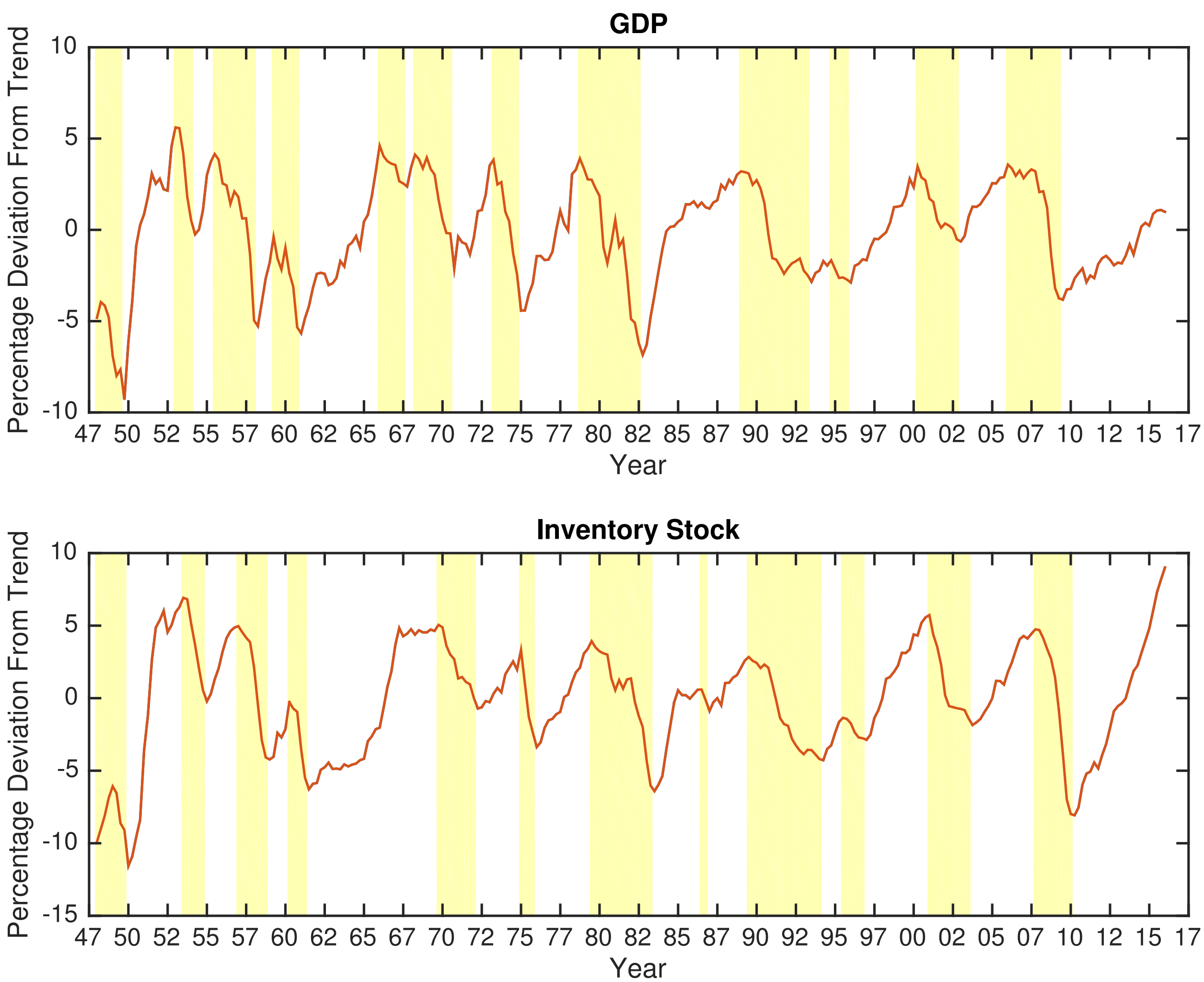


Figure A.1: Business Cycle Dates

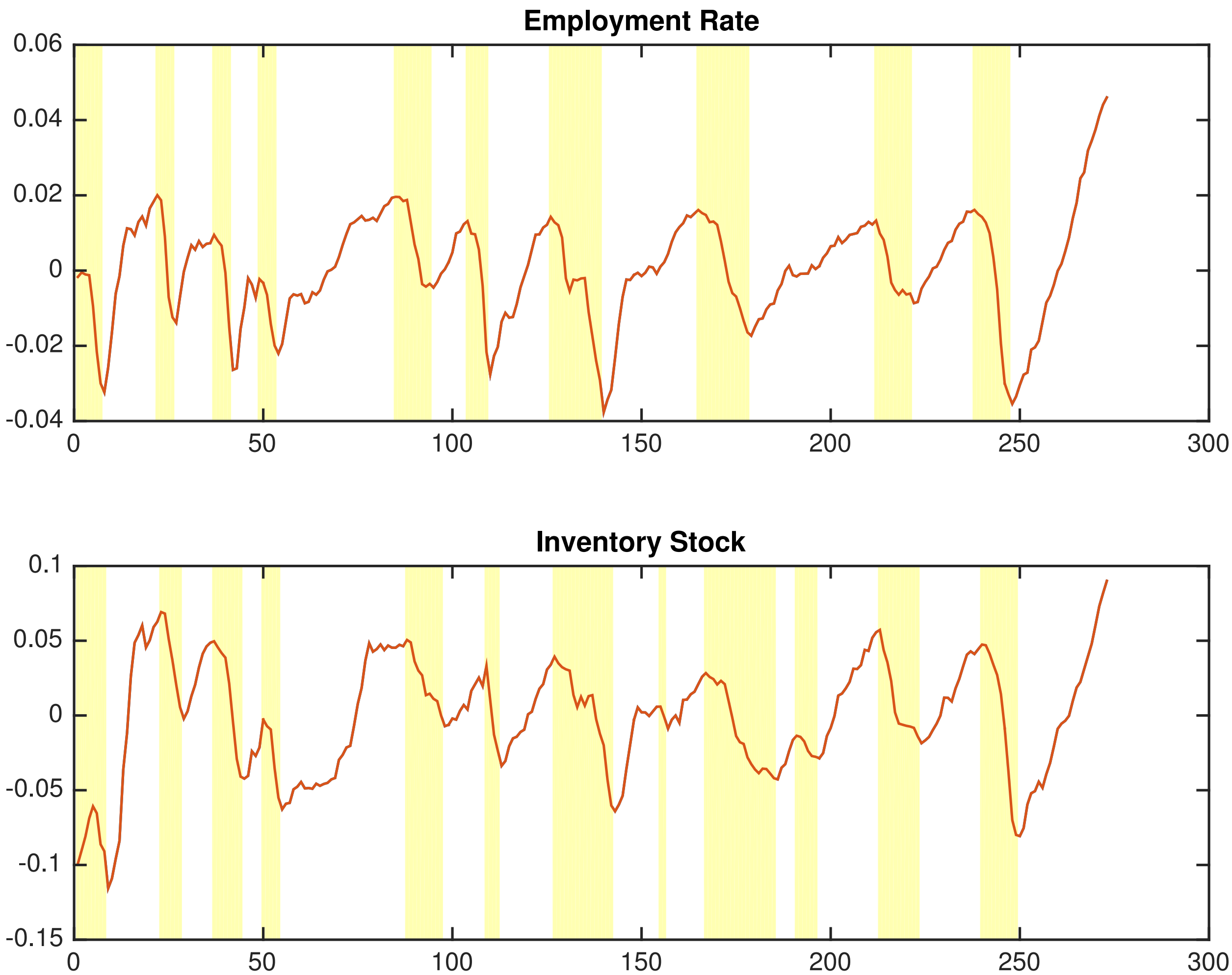


Figure A.2: Business Cycle Dates

Typical behaviors of inventory and a measure of economic activity (GDP and employment rate, respectively) is depicted in Figure A.3 and Figure A.4. It is quite clear that inventory stock, and thus inventory investment lags the indicator of business cycle about 3 to 4 quarters and this fact motivates the inclusion of at least 3 lags in the VAR study.

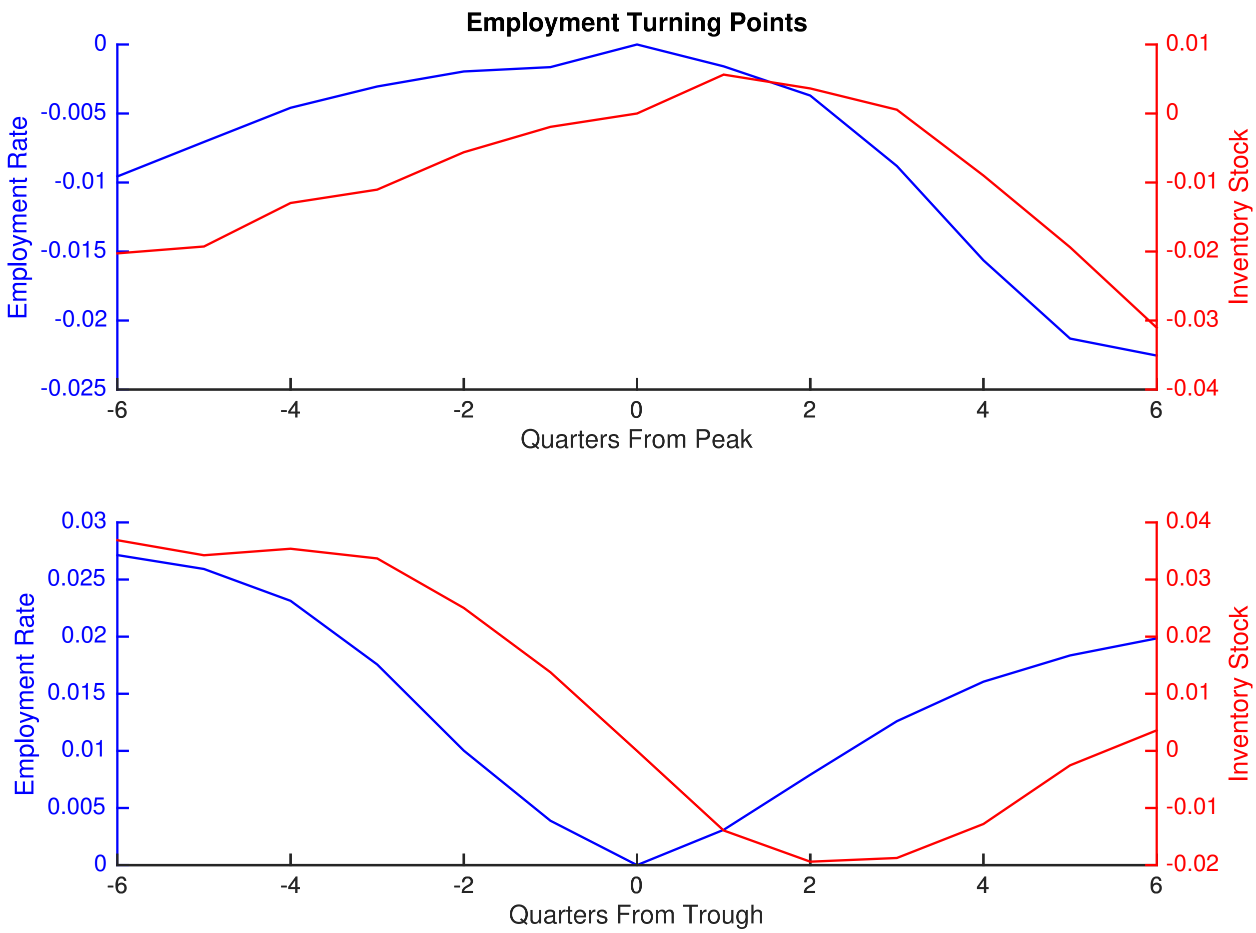


Figure A.3: Business Cycle Dates

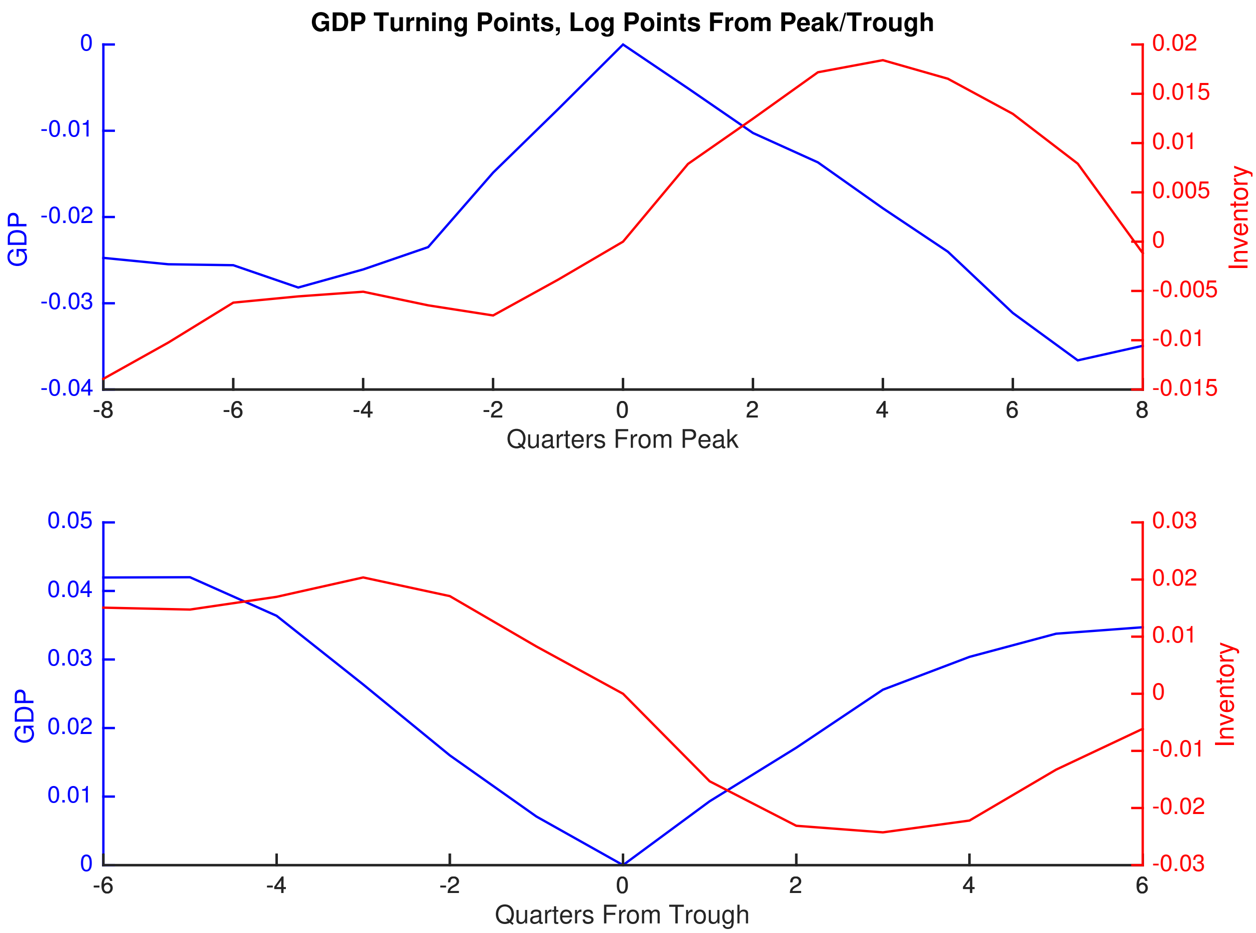


Figure A.4: Business Cycle Dates