Economics 105D – Summer Term II 2009 – Final

Name: ANSWER KEY

Please answer all questions in the space provided after each question (including the sheets following each question). Feel free to use the back of each sheet if you need to. You are not permitted to use any books or notes. You may use a calculator (although it is not necessary, feel free to simplify your answers and leave them in fraction form) and a ruler.

There are 5 questions, for a total of 100 points on the exam, and you have 3 hours to complete the exam. The total number of points for each part is indicated in each question.

Make sure you have 16 pages in your exam (including the cover sheet and extra work pages).

Before starting the exam, you must sign the following statement:

I pledge to obey the Duke University Honor Code during this exam.
Signed: Instructor - Erika Martinez
1. (15 points) Suppose Stephanie has the following utility function over tutoring for Econ105D, \( x \), and arcade games, \( y \).

\[
U(x, y) = x^{1/2} + y^{1/2}
\]

where \( x \) is measured in number of hours spent being tutored and \( y \) in number of hours spent playing arcade games.

(a) (4 points) Let \( p_x, p_y, I \) denote the price of \( x \), \( y \), and Stephanie’s monthly allowance, respectively. Derive Stephanie’s Marshallian demand for \( x \) and \( y \).

**Solution:** The optimality condition requires that

\[
\frac{x^{1/2}}{x^{1/2} + y^{1/2}} = \frac{p_x}{p_y} \implies x = \left( \frac{p_x}{p_y} \right)^2 y.
\]

Using the budget constraint we obtain:

\[
p_x \left( \frac{p_x}{p_y} \right)^2 y + p_y y = I \implies y(p_x, p_y, I) = \frac{I}{p_y (1 + \frac{p_x}{p_y})}
\]

\[
x(p_x, p_y, I) = \frac{I}{p_x (1 + \frac{p_x}{p_y})}
\]

(b) (5 points) (3 points) Suppose that an hour of tutoring costs $20 and an hour of playing arcade games costs $5. In addition, Stephanie’s parents give her $500 each month to allocated between her two consumption goods. How many tutoring and gaming hours will Stephanie consume each month? What is her optimal utility level?

**Solution:** Simply plug the prices and allowance into the demands and utility function to obtain the following:

\[
x(20, 5, 500) = 5
\]

\[
y(20, 5, 500) = 80
\]

\[
U(5, 80) = 5\sqrt{5}
\]

(c) (3 points) Suppose that the price of gaming goes up unexpectedly to $10. Stephanie’s parents are pleased because they believe their daughter will substitute away from games to more tutoring. Are they correct? How much will Stephanie consume of \( x \) and \( y \) after the price increase? What is her new utility level?

**Solution:** Similarly,

\[
x(20, 10, 500) = \frac{25}{3}
\]

\[
y(20, 5, 500) = \frac{100}{3}
\]

\[
U\left( \frac{25}{3}, \frac{100}{3} \right) = 5\sqrt{3}
\]

Stephanie’s parents are correct, she spends more time being tutored than gaming as the price of gaming increases.

(d) (5 points) Stephanie finds out about a promotional offer that allows her to buy unlimited gaming hours for 50% off. However, to qualify for the offer she must purchase a membership to the arcade. What is the highest price that Stephanie would be willing to pay for the membership? (Suggestion: Do not try to simplify the expenditure function.)
Solution: If Stephanie chooses to purchase the membership, the price of one hour of gaming goes down to $5. The maximum price that she would be willing to pay is:

\[ EV = E(20, 10, 5\sqrt{3}) - E(20, 5, 5\sqrt{3}) = 500 - E(20, 5, 5\sqrt{3}) \]

Thus we need to find Stephanie’s expenditure function. To do this simply plug the Marshallian demands into Stephanie’s utility function to obtain her indirect utility function and invert solve for the expenditure function:

\[ E(p_x, p_y, U) = \frac{(p_x + \frac{p^2_x}{p_y})(p_y + \frac{p^2_y}{p_x})}{\left(\frac{p_x}{p_y} + \frac{p^2_x}{p_y}\right)^{1/2}} \]

Plugging in for price and utility, we find that \( E(20, 5, 5\sqrt{3}) = 300 \). Therefore, the maximum price that Stephanie is willing to pay is 200.

2. (20 points) A particular market has a demand curve given by:

\[ Q = 1100 - 50P \]

There are 100 firms each with cost

\[ C(q) = \frac{1}{2}q^2 + 10q \]

(a) (3 points) Find the short-run market supply function \( Q^S(P) \).

Solution: Each firm maximizes its own profits (or minimizes costs) and chooses \( P = q + 10 \implies Q = P - 10 \). Therefore, the market supply is \( Q^S = 100P - 1000 \).

(b) (4 points) Find the short-run equilibrium price, \( P^{SR} \), and quantity, \( Q^{SR} \) for this market. Also find the producer and consumer surpluses. What is the total surplus?

Solution: To find equilibrium price and quantity equate supply and demand:

\[ 100P - 1000 = 1100 - 50P \implies P^{SR} = 14, Q^{SR} = 400 \]

\[ CS = \frac{8 \times 400}{2} = 1600 \]

\[ PS = \frac{4 \times 400}{2} = 800 \]

\[ TS = CS + PS = 2400 \]

(c) (3 points) Now suppose the government imposes a $3 tax on every unit \( q \). This creates a gap between what consumers are paying and what producers are receiving. Let \( P \) denote the price that the consumers are paying. Then firms receive \( P' = P - 3 \) for every unit they sell. What is the market supply function \( Q^{S'}(P) \) in this case? (Hint: it might help if you write the supply function in term of \( P' \) first)

Solution:

\[ Q^{S'}(P) = 100P' - 1000 = 100(P - 3) - 1000 = 100P - 1300 \]

(d) (3 points) Derive the new short-run equilibrium price \( P^{SR}_t \) and quantity \( Q^{SR}_t \) for this market.

Solution:

\[ 100P - 1300 = 1100 - 50P \implies P^{SR}_t = 16, Q^{SR}_t = 300 \]
(e) (4 points) How is the tax burden shared between consumers and producers? In other words, what is the price increase that consumers are facing after the imposition of the tax? What is the total social surplus in this case? Based on your answer is the tax efficient?

**Solution:** After the imposition of the tax, the price that consumers face increases by $2. Thus, in the short-run the consumers are paying $2 of the tax and the producers are paying the remaining $1.

\[ CS = \frac{6 \times 300}{2} = 900 \]
\[ PS = \frac{3 \times 300}{2} = 450 \]
\[ GS = Tax\ Revenue = 300 \times 3 = 900 \]
\[ TS = CS + PS + GS = 900 + 450 + 900 = 2250 \]

The tax is inefficient since it creates a deadweight loss and total surplus decreases.

(f) (3 points) Suppose now that instead of 100 firms, there is only one firm operating in the market and there is no tax imposed by the government. Find the monopoly price \( P^M \) and output \( Q^M \).

**Solution:** The monopolist will maximize profits by setting quantity such that \( MR = MC \):

\[ MR = 22 - \frac{Q}{25} = Q + 10 = MC \implies Q^M = \frac{150}{13} = 11.54 \]
\[ P^M = 1100 - 50 \frac{150}{13} = \frac{283}{13} = 21.77 \]

3. (15 points) Suppose there is an island with 2 lakes, \( x \) and \( y \), and 20 fishermen. Each fisherman can choose to fish on either lake. The islanders are concerned about equity, therefore at the end of each day all the fish caught on each lake is collected and distributed evenly among the men that chose to fish on that lake. On lake \( x \) the total number of fish caught is given by,

\[ F_x = 10l_x - \frac{1}{2}l_x^2 \]

where \( l_x \) is the number of people fishing on lake \( x \). For lake \( y \), the relationship is given by,

\[ F_y = 5l_y \]

(a) (5 points) What will be the Nash equilibrium number of fishermen on each lake and the total number of fish caught on each lake?

**Solution:** In equilibrium each fisherman should be indifferent between fishing on either lake. Therefore, it must be true that:

\[ F_{x}^{L} = F_{y}^{L} \implies l_x = 10, l_y = 10 \]
\[ F_x = 50, F_y = 50 \implies Total\ Catch = 100 \]

Note that if you maximized the total catch on lake A or equated the total catch on both lakes, you got the same answer, but the approach is wrong and getting the right answer is simply a coincidence. None of the fishermen have the objective of equating the total catch on each lake or maximizing the total catch. They are only interested in their own catch.
(b) (5 points) What number of fishermen should be allowed on each lake in order to maximize the total daily catch of fish? What is the total catch of fish in this case?

**Solution:** We need to maximize the total catch subject to the constraint that \( l_x + l_y = 20 \). Therefore,

\[
\max_{l_x} 10l_x - \frac{1}{2}l_x^2 + 5(20 - l_x)
\]

The solution is \( l_x^* = 5, \ l_y^* = 15 \). At these number of fishermen on each lake the total catch becomes,

\[
F_x = 37.5, \ F_y = 75 \implies \text{Total Catch} = 112.5
\]

(c) (5 points) The island official is opposed to coercion and therefore decides to require a fishing license for lake \( x \). If the licensing procedure is to bring about the optimal allocation of fishing such that it maximizes the total catch, what should the cost, \( c \) of a license be (in terms of fish)?

**Solution:** We need to find a licensing fee \( c \) such that \( \frac{F_x}{l_x} + c = \frac{F_y}{l_y} \) when \( l_x = 5 \) and \( l_y = 15 \). Therefore,

\[
10 - \frac{5}{2} - c = 5 \implies c = 2.5
\]

4. (20 points) Consider the following matrix game in which player 1 chooses a row and player 2 chooses a column simultaneously.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2.6</td>
<td>0.4</td>
<td>6.0</td>
</tr>
<tr>
<td>M</td>
<td>3.0</td>
<td>2.2</td>
<td>7.3</td>
</tr>
<tr>
<td>B</td>
<td>6.8</td>
<td>2.4</td>
<td>4.5</td>
</tr>
</tbody>
</table>

(a) (5 points) Does this game have an equilibrium in strictly dominant strategies? If so, what is it?

**Solution:** No, neither player has a dominant strategy in this game.

(b) (5 points) Which of the 9 strategy profiles survive iterated elimination of strictly dominated strategies?

**Solution:** Without loss of generality, we can begin by considering eliminations for player 1 in the first ‘elimination round’. Player 1’s strategy T is strictly dominated by M in the first round. Player 2’s strategy C is strictly dominated by R in the second round. No further rounds of elimination are possible, thus the surviving strategy profiles are \((M, L), (B, L), (M, R)\) and \((B, R)\).

(c) (5 points) What are the pure-strategy Nash equilibrium profiles in this game?

**Solution:** \((B, L)\) and \((M, R)\)
(d) (5 points) Let $\sigma_1$ denote the probability that player 1 plays M and $\sigma_2$ denote the probability that player 2 plays L. Find the probabilities $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$ that constitute a mixed strategy Nash equilibrium.

**Solution:** In equilibrium player 1 mixes to make player 2 indifferent between L and R:

$$0\sigma_1 + 8(1 - \sigma_1) = 3\sigma_1 + 5(1 - \sigma_1) \implies \sigma_1^* = \frac{1}{2}$$

Similarly, player 2 mixes to make player 1 indifferent between M and B:

$$3\sigma_2 + 7(1 - \sigma_2) = 6\sigma_2 + 4(1 - \sigma_2) \implies \sigma_2^* = \frac{1}{2}$$

5. (15 points) There are two firms in the gadget industry. Firm 1 has the cost function $C_1(q_1) = 0.5q_1^2$ and firm 2 has the cost function $C_2(q_2) = 2q_2$. The inverse demand for gadgets is $p = 18 - Q$.

(a) (3 points) If this market were to operate by a social planner, then what prices $\bar{p}$ would be set in order to maximize total social surplus, $CS$ and $PS$? Find the efficient levels of output for each firm, $\bar{q}_1$ and $\bar{q}_2$. How much total surplus is generated at the efficient market outcome?

**Solution:** The social planner will employ each firm such that the efficient quantity of gadgets is produced. It does this by minimizing the total costs of production across the two firms, $C(Q) = C_1(q_1) + C_2(q_2)$, subject to the fact that the total quantity produced is the sum of the individual firms production, $Q = q_1 + q_2$.

The first order condition of this program yields the condition that total cost is minimized when the marginal costs of each firm are equal; i.e. when $q_1 = 2$. Hence, firm 1 should produce two units and firm 2 should produce all remaining units at $MC_2 = 2$. Because it is efficient to price at industry marginal cost, we have $\bar{p} = 2$. At this price, the quantity demanded is $Q = 16$. Hence, $q_1 = 2$ and $q_2 = 14$. At this outcome, firm 1 makes profit of 2, firm 2 makes profit of 0, and consumer surplus is 128. Thus the total surplus is 130.

(b) (3 points) Suppose that the two firms now compete in quantities (i.e., they play a Cournot game). Write down the payoff functions $\pi_1(q_1, q_2)$ and $\pi_2(q_1, q_2)$ and derive the best-response function $q_1^*(q_2)$ and $q_2^*(q_1)$.

**Solution:** Firm 1’s payoff function is,

$$\pi_1(q_1, q_2) = (18 - q_1 - q_2)q_1 - 0.5q_1^2$$

Deriving the FOC with respect to $q_1$ and solving yields the best-response function,

$$q_1^*(q_2) = 6 - \frac{q_2}{3}$$

Firm 2’s payoff function is,

$$\pi_2(q_2, q_1) = (18 - q_1 - q_2)q_2 - 2q_2$$

Deriving the FOC with respect to $q_2$ and solving yields the best-response function,

$$q_2^*(q_1) = 8 - \frac{q_1}{2}$$

(c) (3 points) Find the Cournot-Nash equilibrium output levels $q_1^*$ and $q_2^*$.

**Solution:** Solving the best-response functions yields output levels $q_1^* = 4$ and $q_2^* = 6$
(d) (3 points) What is the market price, \( p^* \), in the Cournot-Nash equilibrium? How much consumer surplus, \( CS^* \), is generated? How much profit, \( \pi_1^* \) and \( \pi_2^* \), does each firm make?

**Solution:** The market price is \( p^* = 8 \). Consumer surplus is \( CS^* = 50 \). Profit to firm 1 is \( \pi_1^* = 24 \) and profit to firm 2 is \( \pi_2^* = 36 \).

(e) (3 points) How much dead weight loss is generated in the Cournot-Nash equilibrium? Identify and calculate the sources of inefficiency.

**Solution:** The difference between total surplus in the efficient outcome and the Cournot-Nash outcome is \( DWL = 130 - 110 = 20 \). Of this 18 derives from inefficiently low production and 2 from the fact that it is efficient for firm 1 to produce only 2 units opposed to the 4 it produces in the Cournot-Nash outcome.

6. (20 points) Robinson Crusoe lives alone on a tropical island where he spends 12 hours a day either fishing or relaxing. If he spends \( l \) hours fishing then he catches \( y = 2 \sqrt{l} \) fish and enjoys \( h = 12 - l \) hours of leisure. Robinson’s preferences over hours of leisure, \( h \), and fish, \( y \), are represented by the utility function \( u = \ln h + \ln y \).

(a) (2 points) Find Robinson’s production possibility frontier (\( PPF \)), \( y^*(h) \).

**Solution:** Because \( y = 2 \sqrt{l} \) and \( l = 12 - h \),

\[
y^*(h) = 2 \sqrt{12 - h}
\]

(b) (3 points) Find the allocation, \( (h^*, y^*) \), that maximizes Robinson’s daily utility subject to the resource and technological constraints he faces.

**Solution:** Robinson’s \( MRS = -\frac{y}{h} \) and his \( MRT = -\frac{1}{\sqrt{12 - h}} \). Hence the tangency condition is,

\[
\frac{y}{h} = \frac{1}{\sqrt{12 - h}}
\]

Substituting from the \( PPF \) gives,

\[
\frac{y}{h} = \frac{2}{y}
\]

or \( y^2 = 2h \Rightarrow y = \sqrt{2h} \). Substituting this for the left side of the \( PPF \) gives,

\[
\sqrt{2h} = 2 \sqrt{12 - h}
\]

Solving this gives \( h^* = 8 \) and \( y^* = 2 \sqrt{12 - 8} = 4 \).

(c) (2 points) In a graph with \( h \) on the horizontal axis and \( y \) on the vertical axis, sketch the \( PPF \) and the highest indifference curve Robinson attains.

**Solution:** See graph below.

Now imagine that there is a labor market and a fish market in Robinson’s one-man economy and the Robinson is a price taker in both markets. Suppose also that Robinson has a split personality: Robinson the firm owner demands labor and supplies fish, while Robinson the consumer supplies labor and demands fish. Let the wage rate be \( w = 1 \) so that the price of fish is in terms of labor hours.

(d) Derive Robinson the firm owner’s:

i. (1 point) cost function, \( C(y) \)
**Solution:** Inverting the production function gives the labor requirements function $l = \frac{y^2}{4}$. Because the wage rate is normalized to 1, the cost function is simply, $C(y) = \frac{y^2}{4}$.

ii. (1 point) supply function for fish, $y_S(p)$

**Solution:** The firm solves

$$\max_y \pi = py - \frac{y^2}{4}$$

The first-order condition is

$$p - \frac{y}{2} = 0$$

Solving for $y$ gives,

$$y_S(p) = 2p$$

iii. (1 point) profit function $\Pi(p)$

**Solution:**

$$\Pi(p) = pys(p) - C(y_S(p)) = p^2$$

(e) (2 points) Recalling that Robinson owns the firm, specify his budget constraint for fish and leisure hours.

**Solution:** Since Robinson has claim to the firms profits we must account for this in his budget constraint.

$$py + h = 12 + p^2$$

(f) (2 points) Derive Robinson the consumer’s demand function for fish, $y_D(p)$.

**Solution:** The consumer solves

$$\max_{(y,h)} hys.t, py + h = 24 + p^2$$

The tangency condition yields $py = h$. Substituting for $h$ in the budget constraint gives

$$2py = 12 + p^2$$

So demand is

$$y_D(p) = \frac{6}{p} + \frac{p}{2}$$

(g) (2 points) Find the equilibrium price, $p^{**}$, and quantity, $y^{**}$, of fish.

**Solution:** Equating supply and demand gives

$$2p = \frac{6}{p} + \frac{p}{2}$$

Solving yields $p^{**} = 2$. Plugging this into supply (or demand) gives $y^{**} = 4$.

(h) (2 points) Use Walras’ Law to find the amount of labor Robinson the firm owner demands and Robinson the consumer supplies in a general equilibrium.
Solution: Because the fish market clears when $p = 2$ and $w = 1$, Walras’ Law implies that the labor market must also clear at these prices. So $y^{**} = \sqrt{l^{**}}$ or $l^{**} = 4$.

(i) (2 points) Briefly discuss how the First-Welfare Theorem applies here.

Solution: The allocation that maximizes Robinson’s utility subject to his PPF is the same as the general equilibrium allocation.